Introduction to Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 04
Cauchy's theorem

So, we will now go to our next objective that is given this components of the stress tensor, how we may utilize these concepts or these components to designate the state of stress, on any arbitrary surface which is neither oriented along x y or z.

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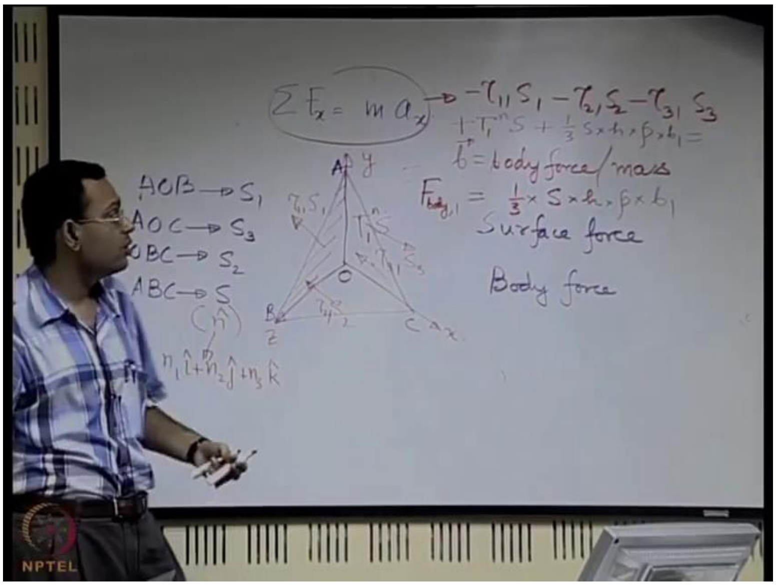

For that what we do is, we consider an elemental volume like a tetrahedron, we give the point certain names for convenience. So, there are surfaces like this the surface AOB let us call it as S 1, surface AOC we call it as S 3, why such 1 and 3 because this index we are trying to preserve for the direction normal of those surfaces.

So, this 1 is for the fact that the direction normal of this AOB is along x. Similarly this 1 is S 2 and ABC let us give it a name S. Can you tell what is the motivation of this taking such a volume? See whenever we are deriving something in the class, it is like it will appear to you that yes it has to be done like that; remember it is not a ritual do not accept anything whatever we are learning in the class as a ritual, always try to ask yourself a question why I have taken such a volume, what is the motivation behind taking such an element. So, if you see this element has 4 phases, out of these 4 phases 3 are the special ones which I have their direction normal either along x y or z, the fourth one is not a special one it is arbitrarily oriented.

So, now by considering the equilibrium of this element, by considering the forces which are acting on it, we will be able to express what is there on that odd surface in terms of what is there on the special surfaces. So, that is what is the motivation behind taking this one. Now whenever we are coming to such an element, our objective will be say to write the Newton's second law of motion for this. So, just resultant force equal to mass into acceleration; question is what forces are acting on this element. So, when we say what forces are acting on the element, we will be classifying the forces in continuum mechanics in 2 categories: one is a surface force, another is a body force; these names are almost self explanatory.

So, when you say surface force it means that these are forces which are acting on the surface or surfaces, which are comprising the volume element chosen. And body force is a force which is acting over the volume of the body; example is body force one of the examples of body forces is the gravity force which acts throughout the body, volume of the body. Surface force pressure is an example, force due to pressure is a surface force. So, whenever we are having forces we will categorize in terms of surface force and body force. So, wherever we have surface force the surface force may be expressed in terms of the traction vector, because the traction vector we have defined in such a way that on a surface it represents the resultant force per unit area.

So, it represents a cumulative effect of all forces which are acting at that point on the surface. So, this particular element has 4 surfaces; let us write the force forces which are acting on these 4 surfaces, and write the Newton's second law of motion along the x direction. So, what we are going to write is resultant force along x is equal to the mass of the fluid element, times the acceleration along x. So, when we write the resultant force will write it in terms of the surface force and body force.

So, first let us come to the surface say AOB. So, on the surface which component of the stress tensor will give a force along x in terms of tau I j.

Student: tau 1 1.

Tau 1 1 right. So, what would be the positive sign convention direction of tau 11.

Student: (Refer Time: 05:41).

Along this. So, tau 1 1 this is the force per unit area. So, what is the area on which it is acting into S 1. Similarly for S 2 what is the force which is acting along 1 tau 2 1 right what would be its direction? Just like this because the its outward normal is along negative 2 direction therefore, the positive sense of tau 2 1 on the surface is along negative 1. So, and this multiplied by S 2; similarly this will be tau 3 1, S 3. There is a fourth surface which is really the back surface here which has its normal neither along x y or z.

Let us say that normal to this is n cap, the normal vector outward normal vector of S. So, this n cap say it has its components like this n 1 I cap, plus n 2 j cap, plus n 3 k cap where n 1, n 2, n 3 are the components of this along x y and z. We are now going to write the force on S. So, the force on S let us say that it is T with superscript n. So, now, we cannot use the tau notation for that, because it is not a special surface tau notation you can write for a special surface where the normals are along x y or zee. So, we use the t notation this is for the ABC. This component wise it is along 1 and the area on which it is acting is S and; obviously, by default we are taking it as along positive x, these are the surface forces what is the body force? Say we call that b is the body force per unit mass.

So, if we find out what is the mass of the fluid element. So, what is the mass of the fluid element? So, let us say that we find out first what is the volume of the fluid element, so for this type of element we can say that it is one third of one third into the area of this ABC times, the perpendicular distance from O to ABC, let us say that perpendicular distance is h. So, we first find what is b 1. So, what is b 1? First the volume one third into S into h, where h is the perpendicular distance from O to ABC, this is the volume mass of this you multiply it by the density say rho is the density. So, this is the mass and sorry this multiplied by the body force per unit mass along x, will give the total body force along the direction 1 or x. So, this is the mass this is the body force per unit mass along x. So, the product is the total body force along x right.

So, we write the newton second law of motion, this particular expression now we write the forces. So, minus tau 1 1 is 1, minus tau 2 1 is 2, minus tau 3 1 S 3, plus T 1 with superscript n into S, that is the total surface force plus the total body force, that is the net force which is acting on it and that net force is equal to its mass into acceleration.

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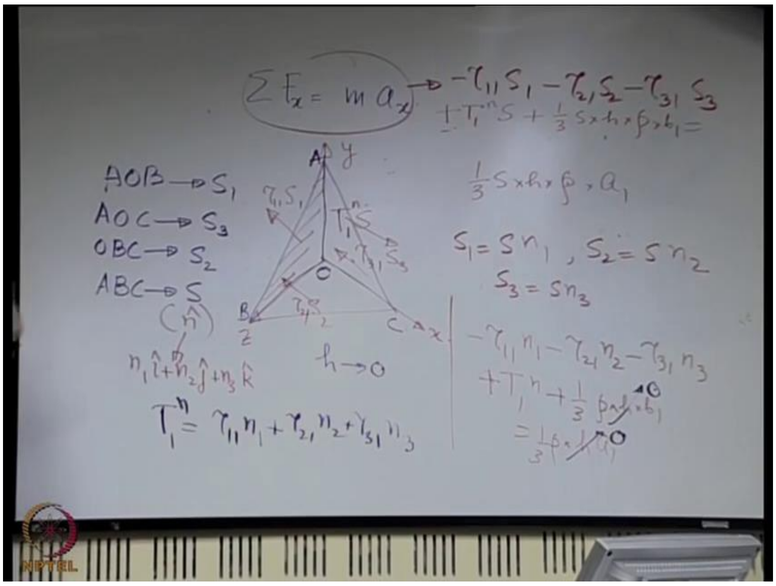

So, that is equal to what is its mass one third S h rho that is the mass let us say a is the acceleration. So, a 1 is the acceleration along x 1.

Student: (Refer Time: 11:21).

Yes.

Student: can you sir will we kept (Refer Time: 11:26) what is surface force and body force.

Physic the physical definition and the mathematical definition is identical here, surface force is a force which is distributed over the surface which is the envelope of the volume that is being considered, and body force is something which is acting within the volume of the body.

Student: within the body.

Yes.

Student: So, like surface force is shear force.

Yes shear force is the surface forces an example.

Student: only shear force (Refer Time: 11:53).

Also normal force just like pressure is a normal component is not a shear component.

Student: actually 1 term is (Refer Time: 12:01) just tau 1 1 raise to S 1.

Yes

Student: it is also acting normally to the body.

But it is it is a force which is which is acting on the surface, I mean whether normally to the body or not it is it is a matter of direction, but the force may act on the surface or the force may be acting throughout the volume of the body, that is how this is classified. So obviously, when we are considering this this is something which is acting on the surface; obviously, it will have a direction right I mean there is no contradiction with that with a surface forces and body force, surface force will have a direction body force will also have a direction.

So, when you have this expression next is you can write S 1, S 2 and S 3 in terms of S how can you write it? See S 1 is like the projection of S on.

Student: (Refer Time: 12:57).

The y y z plane right. So, how do you find out the projection? You find out basically the component of this so called vectorial representation of ABC on the y z plane so; that means, when you want to find out the component you basically find the dot product of the corresponding unit vectors. So, this has a unit vector in the direction of ABC has unit vector in the direction of n, AOB has unit vector in the direction of.

Student: minus i.

Minus i so, but; obviously, here the plus minus you are already taking care of through this sign convention. So, you are not duplicating it once more. So, the dot product of those 2 directions will be in one is this vector another is i. So, the dot product will be n 1. So, in terms so these are all magnitudes, their senses have already been taken care of with plus minus. So, S 1 is nothing, but S n 1 by taking the component of so called S in the direction of the so called S 1.

Similarly, S 2 is S n 2, and S 3 is S n 3. So, in that way if you substitute in this equation you will see that S gets cancelled out. So, what you will get minus tau 11 n 1, minus tau

2 1 n 2, minus tau 3 1 n 3, plus t 1 plus one third row into h into b 1 is equal to one third row into h into a 1. When you have these equations the next consideration that we have to make is something which is subtle, but important to understand. We will shrink this volume to a point such that this entire volume as if converges to the point O, because our end objective is to find out the state of stress at a point in terms of an area chosen around that point. So, we will be considering a vanishing area or vanishing volume. So, to say not a vanishing area, so that everything converges to O; that means, we will taking the limit as h tends to 0. So, when you take h tends to 0 the entire volume will converge to the point O then whatever we describe basically is the description of state of stress at a point O. So, when you take that limit as h tends to 0 you will see very beautifully these terms will tend to 0.

So, in that case you are left with a very simple expression for t 1 that is tau 1 1 n 1, plus tau 2 1 n 2, plus tau 3 1 n 3. You can see that this is a very excellent expression because it relates the traction vector on an arbitrary surface with the components of stress tensor. What are the inputs? The inputs are that you must know the state of stress on those specified orient planes with specified orientations, and the components of the unit vector direction of the arbitrary plane that you are considering everything at a particular location. The other thing is that there is a way of writing these symbolically in a more compact manner.

You can see that here you are having 2 indices. So, the first index is what is varying, the second index is something which corresponds to this one.

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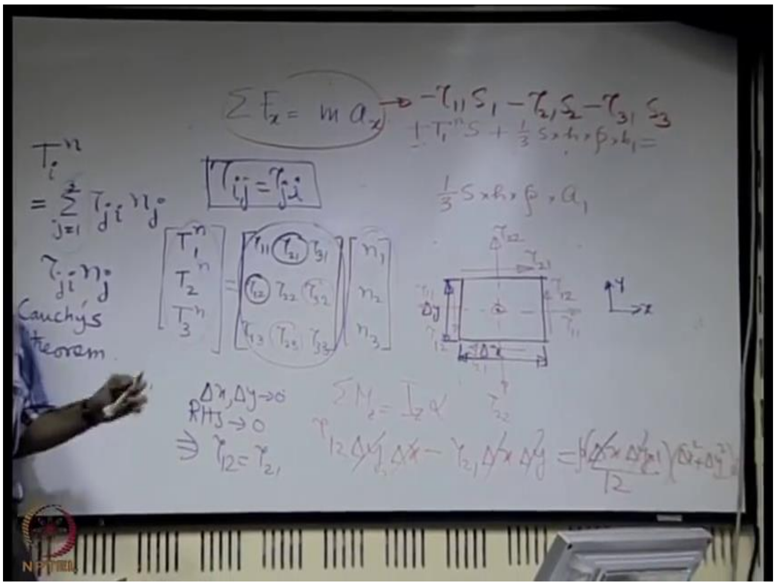

So, you can just generalize it and say right. So, you have 1 index i which is fixed, the other index j which is a variable which varies from 1 to 3. Now because this type of notation is very common, the general rule is again the general notation is that this summation is omitted. So, this becomes invisible. So, this is also written as just tau j i n j how you will know that there is a summation? Whenever there is a repeating index you have to keep in mind that there is an invisible summation in it. So, from now onwards many times we will be using this notation, without using the summation symbol, but have to keep in mind that whenever there is a repeating index there should be an invisible summation over that, this is known as Einstein index notation this was first introduced by Einstein.

So, this type of notation gives a very compact way of writing these terms of writing this traction vector in terms of the components of the stress tensor. So, this is known as cautious theorem this expression. So, what it does is, it expresses the traction vector on any arbitrary plane in terms of the corresponding stress tensor components. It is also possible to write it in a matrix form. So, what you can do, you are having components of the traction vector on a given plane say with orientation n, 3 components you have.

So, this if you just follow these expressions you will see that. So, you can see that this is nothing, but a matrix way of writing the 3 components of the traction vector. So, whatever equations that we have written this is not something new, just put I equal to 1 it will correspond to the first row of this I equal to 2 and I equal to 3. So, what you can see is that here you get those so called nine components of the stress tensor; and these all these nine are not independent we will see, but this is something what it is mathematically doing you can see here look into these quantities what is this this is a vector it has its 3 components n 1, n 2, n 3 this also a vector. So, this is acting like a transformation which maps a vector on to a vector. So, it is a very important characteristic of a second order tensor, that a second order tensor maps a vector on to a vector.

Similarly, like a fourth order tensor maps a second order tensor on to a second order tensor like that as an example. So, tensor is also like a transformation tool or a transformation, which tries to transform 1 vector into another vector if it is a second order tensor. That next thing that follows from this is that are all these independent or all these tau I is independent, or we should be in a position to express these tau I js some of these in terms of the other. So, for that we will quickly do 1 exercise, we will consider now a 2 dimensional element. 2 dimensional element is something where everything is occurring in a plane just for simplicity. So, we are assuming that the third direction is like unity or whatever. So, it has its length like say this is delta x, this is delta y, just imagine that it has faces perpendicular to whatever has been drawn in the figure having all width as 1.

So, let us write the components of the stress tensor on these surfaces. So, very quickly we will write it because we have learnt by this time how to write it. So, this is tau 1 1 , this is what is this tau 2 1, here this is tau 1 1, this is tau 2 1, here this is what will be here tau 1 2 it is along the positive 1 because the normally is outward normal is along positive 2.

Student: (Refer Time:23:40).

Yes. So, these 2 will be reversed. So, I would expect that you always correct it. So, the first index is what direction normal. So, direction normal is what 1. Second index is the direction on which it is acting. So, this is tau 1 2, and this is tau 2 1; here same. Now we are interested about the equilibrium of this element. So, when you are interested about the equilibrium of this element, we will consider the rotational equilibrium as an example. So, rotational equilibrium let us consider that as if it is an element, where we will be writing an equivalent form of Newton’s second law for rotation; as if like we are writing rotation of a rigid body with respect to a fixed axis something like that what the axis is this center. So, as if we are writing a rotational equilibrium equation with respect to an axis which passes through the center O and is perpendicular to this plane of the board.

So, that is why it is a 2 dimensional thing, we are considering a rotation in this plane basically. So, what we can write resultant moment of all forces which are acting on this is what; say we are writing the moment with respect to which axis z axis here, is equal to what tells that it is 0 it might be having an angular acceleration. So, it is I with respect to the same axis which is passing through this 1 and perpendicular to the plane of the board, say z times the angular acceleration say alpha.

So, the resultant moment of all these forces what will be that? So, you will see that tau 1 one and minus tau 1 1 tau 2 2 and minus tau 2 2 they cancel. So, moment contributes of contributors will be tau 1 2 and tau 2 1. So, tau 1 2 and these tau 1 2 they form like a couple. So, it is tau 1 2 what is the area on which it is acting into delta y into 1 which is the width, times the arm of the couple moment delta x for the other 1 it is clockwise. So, minus tau 2 1 delta x, delta y; we are assuming there is no body couple which is acting on it just like body force there could be body couple, fluids usually do not sustain body couple. So, there is no body couple which is acting on it is equal to the moment of inertia is like it is having a dimension of m into length square.

So, m is like delta x delta y into 1 into rho, that is the m if you write it properly it is delta x square plus delta y square by 12 with respect to this axis times the alpha. Keep in mind that delta x delta y are all small and tending to 0. So, if you cancel by considering that these are tending to 0, but not equal to 0, so you are left with what; in the right hand side you have terms because delta x and delta y are tending to 0 you have the right hand side tending to 0, and that will give you a very interesting result tau 1 2 is equal to tau 2 1.

So, in general tau i j is equal to tau j i; that means, tau 2 1 and tau 1 2 are same, tau 3 1 and tau 1 3 are same, and tau 3 2 and tau 2 3 same. So, you are left with 6 independent components in this stress tensor; and you see that tau i j equal to tau j i what are the assumptions on under which it is valid there is no body couple that is the only assumption. It does not depend on whether it is at rest or in motion, this is a very common misconception that people have that is valid only for static systems no it might be accelerating, but it does not matter because in the limit as delta x delta y tends to 0, the acceleration angular acceleration term becomes insignificant and that is how you get tau i j equal to tau j i we stop our discussion today we will continue in the next lecture.

Thank you.