Introduction to Fluid Mechanics

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Lecture - 41

Reynolds Transport Theorem (RTT)

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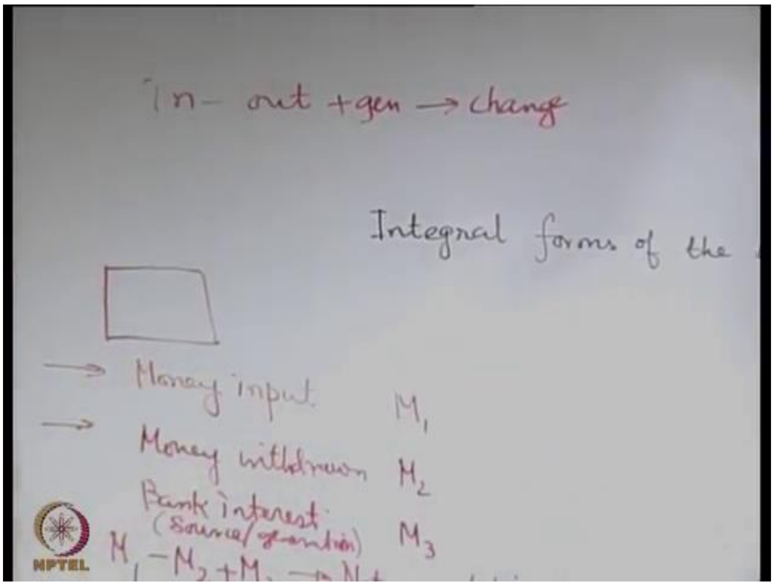

Today, we will start with the new chapter, which is integral forms of the conservation equations. This is the logical extension of some of our previous discussions, where we were discussing about the differential forms of the conservation equations. And some of the differential forms of the conservation equations, where in terms of the conservation of mass like we discussed about the continuity equation and conservation of the linear momentum may be expressed to the Newton's second law of motion that was the Euler's equation of motion for an inviscid flow. And we saw that subsequently it could give rise to a form of a mechanical energy conservation also, but most of those forms that we disused where of differential in nature. And we will now see we will try to look into the feature of integral forms of these conservation equations.

But before going into that we must try to develop a feel of what we mean by these conservation equations, and how they are important in fluid mechanics. So, when we talk about a conservation equation, we are talking about certain mathematical form which represents the physical meaning of conservation of something. And we will try to see that what is the basic physical principle that gives rise to the sense of conservation of whatever may be mass, momentum, energy or whatever and how we can express that in an equivalent mathematical form, so first of all we will try to get a feel of what is a conservation principle.

So, when we talk about the conservation principle, it may not be a bad idea to discuss about it in a bit of a more general frame work that is not very specific to may be mass or momentum or whatever. Let us take an example which is totally deviated from fluid mechanics, so that it is not as boring as many of the issues of fluid mechanics let us talk about the conservation of money. So, money something which excites most of you and so I hope that you will listen to whatever example that we are talking about. Let us say that you have gone to a bank to open a bank account. So, your bank account starts at a given instant where let us say this is like a symbolic representation of your bank account. So, you have put some initial money in your bank account, and the back account operates from then onwards.

Now, say you are working in a job, earning a fabulous salary, but you are having such a great amount of money, otherwise that you do not care about what is going on in your bank account. One fine morning, you feel that well you need to see what is there, I mean let us think about traditional way of banking, of course these days you go to like you log onto your computer to see what is there in your bank account and you do lots of transactions. Let us say you physically go to the bank. So, we physically go to the bank and you put a query that like how much is there in your bank account, because you want to withdraw some money also not that you need that money, but you want to just make your account keep your account in a regular form.

So, you have seen that what is there in the bank. Now, you what you see initially there was some money that was put in the bank. Now, when you have gone to the bank and even if you have not enquired of what is the total amount of money that is there may be you have an estimate of what is there, and you have withdrawn some money. So, you have some money withdrawn. And if the bank is good enough, it depends on the economy, but poor at the economy the bank is better for you to give you a good interest. So, the bank at the end, you will find that whatever money that was originally input or may be was being input month after month by your employee as your salary, and the money that you are withdrawn it is not that they are just the 2 important constituents, you also have a bank interest.

So, there is a bank interest. So, bank interest. So, if you have originally put money input of say M 1 not originally, but M 1 first I mean may be month after month after it has got accumulated total money amount input say is M 1 money. Withdrawn if it is M 2, say the bank interest is M 3 then at the end of the transaction and everything that you are coming out of the bank may be you are happy or not so happy whatever what remains in the bank it is. What it is M 1 was the original input you had withdrawn M 2 there is a bank interest, which is favour of you, so this is the net accumulation.

This is the change in your bank account that you see maybe before visiting the bank, and after visiting the bank. So, this net accumulation is the sort of like coming out of a conservation of money here. So, some money was the input apart of that was withdrawn something has been so called bank interest like a is like a generation of source, source or generation does not mean that money is automatically generated bank of course, invest it in different ways and that is how you get the interest. But I mean from your point of which is like as if there is a source of money or generation of money.

So, you have an input, you have something which has been withdrawn and there is an interest. So, the net effect of that taken algebraically with plus or minus is the net accumulation that is there in your bank account. So, these talks about principle of conservation of say money in a bank account say the bank account is like a control volume. So, across which there is something which is entering and something which is leaving, we have talked about mass entering, momentum entering, but here it is like money entering and money leaving.

But if you now abstract yourself bit from money to something which is a big generic one then it is like something has entered. So, something has entered minus something has left in minus out. So, in is like input, out is like withdrawn, then something is generated this is like interest is leading to a net change. So, this is very simple you cannot even think of simpler way of looking into conservation, and believe me or not whatever equations of integral or differential form that we write in fluid mechanics representing conversation of various quantities fundamentally follow these principles. So, we will try to see that how from these fundamental principles, we can develop or we can derive some of the very important conservation equations in fluid mechanics.

When we want to do that, we need to keep in mind that there is a general way of looking into this conservation as represented by the simple statement. So, our objective will be to express this simple statement in a somewhat mathematical sense, and how we do that there is a very important theorem that tells us that how to go about that expression.

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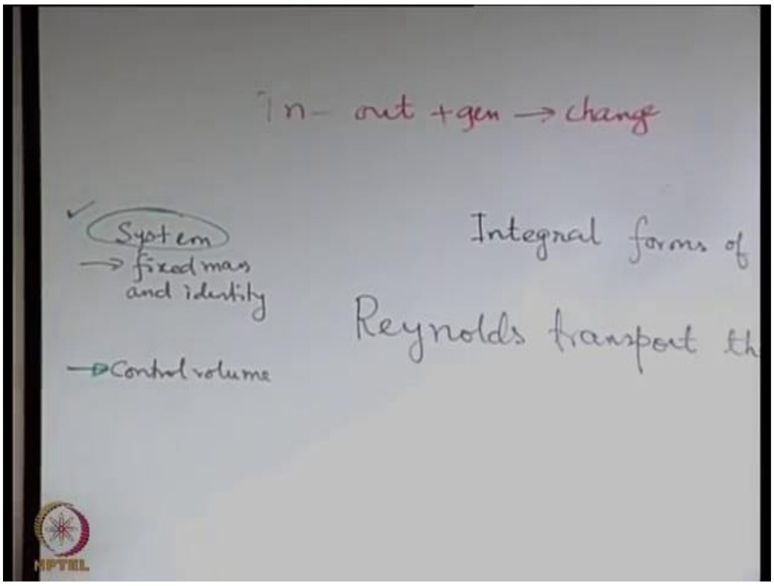

And that theorem is known as Reynolds transport theorem. Reynolds transport theorem is so powerful that if you know how to make use of this theorem, you can derive any conservation equation in fluid mechanics by using this theorem. So, this is very, very powerful. I mean exploiting these it is really possible to derive any equation in fluid mechanics. So, one will the tempted to look into the perspective of this theorem that is what this theorem is essentially trying to do, so before going in to the theorem let us try to understand the motivation behind the theorem or in particular the Reynolds transport theorem here.

So, if you recall we have discussed many times that the difference between a traditional way of looking into mechanics say particle mechanics and fluid mechanics is the reference frame mostly like a in particle mechanics, you looking for a Lagrangian reference frame; and in fluid mechanics mostly we are discussing in the context of an Eulerian reference frame. The reason is obvious that fluids are continuously deforming and it is virtually impossible or very difficult to track individual fluid particles and see that how they are evolving.

So, the approach that fix mostly with fluid mechanics is a control volume approach. On the other hand, all the basic equations, which have been classically developed in mechanics have been not based on a control volume, but based on something which is of a fixed mass and identity and that we call as a system that also we have discussed earlier. So, we have something has a system which is of fixed mass and identity, and we have something as a control volume, which is an identified region in space across which any mass, energy or whatever can flow. And the apparent travel is that all our conservation equations classically have been developed for a system, but we in fluid mechanics try to express that in terms of the corresponding phenomenon over a control volume. So, we require a transformation from a system approach to a control volume approach, so that the same type of equation let us say Newton's second law, which is there, originally defined for a system we want to express and equivalent form for a control volume.

So, we require a theorem that gives us a kind of transformation law in perspective of what happens across the system with respect to what happens across the control volumes. So, in very brief transformation from system to control volume or maybe vice versa from control volume to system and that is given by this Reynolds transport theorem. So, the motivation of these is to get general formulation which will try to give us a guideline of how to have a transformation from a system approach to control volume approach. So, to do that, we will start with like very simple way of looking into the theorem and maybe we start with the sketch of what happens to a system and what happens to a control volume.

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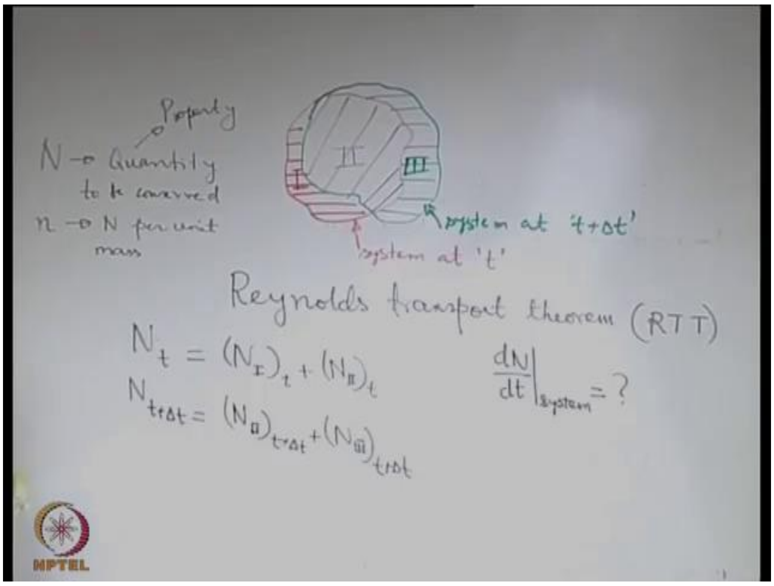

So, let us say that we define a system with its system boundary like this. So, this is an arbitrary way of just sketching a system. What it means that there is something may be some fluid inside this, this is the boundary, and the fluid particles, which are there insides this, these are identified, so that is the system. Let us say that this is the system boundary at time t.

Now, let us consider a small time interval of delta t; over the small time interval of delta t. This system boundary has now occupied a different configuration in reality if the delta t is small these configuration which is there with the green colour is almost merging with that of what was the configuration at time t drawn with the red colour. Just for distinguishing these 2, we have really amplified the change here in the figure, but keep in mind that these 2 envelops of the system which are the so called system boundaries will be actually almost merging on one on the top of the other as the delta t time interval tends to 0. So, this we call as system boundary at time t plus delta t, keeping in mind delta t is very small. So, this is just to isolate the 2 system boundaries that we have drawn in a magnified way, but please keep in mind that these 2 are almost coincident not that they are coincident, but they are virtually falling one on the top of the other.

Now, when we have drawn a sketch like this, you can clearly see that there are three important regions in this sketch. And one important region straight away is the common intersection between the 2 configurations. So, let us identify that mark that in this way.

Then there are other parts like you have this are one part, where it is solely belonging to the system which was there at time t; and there is a third part which solely belongs to the system at time t plus delta t.

So, this three zones just for our own convenience, let us give some name let us say the name of the first zone, which is marked with the red one is 1. The next zone is 2, and the third one that is to the right is 3, these are just three different volumes that we identify for our own convenience of demarcation. Now, what we can say? Let us say that we want to write a conservation law for some quantities. So, we are not really committing what is the quantity, keep in mind the quantity that we are interested to conserve may be mass, may be momentum, may be energy or whatever, but let us say that we are interested to conserve some quantity.

So, let us say that capital N is the quantity that is conserved. Quantity to be conserved means it is satisfying the basic conservation principle that we discussed through an example before this. And let us say that small n is capital N per unit mass. So, capital N is like it is a total and small n is that expressed per unity mass and that total may be any property. So, when you say quantity, it is better to like call it property, and what property that we will see. Now, when we say a property that we need to conserve and we when we express the property also as per unit mass that means, we are trying to implicitly make a statement that capital N depends on the total mass of the system. So, it is not any property, but some property which depends on the total mass, and that we call as the sort of extensive property that is the property that depends on the extent of the system. And when we express it per unit mass, we call it a specific property that is the property expressed per unit mass.

So, when we look for such a property we will look into examples to see that what could be properties, but right now we are very, very abstract in a mathematical way of looking into this. So, when we say that capital N is the property that is there. So, when we say capital N at time t. So, capital N at time t is nothing but whatever is capital N at which is being occupied which is because of the volume occupied by the region marked as 1 at time t plus capital N for the region which is occupied by the marked 2 at time t. So, we can say that N at t is N of 1 at t plus N of 2 at t.

So, when we are describing the N, we describing the N in terms of the system that you can clearly see the reason is that when you are calling it that is it is dependent on the total mass of the system. So, we must have an identified mass. So, these fundamentally is originating from a system concept that you have a system with identified mass, there is some property which depends on the total mass and the total property is the some of the properties over the regions 1 and 2 very straight forward.

Now, what happens to N at time t plus delta t. This at t plus delta t it occupies the regions 2 and 3. So, it is N at 2 at time t plus delta t plus N at 3 at time t plus delta t. What is our interest, our interest is; what is the rate of change of N? So, we are interested to find out what is the d N d t of the system this is our objective.

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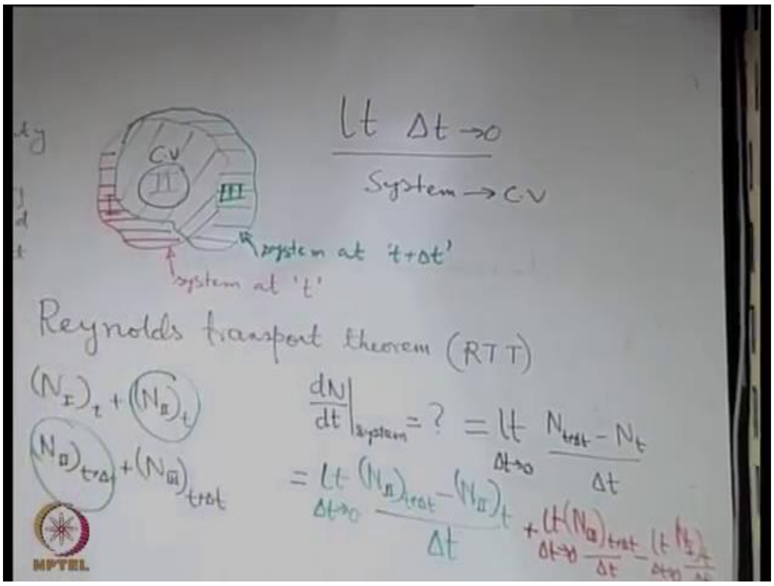

And for that we can write it in the fundamental definition of a derivative as limit as delta t tends to 0 N at t plus delta t minus N at t divided by delta t. So, the basic definition of the derivative becomes applicable here also, there is no reason that it should not be applicable.

Now, we are interested to write these term in a bit of more explicit way. So, when we write this N at t plus delta t and N at t, you can clearly see that there are again different types of terms. So, when we subtract N t plus N t from N t plus delta t, you have to keep in mind that this terms N at 2, it is a very special term because it refers to the same region, but different times. So, these 2 terms may be grouped together to give a very special meaning of what is the change over and identified region. On the other hand, the other terms which are remaining they are having a different interpretation and we will see, what is the interpretation?

So, mathematically we will write it or we will express this or break it up into 2 different terms. So, the first term will be limit as delta t tends to 0, N at 2 at time t plus delta t minus N at 2 at time t divided by delta t then plus the other terms N 3 at time t plus delta t by delta t minus limit as delta t tends to 0 N 1 at time t divided by delta t sort of that. So, the next step that what will be doing is very important conceptual interpretation of what is there as the limit as delta t tends to 0.

So, let us try to figure out a case when you have the limit as delta t tends to 0. So, this is a very important conceptual pyridine. So, limit as delta t tends to 0 is not just a mechanical way of like evaluating the limit and therefore getting a derivative. When in the limit as delta t tends to 0, we have already hinted that then the system boundaries at the 2 time instance are almost coincident that means, in such a case if you call the region 2 as a control volume this becomes identified. So, you may focus a control volume is what it is an identified region in space over which you may focus your attention and see what is the change that is taking place.

So, in the limit as delta t tends to 0, the uncertainty in the configuration of the system at the different time instance is not that strong because the time interval is so small that it is almost like identified that where the things are. And let us say that somehow you have identified this region 2 as your control volume. So, if you identify the region 2 as your control volume in the limit as delta t tends to 0, your system tends to the control volume that you have to keep in mind.

So, always not system is a control volume, but in the limit as delta t tends to 0, again it is not equal to control volume. The system tends to the control volume because the configurations are almost merging on the top of the other. The whole idea is that using that limit, we will be able to express what happens in the system in terms of what happens across the control volume. So, in the limit as delta t tends to 0, system tends to control volume. And what is the control volume that we have identified this region 2 is the control volume that we have identified. Keeping that in mind; let us try to simplify the expression for d and d t for the system further.

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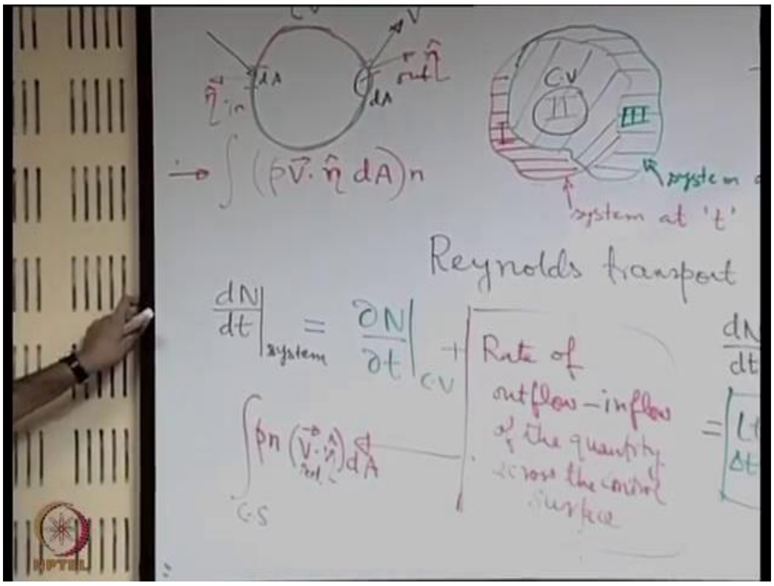

So, d N d t for the system is equal to now let us try to write the different terms mathematically. So, let us first concentrate on the first group of terms. So, what does it represent, it represents the time rate of change of N in the control volume, because 2 is the control volume. And we will write it as the partial derivative of N with respect to time within the control volume. Why partial derivative with respect to time, N is in general of function of position and time you have fixed up the position. So, at a given position which is like 2 given symbolically by 2, you are trying to see that what is the rate of change with respect to time that is why a partial derivative.

Now, this is not the only term there are 2 other terms which are remaining and instead of writing those 2 mathematically at this stage, let us try to understand physically what they are. So, that physical understanding we will try to write in some sense here. So, physically in the limit as delta t tends to 0, what happens then what does the fluid which is being located at one at time t, the rate of change of that what does it represents in the limit as delta t tends to 0? Say this is a very small region, so this represents some fluid which is being located on the surface of the control volume. So, in the limit as you tend t tend to 0 these volume will spring to almost like located on the surface of the control volume.

So, at the surface of the control volume, there is some fluid; and in the limit that you have delta t tends to 0, then the term corresponding to one at t what does it represent. It represents a rate process because you are having division by delta t in the limit as delta t tends to 0. So, per unit time it is representing something. So, per unit time is a rate process. So, when the system tends to control volume, these volume effects almost tends to a surface effect. So, at the surface of the control volume at the time t, it represents that there is some rate of transport or the rate of influx of the quantity that you are looking for. So, it is like the rate of inflow of the quantity across the surface of the control volume that is called as the control surface, surface of the control volume.

Similarly, the term which is there for the region 3 in the limit as it delta t tends to 0; this represents sort of the fluids with the property which is ready to leave the control volume at that instant. So, when it is divided by delta t in the limit as delta t tends to 0, then N 3 term that sort of represents the rate of out flux the rate of outflow of the quantity that you are looking for. So, this when you have the difference between these 2 it is nothing but rate of outflow minus inflow of the quantity across the control surface, so that is the physical meaning of the last 2 terms.

So, this first thing, you have to appreciate that these are not volume metric phenomenon these are surface phenomena. And this surface phenomenon occurring as a consequence there is some property which is entering the control volume, there is some property which is leaving the control volume. And it the leaving one is like with the positive sign and entering one is with the negative sign. So, it is like the net rate of out minus in, just in a very qualitative way. And of course, we will try to write it in more mathematical way in the subsequent steps.

Let us try to do that. To do that what we will keep in mind is that we have already defined that small n is the capital N per unit mass. Now, let us say that there is a boundary located on the control volume, let us draw the control volume separately. Let us say that this is the control volume. And we are interested to develop a mathematical expression for the rate of outflow of the quantity. So, how we do that? So, rate of out flow depends on what, here we are talking about fluid flow. So, if there is a property, the property is being transported by the fluid. So, if the fluid enters, it takes some property with itself; if the fluid leaves, it also takes some property with itself.

So, let us say that you identify small area on the surface over which the fluid has a velocity v. So, the fluid let us say is going out of the control volume over the surface it is a maybe it is a part of that surface over which it is leaving. So, it is a part of the out flow boundary. So, out of the total boundary of the control volume, there may be some part over which there is no inflow or outflow may be its like a wall, there may be some part across which fluid is entering or leaving, maybe it is like a hole. So, this is such a place across which the fluid is leaving as an example.

So, the fluid is leaving with the particular velocity and it is arbitrary some arbitrary velocity vector. The area that is being identified we have seen it many times that area has an important directionality. So, this area has to be identified with unit vector n. So, do not confuse these with the other n that we put as a symbol for the property per unit mass. So, this is unit vector or maybe you may call it eta, just to avoid the same symbol. Now, what is the total rate of property that is coming out of this one, we have to keep in mind that small n is the property per unit mass. So, if you find out what is the rate of flow of mass over this multiply small n with that that will be the rate of transport of n through this flow of mass.

So, first off all our objective is there for to find out what is the rate of flow of mass over these. What is that? So, for the flow of mass what is important is; what is the component of the velocity normal to the area? So, that is given by the dot product of V with n. So, V dot n is the velocity component normal to the area that multiplied with d A is the volume flow rate through the area that multiplied by the density row is the mass product to the area and that multiplied by n which is the property per unit mass gives the rate of flow of the property through this area. And if you integrate it over the area over with fluid is flowing out then such an integral this is an area integral that should give the rate of out flow.

Now, what about in flow, let us see that is it difference from these or is it same as this let us take a different case, maybe just cased on the same control volume, where you identify again a small area d A, now fluid is entering the control. So, the fluids are entering the control volume, and let us say that we identify the direction normal eta in this way and other things hold similarly. So, what is the rate of inflow of the quantity through this? Again it goes by the same principle. And if you evaluate this algebraically, you see a difference between the outflow and the inflow. What is the difference, difference is here v dot n will be positive, here v dot N will be negative; that means, when we have an outflow minus inflow, algebraically we may treat it in the same way.

We will just use this term automatically that positive dot product will tell us that it is outflow, and minus will tell us what is inflow.

So, we do not have to separately treat the outflow minus inflow it is just like a net flux. And therefore, this total term we may express in what form integral of rho n v dot n dA integral over the control surface this is the total integral control surface is the surface of the control volume. So, wherever there is some velocity, we will put that velocity the velocity may locally vary along the surface that is why this integral is there. Whenever v dot n is positive, it is out flow; when whenever v dot n is negative or v dot eta either v dot eta is positive it is out flow; whenever it is negative that is inflow. So, we need not have a separate consideration for outflow minus inflow. This algebraically takes care of everything.

The other point and which I believe is the very important point is this V is not the absolute velocity of flow, because the flow depends on what is the velocity of the fluid relative to the control volume. Let us say the control volume constitutes tank, and water is coming out of the tank through a hole. And let us say the tank is moving with the velocity say 1 meter per second in a certain direction, and the fluid velocity also 1 meter per second in the same direction. So, there is no net flow of the water relative to the tank. So, there is no mass that comes out. So, this V that what we have put here it should not be the velocity of the fluid in an absolute sense, but velocity of the fluid relative to the control volume. So, this must be amplified with v relative. So, all these V is that we have talked about these are V relative and involved these v relative means velocity of the fluid relative to the control surface.

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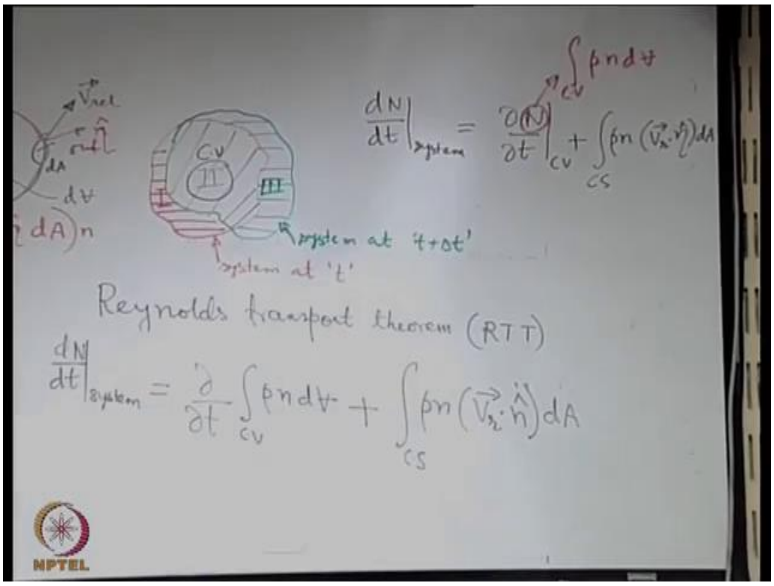

So, you have now seen that we may write the expression of these system derivative in terms of the control volume derivative in bit more compact with than the previous state that is we may write d N d t for the system is equal to this for the control volume plus integral of rho n V r. We just write in short V r as V relative v r dot with eta d A over the control surface. We may simplify this bit more, if we want by considering that what is capital N, capital N is like let us try to simplify the first term in the right hand side. So, if you have a control volume the control volume has different small chunks of volume element let us say d V is a small chunk of volume element in the control volume, we are using this symbol for volume to distinguish it from velocity. So, you have small volume element.

So, what is the total property of this volume element, small n is the property per unit mass. So, if you multiply that with the mass what is there in this volume that will be the total property within this volume; and that integrated over the control volume will give the total capital N of the control volume. So, the capital N of the control volume, this is like what, this is the partial derivative with respect to time of capital N of the control volume. So, what is the capital N of the control volume this integral of so first what is the elemental property that is rho small n d V, this is the volume integral you have to remember. So, this is these are symbolic. So, like all the; we are expressing it with just a single integral sign it depends on how what kind of coordinate system you are using and so on. So, if in a more mathematical way, this should be a double integral, this should be a tribal integral like that, but these are symbolic representation that you keep in mind. So, this is like the total property capital N within the control volume.

So, the final expression, we may write in this way that is d N d t of the system we may express very neatly in terms of the rate of change within the control volume. So, we can say that d N d t of the system is equal to the partial derivative with respect to time of integral rho n d v of the control volume plus. So, what objective we have achieved by this excise we have now been able to express the rate of change of some quantity in a system in terms of the rate of change within the control volume, and that 2 are adjusted with each other by a term which physically represents the net rate of flow across the control surface. So, this sort of represents the conservation. And this is known as Reynolds transport theorem.