Introduction to Fluid Mechanics
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Lecture – 37
Problems and Solutions

We continue with the example that we were discussing in the last lecture, that there is a tank, and through the bottom of the tank, the water is being drained out, and height of water in the tank is therefore, is changing with time. Our objective is to find out how the height changes with time.

Now we were discussing about what is the significance or impact of the unsteady term; that is retained or that should not be retained or should be retained is our doubt in the Bernoulli's equation. Now if you try to approximate it in some way. See in engineering we try to get a feel of the order of magnitude. So, we may try to approximate it by a certain term, which should be like or derivative of velocity with respect to time, times some height

So, let us say that if this dv dt was a constant, if it was a constant, not there it is a constant if it was a constant, it could have come out of the integral and then it would have been some equivalent constant dv dt times s 2 minus s 1. So, s 2 minus s 1 may be roughly like the height. If you take a streamline which is coming, I mean which is which is just along the axis then it is exactly equal to h, but you cannot just write it as some equivalent dv dt into h, because v is changing with time in an unknown way. So, you do not have really an equivalent constant dv dt.

But you may make a kind of approximation, you can say that I approximate this dv dt with dv 1 dt, why. If you see that except for very close to the outlet the streamlines are almost parallel to each other, and when the streamlines are almost parallel to each other, it represents case when that v is not varying very much. See why v is varying; see there is a flow rate confined between this. So, when other stream line the distance between these two streamlines remains the same, you have say a 1 b 1 equal to a 2 v 2. So, a 1 is like this a 2 is like these both are like as the cross sectional areas with the streamlines as envelops. In fact you can have a large number of streamlines their envelope will look like an imaginary pipe or a tube; that is known as a stream tube.

So, it is a collection of streamlines making an imaginary tube within which the fluid is flowing. So, if you consider such a tube, you can always see that the extent of that tube, that remains almost the same, till you come to the exit where it is really accelerating, because now the area available to it is so small that it has to get adjusted to itself. So, when the area is very small and it has to get adjusted to itself; that is only a small portion in comparison to the tank extent. So, if you approximate this dv dt with dv 1 dt, it is wrong, but it will give us some picture or some idea of what is the effect of, what is the impact of this term.

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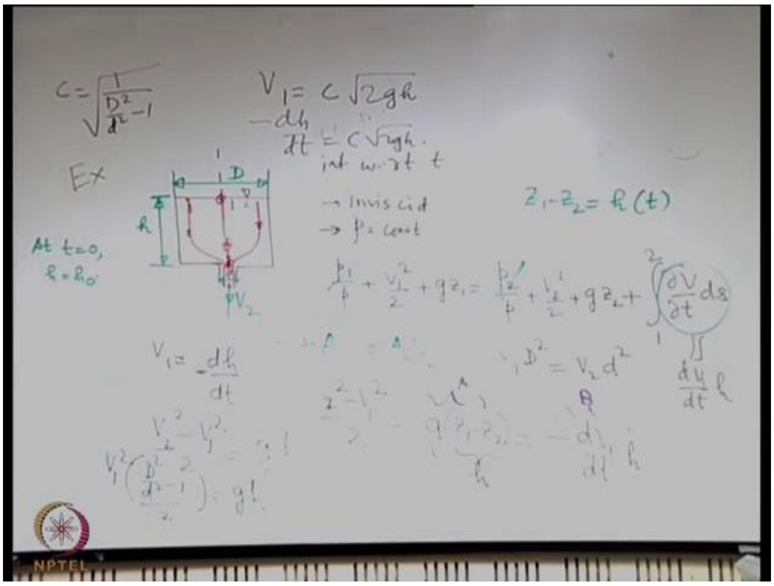

So, if you make an approximation that this is equal to dv 1 dt times h. You have to remember that both are functions of time, h is the function of time, v 1 is also a function of time. So, if you write this equation in a bit different way, you can write say v 1 say, v 2 square minus v 1 square by 2 plus or minus g z 1 minus z 2 is equal to minus of and z 1 minus z 2 is equal to h which is itself a function of time. So, these are valid locally at a each and every interval of times, at that time you have a dv dt and you have an h. now if you try to compare term say we want compare these term with this term. So, if these 2 terms are compared, then let us say this is a term a and this is term b. So, when can you neglect term b in comparison to term a, when you have this mod of this divided by mod of g h, when this is much less than 1, then b is much less than a.

So, if this is the condition. Well h is something which you do not consider locally, because this is like a h is always a constant. I mean always a local constant; that means, whatever h is a function of time here in terms a, same h is there in term b. So, only; that means, you have you are comparing dv dt with g. So, the rate at which the changes of velocity of the pre surface it is there; that is it is a sort of acceleration. If it is comparable with the acceleration due to gravity, then you cannot drop this term, and then you should retain this term at least frame a differential equation, it cannot be solved analytically, but if this is the case which is true for most of the practical cases, then it is possible to drop this term.

Now, second important point is, irrespective of whether you drop this term or not a 1 v 1 equal to a 2 v 2 is what you are always using. The reason is straightforward; the origin of this does not come from steady flow. Although this is valid for steady flow, it does not mean that its cannot be used for cases when the flow is unsteady, because the fundamental way in which it was derived from what, from the continuity equation, first by dropping the partial derivative of rho with respect to time equal to 0.

So, if rho is a constant partial derivative of rho with respect to time is 0. It may still be unsteady flow, because the velocity may be function of time, but rho not being a function of time, was the first thing to drop the first time in the continuity equation, that derivative with respect to time. For the other terms then what how we came up with this we integrated this that differential form of the continuity equation, and then if there say rho at the inlet and exit sections are equal again if rho equal to constant that is valid then you have a 1 v 1 equal to a 2 v 2. So, a very important thing is for a 1 v 1 equal to a 2 v 2 to be satisfied, it is not necessary that it has to be a steady flow. Only thing rho should not change, that is a very important thing that you have to keep in mind. So, even when it is varying with time you can use that.

Now, let us say that this is the case, so that we can drop the term v. So, if we can drop the term v, then you can write v 2 square minus v 1 square by 2 is equal to g h. Now what is v 2? You can express v 2 in terms of v 1. So, v 2 is v 1 into V square by a small v square. So, it is v 1 square D square by small v square minus 1 by 2 is equal to g h. And the remaining work is very straightforward, you can find out. So, v 1 is of the form, some constant into root 2 g h, where that constant is basically D Square by D Square minus 1 by that, square root of that

See these gives, I mean this gives a contradiction. What is the contradiction? When small d is very small since you consider the limit as small d by d tends to 0 that it is a very big tank of a large cross section area and there is a very small hole, through which the water is coming down. Then how does this work? Yes, how does this work?

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C is almost 0, if c is almost 0 then d 1 is almost 0. I mean practically it is it is true that if it is a tank of very large area and if there is a very small hole, the velocity at which the free surface is coming down is not perceptible it is very small. So, that is, and let us not bother about that too much, let us just try to complete this one by writing this as minus d h dt is equal to c root 2 g h. Now if we integrate with respect to time, you can find out how h varies with t, this is the very simple work. Now try to relate these with the kind of, again formula that you have used earlier in your studies. So, let us think that this tank, I mean this whole is not located here, but located at the side; that this is a different example just I am drawing in the same figure to say (Refer Time: 11:00).

So, let us say that now this height is h, which is changing with time. So, there is no hole here, but there is some hole here. There is a nozzle that is fitted and water is coming out. So, when you are doing that, the way in which most of you have done, is like you have assume the velocity that which the z is coming out is root 2 g h. This is known as Torricelli's formula. So, how you have arrived at that equation, you have used Bernoulli's equation between 1 and 2, at that time we were not very careful about whether they are along stream line or not, just out of pleasured you have applied between two points, and then when you have applied between two points, you put v 1 equal to 0, you put p 1 equal to p 2. The difference between the two heights h, and so v 2 will come root 2 g h right.

So, what are the approx, what are the assumptions under which that is valid that is, not a very bad formula, Torricelli derived it long back. I mean in a historical perspective it is a great development, because nowadays we can speak big words, but the subject when it was fundamentally developed, this itself was not a very trivial matter to resolve. So, then when a Torricelli found out this expression. What were the assumptions in in which this expression you expect to work still. So, one of the things was taken as v 1 equal to 0; that means, v 1 equal to 0 when, when D is much greater than small d. So, d 1 is approximately tending to 0.

The other approximations are, that you are having a streamline like this with respect to which you have the points 1 and 2, and the unsteady term does not appear in that analysis, and it is assume to be an inviscid flow. The greatest deviation from reality is, because of the assumption of the inviscid flow. So, that is one of the very important features that we have to keep in mind. So, with that assumption this formula is not illogical, but a very important thing is we must keep in mind that some of those assumptions are to be questioned.

One of the important assumption is like D is much greater than small d, which is true if it is a very large tank and from that there is a small hole through which water is coming out, and the dropping of the unsteady term, and we have discussed that, I mean when how this unsteady term this particular term in what conditions it may be dropped or not. So, this is a very simple problem. But if you try to look into this problem very careful it will give you a lot of insight, on the use of Bernoulli's equation under different conditions.

And I would encourage you to think about it more deeply, under what conditions different terms are important in in different ways. Not just satisfied with finding h of the function of time, but to write the differential equation of maybe say v 1 as a function of time in a very simple case, and in the most general case, and try to compare them; that is what are the terms that are making them to be different. We will consider another example in the unsteady Bernoulli's equation, in the use of the unsteady Bernoulli's equation; that is given by the next problem.

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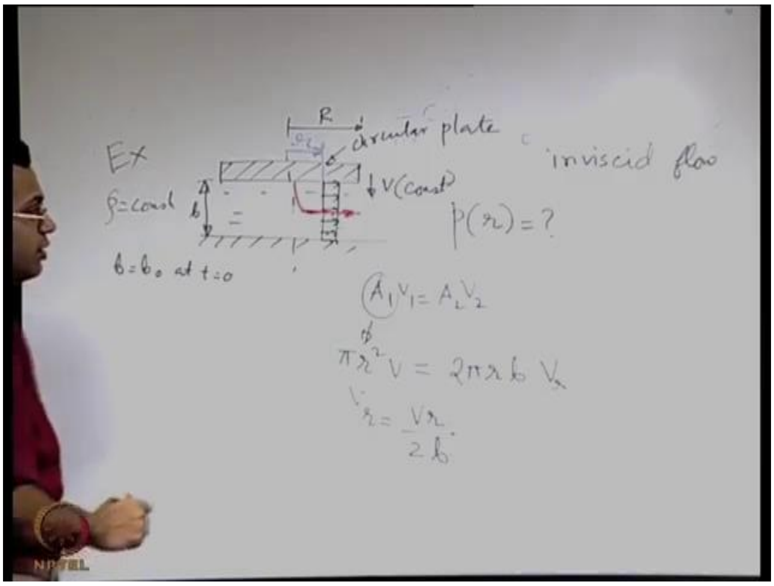

Let us say that you have two plates, these are circular plates. So, we have solved problems with rectangular plates, just for a change let us consider that it is a circular plate. So, this is, like this plate is coming down with a uniform velocity v, this is a circular plate, the radius of the plate is r, and say the we are considering a coordinate system, the local coordinate as small r. So, small r is the local coordinate at a radius r.

Now, with this we are interested to see. So, the bottom plate is stationary, there is some water with rho equal to constant, and when this plate is coming down what is happening, water is squeezed out of the place, because whatever water was there, say originally this was b naught. So, b equal to b naught at time equal to 0, but as this is coming down this b is changing, b is decreasing. So, where will that water go that water will be squeezed out radially, to make sure that the continuity is maintained? We are interested to find out how the pressure varies with r. Assume inviscid flow, and flow is constant that we have already defined, or we have already assumed.

So, as we have seen that in in in all these cases it is important to get a feel of the velocity profile. So, if it is a inviscid flow, the velocity variation over the cross section, over the section is not there. So, the velocity is uniform over each section, but this uniform velocity is changing with radius. So, how you can find out it? You have to think that what is the rate at which this is pulling water downwards, this is the same rate at which this being squeezed out.

So, if you consider a local radius r, what is the rate at which this is coming down? So, when you write a 1 v 1 equal to a 2 v 2, question is how do you write v 1 v 2 a 1 and a 2. What is v 1 v 1 is the rate at which? So, it is like an artificial flow imposed by the movement of the top plate. So, that flow velocity is given by v 1. So, what is that? A 1 into; so what is a 1, see if you consider only up to a local radius of small r. So, a 1 is.

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Pi in to small r square.

Student: (Refer time: 10:20).

So, a 1 is pi into small r square, what is v 1. V 1 is v, because it is a uniform rate, this is this is uniform, this is not a function of time this is constant, is equal to what is a 2.

Student: 2 pi 1 pi r.

2 pi r into b, b is a function of time into v 2, or v as a function of r, let us write v r, just to emphasize that it is v at a radius r. So, you can write v at a radius r is equal to v divided by v r by 2 b. Now so this is the velocity at a radius r. Next we are interested to find out the pressure.

So, if we are satisfied with inviscid flow, and rho equal to constant, we can consider a stream line that connects two points, any two point say 1 and 2. So, the stream lines, how the stream lines will look. So, the stream lines will virtually look like this. So, the flow is being squeezed out in this way. So, that is how a stream line will look. So, let us take, any two points located on the stream line, and write the Bernoulli’s equation between those two points located on this identified streamline, but because it is an unsteady flow, we need to retain the unsteady term in the Bernoulli’s equation.

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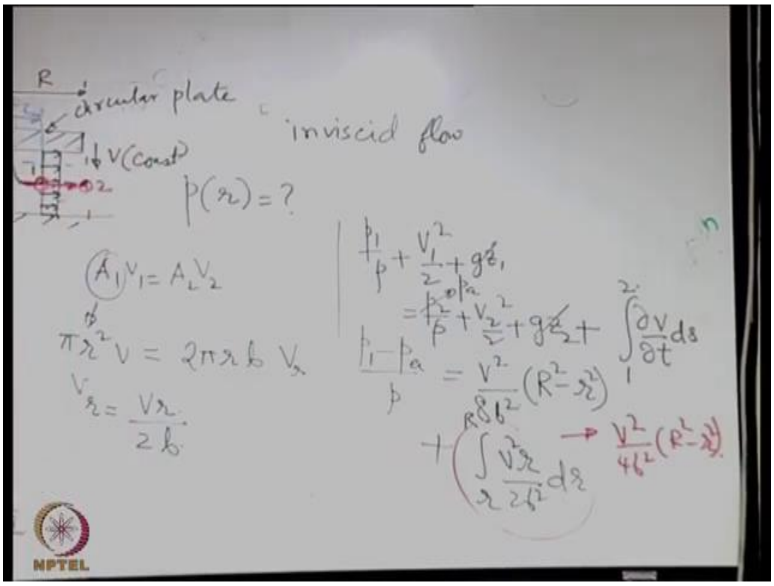

So, c 1 by rho plus v 1 square by 2 plus g z 1 is equal to p 2 by rho plus v 2 square by 2 plus g z 2 plus. Let us say that we apply that between two points; one point is located at r equal to small r, and another point 2 is locate at r equal to R.

So, when you have such a case, you are getting rid of many things; one is between the points 1 and 2 there is no difference in height. So, this of course, if this gap b itself is narrow then, even if there was a change in height, because of taking the points 1 and 2 not exactly along the same line that term itself is not that large, but if you take them along the same horizontal line, they are identically the same.

Then you are interested to find out p 1, and p to that p 2 is the atmospheric pressure. So, because it is it is at the exit plane. So, you are interested to write p 1 minus say p 2 is p atmospheric p 1 minus p atmospheric by rho is equal to; now v 2 squares minus v 1 square by 2. So, v 2 square minus v 1 square by 2 is v square by 4 b square into R square minus small r square, because v is having only this component, then plus this term, so by 2 will be there right. So, 8 b square, then plus let us calculate the third term.

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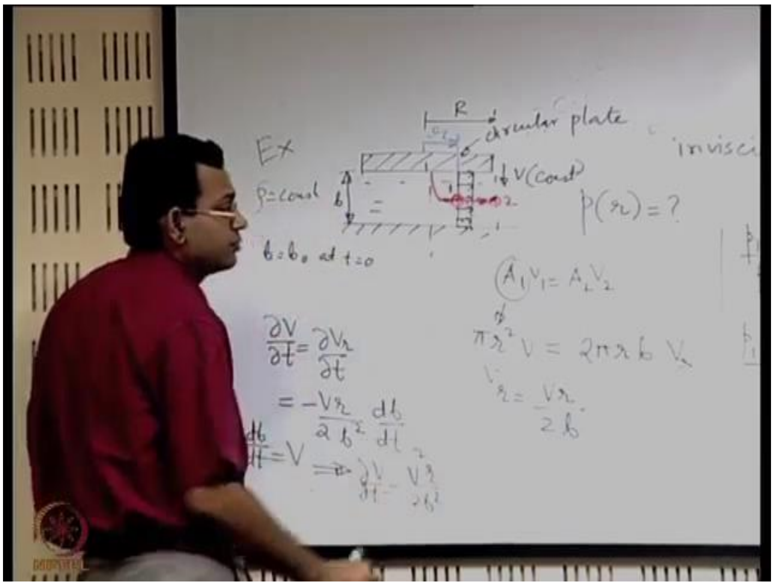

So, what is the partial derivative of v with respect to d; that is the partial derivative of v r with respect to t; that is the only v component that is there? Which is the function of t here? V is a function of t here. So, this will be minus v r by 2 b square into d b dt, and minus d b dt is equal to v.

So, minus d b dt is equal to v, just like the previous tank problem that we were considering. So, this term becomes d square r by 2 b square. So, that you can substitute here, and d s will be d r, because you have chosen your stream line in such a way that the change in s is like change in r. So, this is from integration from small r to R v square r by 2 v square d r. Very straightforward to complete it, it becomes v square by 4 b square into R square minus small r square.

So, at a given instant, you can see the pressure at the radius small r is varying with time, because b is a function of time. So, this only at a given instant you can say. At different instance you have different values of b, and you can find out what is the value of b at a given time how, because you know that db dt is minus v. So, b equal to b naught minus b t. So, if you are given a particular time, this will give you b is equal b naught minus v t. So, if you are given a particular time, you can find out what is the value of v at that time. Then you may substitute the value of b at that particular time to get the pressure at a radius.

So, you can clearly see that the unsteady Bernoulli's equation how it can be utilized.