Introduction to Fluid Mechanics
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Lecture – 32
Bernoulli’s equation-Part-I

So, to do that what we will do? We will leave this example and go back to the Euler’s equations of motion along the different directions. So, we have written the Euler’s equation of motion along x which is there in the board, similar equations are there along y and z. Now what we are interested to we are interested to write or to find; what is the difference in pressure between 2 points?

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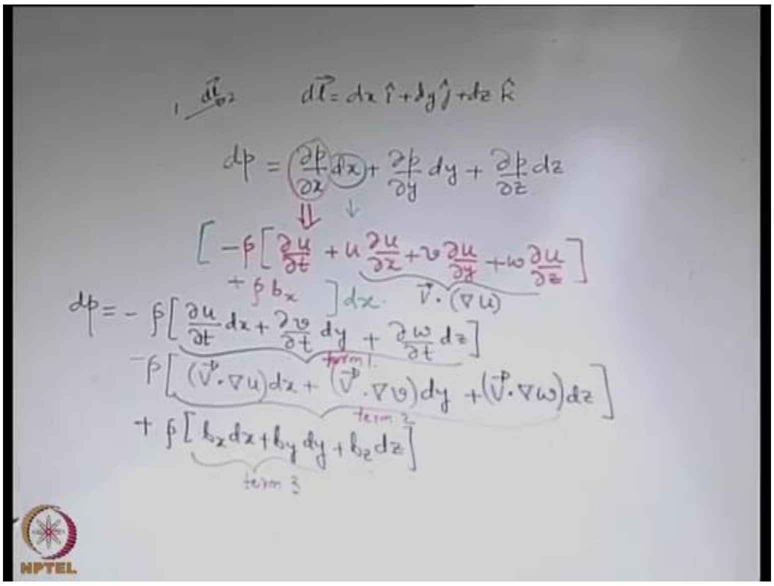

So, let us say that we have 2 points - 1 and 2 which are quite close. So, that they are connected by position vector dl d l is given by dx i plus dy j plus dz k. This is the position vector that we are looking for. What is our interest to find out, what is the difference in pressure between points 1 to 2? So, whatever we did in the previous problem a bit more informally, we will now try to generalize it for a very general case that what happens in that case.

To do that we will note that if you want to find out the difference in pressure between the 2 points here pressure is a function of what? X, y and z, so you can write these as the sum of these 3 partial derivative terms. Each of these terms we can substitute from each component of the equation of motion. So, the first term you can substitute from the x component of the equation of motion which is written below. So, let us write that this will be minus of, so this you are writing now the plus of this one; that means, this term will become minus the right hand side. So, that will become minus of rho then plus a body. So, plus rho b x dx, b x with dx multiplication will come separately we are just isolating the d p dx term. So, we are not writing the dx together with this. So, if you also consider the dx term together with this then it will be the entire thing multiplied by dx.

Now we will try to write it in a compact form because like it is possible to utilize some of the very well known identities of vector calculus to simplify it. So, what we will do is we will write this particular term in a vector calculus notation. So, we can write this as v, v dot with? Gradient of u, right, so then you will get these terms.

Keeping that in mind that other terms will also give similar expressions like what will change for the second term in place of this u it will be v, in place of this u it will be v, in place of v x it will be v y like that. So, it is very very analogous and we can write the general expression for d p as minus rho. Now we will collect all the terms we will keep all the terms of similar type together, this is one term then next we will write that acceleration term that is the convective component of the acceleration term is the temporal component then minus rho and the body force term.

So, these 3 types of terms are there, we will just for the writing convenience we will call it term 1 and there is logic behind that these terms are containing expressions of similar nature. So, we can simplify them in groups. Let us write or let us try to simplify terms 1, 2 and 3 separately, we will do that keeping in mind that the term 1 is the transient term and when we are simplifying we will be keeping in mind that we will be utilizing the vector dl which connects the 2 points which are close to each other.

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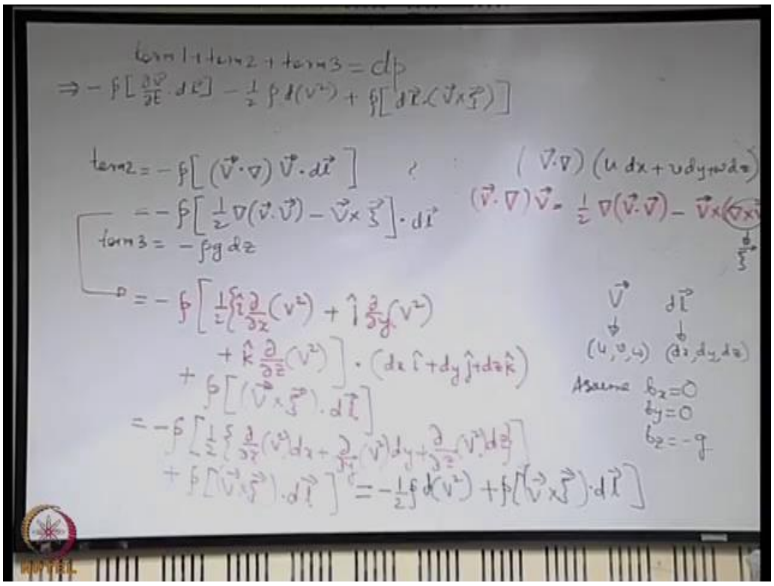

So, the term 1 is minus rho now can you write it in terms of the 2 vectors v and dl remember v has components u, v, w, dl has components dx, dy, dz. So, if you write for example, like this dot with dl then that expression and this is the same it is a dot product of this with this of course, you have a partial derivative of that term. So, it is just writing the same expression in a vector form.

Let us write the term 2, how do you write the term 2? Minus rho again let us try to write using the vectors v dot let us check whether this is all right or not, see and then we have to keep in mind that this is a scalar term right. So, first of all whatever is a vector operator it should give back a scalar. So, you have one dot product and the dot product and the product of that is expected to give back a scalar, you can just check let us check that. So, you can write this as v dot del then u dx plus v dy plus w dz. So, it becomes of the same form as that given by the term 2. Now it is possible to simplify the v dot del v using a vector identity what is that? So, v dot del v is equal to; one important thing we will see that whether the bracket is to be put here or after the v and till this is not complete. So, let us tentatively write it.

This is a very well known vector identity. Now you see that what is this v dot v is a scalar the gradient operator operating on the makes it a vector and this is very clear this is a vector. So, this should be a vector. So, when you have v dot del this is a scalar, but this being a vector keeps it a vector. So, whenever you write an identity these are certain common sense things that you should check because depending on what you operate the same thing may look may become a scalar and vector very easily, depending on how you put your cross products and the dot products.

Now why we are putting in this particular form is because here you get the vorticity vector and we were finding out that the condition of rotationality or irrotationality has some influence on the pressure difference between the points and these vectors only is responsible for whether it is rotational or irrotational. So, we will put that simplification here we will put this as minus rho half in place of the call of the velocity vector we will write the vorticity then dot dl.

For the term 3 what we will assume it is again a very general term what we will assume that the gravity is the only body force which acts along the negative z direction as we considered in the problem that we discussed just before this. So, what we will assume that b x is 0 b y is 0 and b z is minus g because that is the common thing that we encounter in many problems, but if there are other components of body force you know that how to simplify like you can just put the corresponding components here.

So, the then that will become term 2, will just become minus rho g dz. Since it has just only one scalar component it is not useful to write it in a general vector form it will not give us back many things. So, d p is the sum of term 1, term 2 and term 3.

We can simplify the term 2 and term 3, further let us let us try to simplify the term 2 one more step. So, minus rho let us now consider the dot product of these with d l. So, half what is v dot v? v dot v is V square, where this capital V is the resultant velocity. So, that we are writing is dot with this one sorry that is the first term and you also have a term plus rho v cross vorticity vector dot dl, you can recognize that it is like a scalar triple product of 3 vectors like a dot b cross c. So, we will keep that simplification for a moment and just consider the first term. So, what does the first term look like? Half that is the first term I mean first term of the term 2 and then rho, you can clearly see that the first term of the term 2 will become what? It is like it will become d dx of v square it is sum of the 3 partial derivatives will give the total 1.

So, this will become at the end the simplified form minus half rho d of v square. So, this is like not d dx just the total d. So, this is partial derivative with respect to x into dx, this is partial derivative of y into dy and that with respect to z into dz. So, that is give that is giving the total d plus whatever term that is remaining. Now let us put back all the terms together in the equation. So, what is our equation our equation is term 1 plus term 2 plus term 3 is equal to 0; that means, minus rho that is the term 1. So, let us sorry d p not 0, it was d p then term 2 in place of term 2 we will write minus half rho d v square plus rho then let us write dl dot b cross zeta that is the term 2 and term 3 is minus rho g dz is equal to d p. This is a compact form and it is possible to simplify it even further based on certain special cases.

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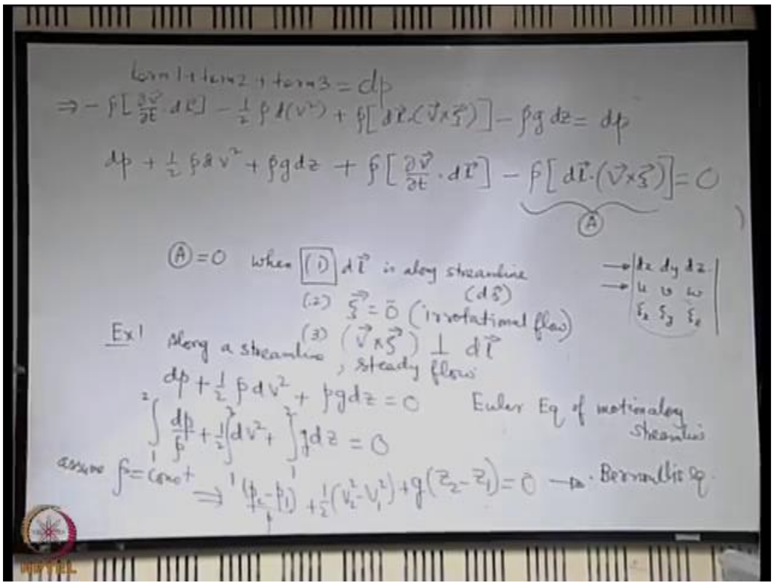

So, what special cases we will be interested in let us see. What special cases maybe let us take all the terms in the same side, so you have d p plus half rho d v square plus rho g dz up to this you can find some similarity with the Bernoulli’s type of equation that you have encountered earlier what you are getting also some 2 extra terms, let us write those extra terms. So, plus then minus rho this equal to 0 because these 2 terms are like beyond what you have encountered many times we will try to put more attention to the last 2 terms. We will put the first important attention on the last term because this particular term in a case when it is a steady flow this trivially goes away. So, there is no big controversy or there is no big uncertainty in that that is quite understandable.

But the last term there are many possibilities when the last term can become 0 what are the cases. So, if you just write it in a determinant form when you are having such a scalar treatment product you can write it in terms of determinants where each row of the determinant will represent the components of the vectors taken in the particular order. So, you have like dx for dl you have dx dy dz, for v you have u v w for the vorticity you have.

Now let us consider a case when this say rho 1 just look it into mathematically say rho 1 is a scalar multiple of the rho 2, when it is possible?

Student: (Refer Time: 20:18).

When the direction of dl and the direction of v are the same then one will just be the scalar multiple of the other because direction wise they are representing vectors oriented identically. So, when that is possible what is the special dl for which that is true, if it is located along a stream line. So, if we consider this as like a term A. So, we will identify in certain cases when A becomes 0. So, A equal to 0 when certain cases one is dl is along streamline let us call the streamline direction as d s, s for stream wise coordination.

When that is the case we do not care whether it is an irrotational flow or not, it does not matter whether it has components 1 0 components of the vorticity vector yes.

Student: When it is a streamline flow then how can we say that (Refer Time: 21:29).

There is nothing called as streamline flow first you have to understand, there is a streamline in a flow, there is nothing called streamline flow, next.

Student: (Refer Time: 21:39) dx, dy, dz, multiple of;

This is what, this is the length element this is the line element that you are considering this is the component these are the components of the velocity vector what is the definition of a streamline, such that tangent to the streamline at any point represents the direction of the velocity vector. So, tangent is this direction dl, a small elemental direction and this is the velocity vector direction. So, if they are located in the same direction; that means, they are parallel vectors; that is the definition of a stream line nothing extra.

So, if dl is located along a streamline then we do not care whether it is a rotational or irrotational flow, but if it is not then if the vorticity vector is identically equal to 0 then a will become 0, no matter whatever is like, no matter whether dl is located along a streamline or not. So, vorticity vector is a null vector this is irrotational flow. So, you can clearly see that if it is a steady and irrotational flow these 2 terms go away and then sum of these 3 is 0; that means, if you integrate that the integration will give a constant of integration and that is what we actually saw in the example the problem that we discussed before going to this derivation.

There is a third case I mean there could be many such cases, but a third case say you have the V cross this vorticity vector is perpendicular to dl, these 2 cases are more common cases that you encountered this is not a very common case you encountered, what this mathematically you cannot rule out. You have a vorticity vector you have a velocity vector you can find the cross product and you can find the element in a direction which is oriented along that cross product and then if you take such an element then for such an element also for steady flow it will appear that the Bernoulli type of equation is valid. So, this is not a Bernoulli type of equation this is in fact, one step before that where we do not make any explicit assumption on how the rho or the density varies. So, this is still Euler equation of motion. So, this is a more general way of writing the Euler equation of motion where you are considering all the individual components and trying to write that in a vector form, but at least we can understand that this term becomes 0 under what cases.

So, let us say that we are considering one such case, let us say that we take an example we are considering along a stream line that is what do we mean by along a streamline; that means, we are interested to find out these changes. So, this relates what? This relates change in pressure, change in velocity, change in elevation with respect to a change in position vector from point 1 to point 2. So, when we are considering along a stream line; that means, we are interested to evaluate that change by moving along a streamline. So, never consider anything like a streamline flow again I am repeating there is nothing called as streamline flow, in flow there are streamlines but it is not a streamline flow. So, when you have a streamline we are looking for the difference in like this variables along a streamline. So, when you have along a streamline and let us say steady flow as the first example, this is example 1.

So, then what you have then A term will become 0 this term because of steadiness will become 0. So, you have d p plus half rho d v square plus rho g dz equal to 0 this is known as Euler equation of motion along a streamline.

Now, this is valid both for compressible as well as incompressible flows, you are not yet committed of how the density changes. So, now, we are interested to see that how the density changes, to do that we will let us say we will write it in this form d p by rho plus half d v square plus rho g dz and try to integrate it. So, we will try to integrate it, rho will not be there because rho we have already divided by rho. So, when we try to integrate it what are the points over which we are integrating?

We are integrating with respect to 2 points 1 and 2 which are located on the same streamline because we have considered along a streamline that is we are considering this particular case which has made the term A equal to 0. So, when we do that this equal to 0 that is still valid for any type of flow compressible or incompressible. Now you make an assumption that rho is a constant, assume that is a special case of an incompressible flow. So, then what you can write, you can take the rho out of the derivatives. So, you can write p 2 minus p 1 by rho plus half v 2 square minus v 1 square plus g into z 2 minus z 1 is equal to 0. This is nothing but the Bernoulli's equation that is p 1 by rho plus v 1 square by 2 plus g z 1 is equal to p 2 by rho plus v 2 square by 2 plus g z 2. So, it is like it is in fact, the Bernoulli's equation.

Now, you tell that what are the assumptions that we followed in deriving this. So, this is the Bernoulli's equation. We will come into the physical significance of this Bernoulli's equation in the next lecture, but let us at least try to identify that what are the assumptions that we utilized to derive this. So, what are the assumptions?

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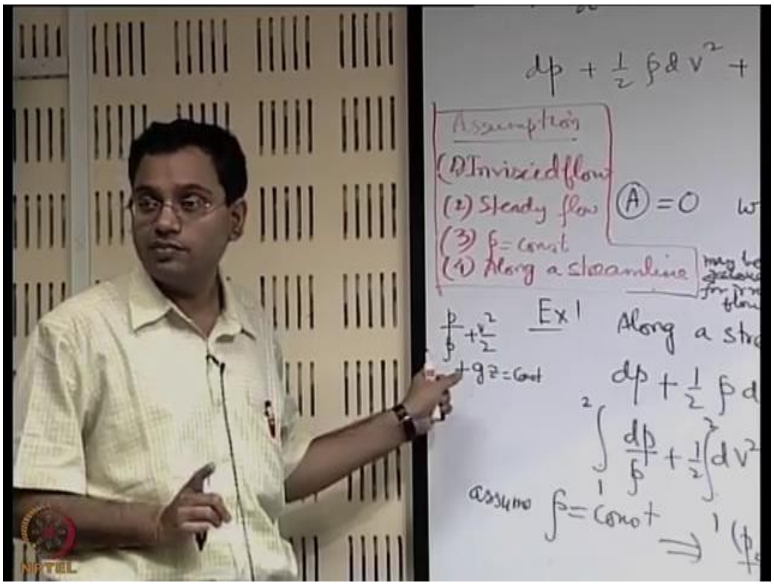

Student: (Refer Time: 28:15).

So, first start with the most basic assumption when we wrote the equation of motion what we assumed.

Student: (Refer Time: 28:24).

Only gravity is the only body force it is, but it is like it is very very explicit, what is not so explicit is inviscid flow. So, inviscid flow is very important. Then steady flow, density is constant, it is a special case of incompressible flow.

Student: Irrotational flow.

Not irrotational flow we have not taken this condition 3, when we take irrotational flow we get a more freedom then we need not be restricted along a streamline, but when we are considering along a streamline then it need not be irrotational if it is irrotational fine if it is not irrotational still ok, so along a streamline that is what we considered in this example. So, these are the four ones that these are the four assumptions that we have considered in deriving this.

Now these are the assumptions that we commonly use because commonly we utilize the Bernoulli’s equation along a streamline. At the same time we must understand that these are not always the cases inviscid flow is the most important thing, now can you tell that if you are thinking about Euler's equation along a streamline out of this which assumption is not necessary, say the Euler's equation of motion along a streamline.

Student: (Refer Time: 30:02).

Density equal to constant is not necessary. So, density equal to constant is the additional assumption beyond the Euler's equation before. So, after you make that assumption you have to also keep in mind that I would say the most important assumption is inviscid flow because many times we tend to apply the Bernoulli's equation in cases when viscous effects are very much present, maybe many times you have solved such problems in your earlier high school exercise problems to solve like to get the velocity pressure and so on we will see that that is not fundamentally correct in some cases you can get rid of that and still get some qualitative picture we will see that when and when not, but fundamentally it has to be inviscid flow.

Steady flow is for this version of the Bernoulli's equation, but you can also have an unsteady version of the Bernoulli's equation that we will see later on maybe in the next class that where we will retain this term and we can write a Bernoulli's equation by considering even the steady flow along a streamline unsteady flow along a streamline. So, only for this version it is steady flow and that is the standard Bernoulli's equation, but we also have unsteady Bernoulli's equation. So, for unsteady Bernoulli's equation the steady flow assumption is not required, rho equal to constant is always required because you are taking rho equal to constant and taking out of the integral and along a streamline is required for this special case when you are not bothered about whether it is irrotational or not, if it is irrotational then this need not be the case, so maybe relaxed for irrotational flow.

So, what is the summary? The summary is if it is an irrotational flow and other conditions are satisfied that is inviscid steady and rho equal to constant you can write p by rho plus v square by 2 plus g z is constant need not always be along a streamline. So, this is constant no matter whether you are considering the points 1 and 2 anywhere in the flow field that is very important. So, points 1 and 2 maybe located anywhere in the flow field still this equation is satisfied if it is an irrotational flow, if it is not an irrotational flow then 1 and 2 have to be located along the same streamline. So, these are very very important fundamental assumptions that go behind the Bernoulli’s equation. We will stop here today, we will continue again in the next class.

Thank you.