Introduction to Fluid Mechanics
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Lecture - 31
Euler's equation

Today we are going to start with the Dynamics of inviscid flows.

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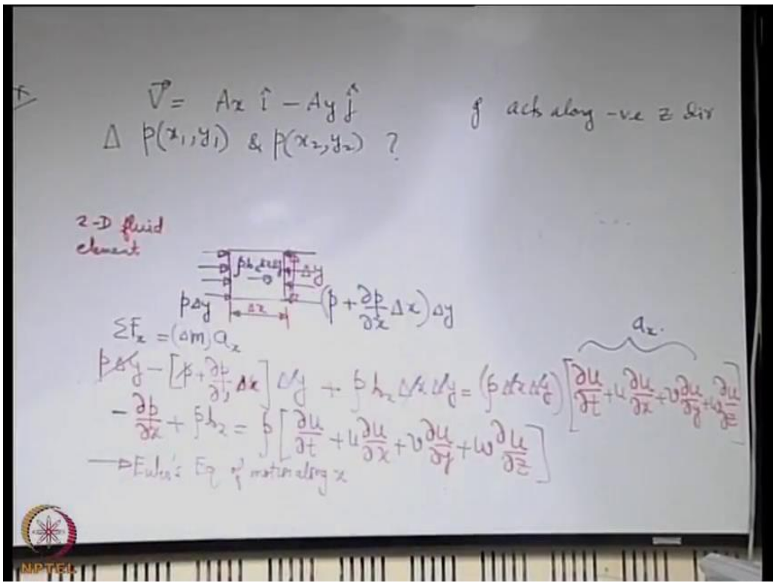

In the last chapter we were discussing about the kinematics. So, we were not discussing about the forcing parameters which are involved to influence the flow. We have discussed about the motion, now we are going to see that; what are the forcing parameters which influence the motion, and how they are related to the motion? When we talk about inviscid flows what we essentially mean is, initially we will viscous about cases where viscous forces are not present. It is a simplified situation of the reality, but at the same time it will provide us with a lot of important insight which we will use later on when we will be discussing about the dynamics of viscous flows. So, when we will be considering or focusing our attention in this particular chapter, we will be considering cases where viscous forces are not there or negligible.

To start with the discussion on this, what we will try to do, we will try to write the equation of motion for a fluid element where viscous forces are not present. When viscous forces are not present, the kinds of forces which are there are the surface forces in terms of the normal components which are manifested through pressure. And some body forces which may be like the gravity forces. Keeping that in mind let us say that we want to write the equation of motion for a fluid element, let us say that it is a 2-dimensional fluid element. It need not always be 2-dimensional, but if you are writing the equation of motion along a particular direction then like for simplicity we can take it as a 2-dimensional one for illustration.

So, let us say that we take a 2-dimensional fluid element as an example. Fundamentally it is always 3-dimensional, so the third dimension you may consider as one or some uniform third dimension. Let us say that these dimensions are delta x and delta y, we will quickly identify what are the forces which are acting on the fluid element only along x. We will identify forces along x because we are interested to write the equation of motion along x, so other forces will not show. So, it is not a complete free body diagram only the x component of forces will be shown.

So, here you have force due to pressure. So, if p is the pressure here, then p times delta y maybe times 1 where 1 is the width is the force that acts on the left face due to pressure. Force that acts on the right face due to pressure is what? We have encountered such situation earlier, so p plus this into d x times delta y into 1. Along x these are the only surface forces because other forces will have surface forces along y. Body force may be there; let us say that b x is the body force per unit mass. So, rho into b x into delta x into delta y is the body force component along x, because rho into delta x into delta y is the mass of the fluid element.

So, we can write the Newton’s second law of motion for the fluid element. That is we can write resultant force along x is equal to mass of the fluid element times acceleration along x. Maybe you can write delta m it is a small mass to acknowledge that. So, we will try to simplify this expression. Resultant force along x is p into delta y minus p plus this one with respect to into delta y, this thing then plus rho b x delta x delta y is equal to; the mass of the fluid element is what rho delta x delta y times acceleration along x, what is acceleration along x? This we have discussed in the kinematics. So, what is that? This is the acceleration along x. This we have derived in the kinematics.

See when we were discussing about the rigid body type of motion of fluid elements then we did not use this expression, we were using an expression as if the entire fluid is having a particular acceleration disregarding the deformation within it. So, now the different gradients of velocity will become important, which was not there or which we kept ourselves abstracted off when we just wrote some acceleration, when it was moving like a rigid body. Now we are more detailing it. So, we are looking into the detailed expression that reflects that acceleration. So, this is acceleration along x.

Now, you can cancel various terms. So, first this term will go away and then like you will have; let us just correct it a little bit it was delta x y we did not consider it d x delta x. Just change this d x to delta x because we took our element as delta x. And then we can cancel delta x into delta y from all the terms because these are small tending to 0, but not actually equal to 0. So, what we are left with? We are left with a simplified expression plus rho b x is equal to rho.

This very simple expression is also known as Euler’s equation of motion along x. Similar expression, we can write for the motion along y and z. We are not repeating it because it is very trivial. Now what does this equation of motion contain? If you look into it is fundamentally like Newton’s second law of motion where viscous forces are not considered. So, this right hand side is something like the mass into acceleration, left hand side is the effect of the force which is acting. So, one force is because of the pressure gradient and another force is because of the body force. So these 2 forces are considered. So, it is just a different way of writing Newton’s second law of motion for a fluid where viscous effects are not present, and any other force other than this body force or this particular form we are not considering it.

Let us take an example to illustrate that how we can make use of these. The example is like this; let us say that you have a velocity field v given by say Ax i minus Ay j, where A is constant number and the dimension number to adjust the dimensions of the 2 sides. We are interested to find out what is the difference between pressure at 2 points given by x 1 comma y 1 and x 2 comma y 2. It is given that g that is the acceleration due to gravity acts along negative z direction.

So, the question is, what is the difference in pressure between these 2 points? The problem is very simple, but it will at least give us some idea of how to make use of this expression. A and b are not functions of time, so it is a steady flow field. Let us write this equation say along x for this one.

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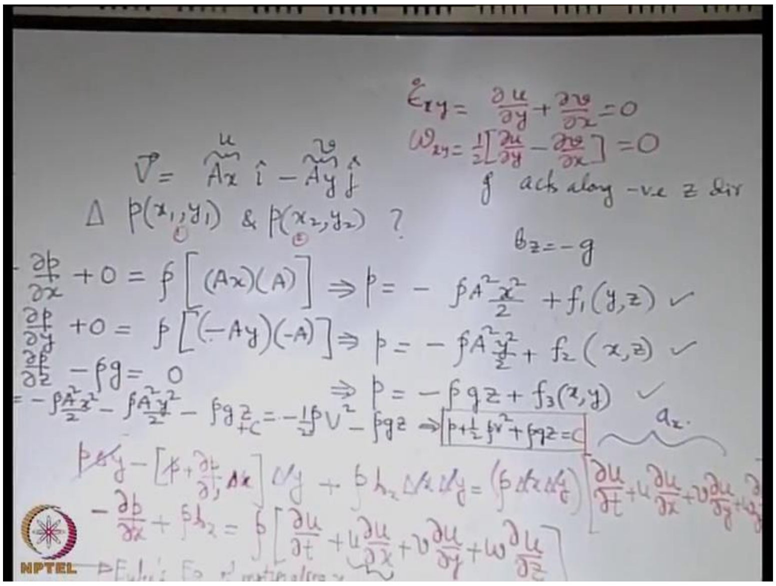

So, if you want to write this Euler’s equation along x, so we have minus of partial derivative of pressure with respect to x plus what is d x, there is no x component of body force; body force only acts along negative Z. So, this plus 0 is equal to rho. Velocity field is not a function of time here; A is a time independent constant that is given. So, the time derivative will be 0. So, what is u and v here, this is u and this is v with a minus sign of course includes the minus sign.

So, u is Ax into A that is this term. The other terms are not there because u does not have any dependence on y and z. So, the other terms are not there. So, this is the equation of motion along x. What will be the equation of motion along y? Just it will be similar to this; there is no body force along y and rho right hand side what is going to happen this u is only going to be replaced with v. So, the term that will remain relevant is only v into partial derivative of v with respect to y. So, it is minus Ay into minus A.

Let us consider the z component. Now you have a body force along Z, what is that? So, v z is equal to minus g. So, minus rho g is equal to the right hand side u will be replaced by w, and there is no w component of velocity; it is a 2-dimensional flow field. So, it is 0. So, it is possible to integrate these expressions to find out how p varies with x y and z. So, let us integrate that, let us say we integrate this one with respect to x. So, we get p here as what? Minus rho A square x d x will become x square by 2 plus function of.

Student: Y.

Y and z right; for this it will become p equal to very similar minus rho A square now y square by 2 plus of function of x and Z. And what will this give p equal to minus rho d z plus a function of x and y. All these 3 expressions are representing the same pressure field. So, you can compare these to get these 3 functions. So, let us compare these and get the 3 functions. If you compare what functions you get; what is f 1? F 1 is the function of y and z. So, minus rho A square y square by 2, then minus rho g z this is f 1; f 2, function of x and z so minus rho A square x square by 2 minus rho g z; and f 3 minus rho A square x square by 2 minus rho A square y square by 2.

So, the expression for pressure becomes minus rho A square x square by 2 minus rho A square y square by 2 minus rho g z. So, this you can write minus half rho- A square x square plus q square y square is u square plus v square so that is the square of the resultant velocity. So, let us say capital V square minus rho g z; that means p plus half rho v square plus rho g z equal to 0. Now when we write this there is a lack of generality here, what is the lack of generality? When we consider this f 1, f 2, f 3, we did not consider a constant.

So, fundamentally we should also have a constant there, where that constant maybe eliminated depending on a choice of a reference frame but that is not done apriori; that is after you get the general expression then only that is possible. So, a very important thing here is that each of these should be augmented with a constant. So, what it means is that this plus c will be there, so this will become in place of 0 it will become some constant.

This looks very familiar to you it is like a Bernoulli's equation. Now do not get a wrong impression here that the Bernoulli's equation is always valid and that is why you can write it in this form. There is a specialty of this problem, because of which the Bernoulli's equation gets valid between any 2 points 1 and 2. So, this is the point 1 and this is the point 2. Solving the problem is trivial you can find out p 1 minus p 2 y is substituting the velocities respectively at x 1 comma y 1 and x 2 comma y 2; that is a straight forward exercise.

Student: Sir, see directly compared the (Refer Time: 18:20) rho f 1, f 2 and f 3.

Yes.

Student: Who directly you (Refer Time: 18:23).

The direct comparison of the equations is possible because they represent the same pressure.

Student: Direct comparison is possible, but after comparing them you just (Refer Time: 18:31) f1, f 2 and f 3.

This is why observation. See, I mean when you write, when you say that these 2 are equal or these 3 are equal it should be such that it does not contain any function of x y or z which falls beyond the functions written here. I mean there are certain things which you can do just by common sense, and this is one of the big things in mathematics which you can do by a little common sense. That is what is expected when you solve such problems.

Now, when you come to this conclusion that it is like a Bernoulli’s equation in fact it is of the same form, and we therefore can apply it between any 2 points 1 and 2 it is not a general conclusion; that we must remember. And we will do it vigorously to show that when it is valid and when it is not valid. This is very very important because all of you are y very habituated in using Bernoulli’s equation anywhere and everywhere you like. So, we will try to restrict you so that you do not apply it anywhere and everywhere. And we will see that when it is applicable and when it is not.

But before that this problem at least tells us that this is a very easy problem and it demonstrates that in this case it is possible to apply it between 2 points 1 and 2. So, what is the specialty of this problem? Let us look into it. See for every problem there is one aspect that is how to solve a problem that is fine, but there is a greater aspect how to develop a more detailed insight on what the problem is about. So, we are now trying to do that; problem is solved, but it is not enough. Let us see that what insight it gives us.

Try to find out what is the rate of deformation and angular velocity of this flow. So, if you recall that if you want to find out say rate of deformation epsilon dot x y; what is that? So, in this case what is the value of this? Identical equal to 0; angular velocity in x y plane half of this one, because each of these terms are 0 you have this also identical equal to 0. So, what does it show? It shows that if there is a fluid element located in this flow field it does not have any shear deformation it does not have any rotation that means, if its edges were originally parallel to x and y those will remain always parallel to x and y. If it is incompressible what will happen, it might stretch along say x; so it should reduce its length along y so that the volume is preserved.

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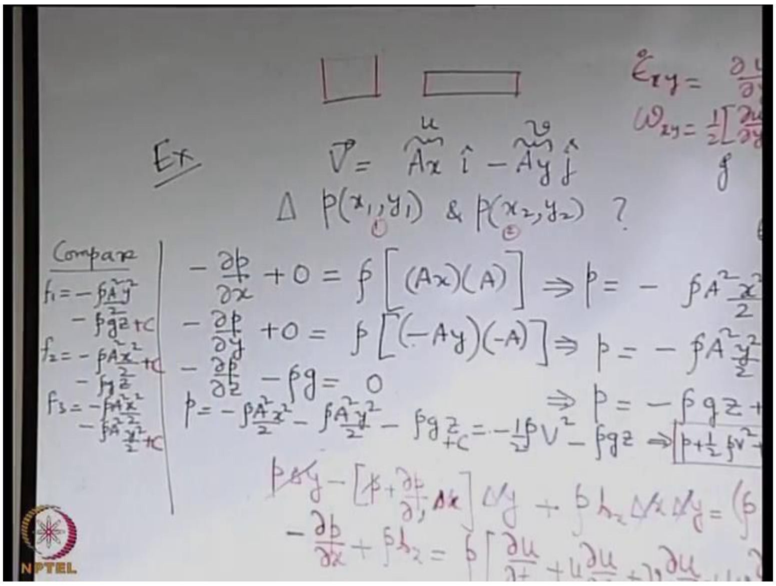

So, it is like if the fluid element was originally like this maybe it will become once like that and it will change its configuration in such a way that angularly there is no change. Only there are changes in linear dimension but ensuring incompressibility, because it also represents an incompressible flow field. That you can check by checking the divergence of the velocity vector is 0. So, it is an incompressible flow. So, that is one important observation. The important observation is, it does not have any shear effect.

Next is it does not have any angular velocity. So, it is like an irrotational flow, because irrotational flow has no angular velocity or no vorticity so to say. Now let us try to see that what will be the equations of the stream line in this case. So, we are interested to find out the equations of the stream line. It will give us even a deeper insight. And we will relate it to one of the movies that we saw in our previous lecture.

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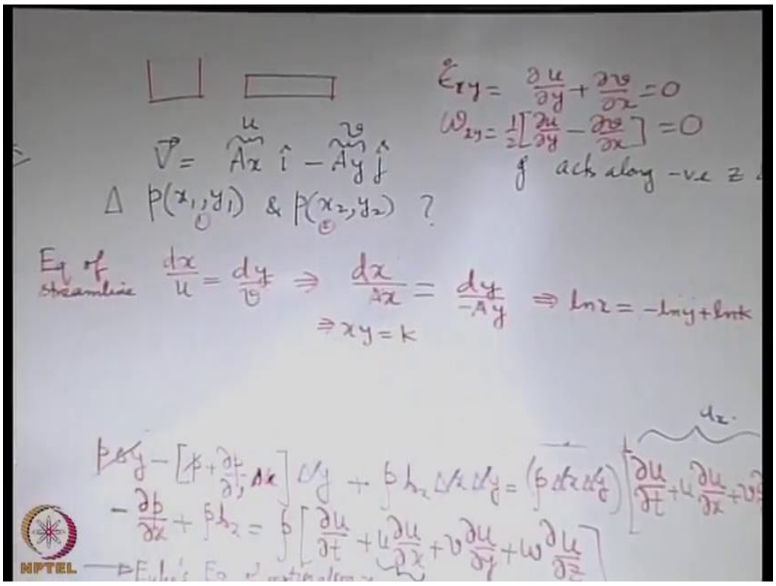

So, if you write the equation of the stream line it is d x by u is equal to d y by v. This is the equation of the stream line. So, you have d x by Ax is equal to d y by minus Ay; A is not equal to 0 you can cancel that and if you integrate it you will get l n x is equal to minus l n y plus a constant let us say l n k. So, this gives an equation of the stream line of the form x y equal to k, which is like a rectangular hyperbola.

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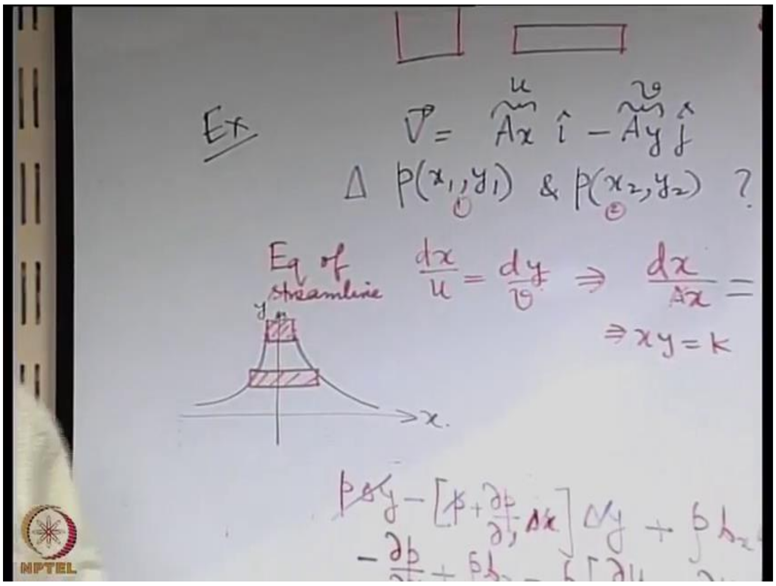

That means, if you consider this as x axis and may be this as y axis, it is possible that you have your stream lines in this way. So, if you have a fluid element originally like this, maybe that fluid element is coming down along the stream line; I am just trying to make you recall one of the movies that we showed to you. That the fluid element is coming down like this with no angular change, no rotation, no shear deformation, but only the lengths of the respective edges are getting altered. So, it is a case of pure linear deformation no angular deformation.

And then in such a case we have 2 things satisfied: one is there is no effective viscous effect because the viscous effect comes through what, viscosity into the rate of shear deformation. So, if the rate of shear deformation is 0 it does not matter whether viscosity is 0 or not. So, inviscid effect is not always through viscosity equal to 0, it maybe the rate of shear deformation equal to 0, because eventually you are interested about whether the shear stress is 0 or not. If the shear stress is 0 it does not matter whether it is 0 because of mu equal to 0 or because of rate of shear deformation equal to 0, here it is 0 because the rate of shear deformation is 0.

So, it does not have any effect of viscous shear, it does not have any effect of rotationality. For it is effectively like an inviscid and irrotational flow, and for such a flow we will show later on that you can apply Bernoulli's equation between any 2 points in a flow field disregarding where they are.

And we will now go into a more vigorous way of establishing this very important consideration.