Introduction to Fluid Mechanics

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Lecture - 30
Circulation, Velocity potential

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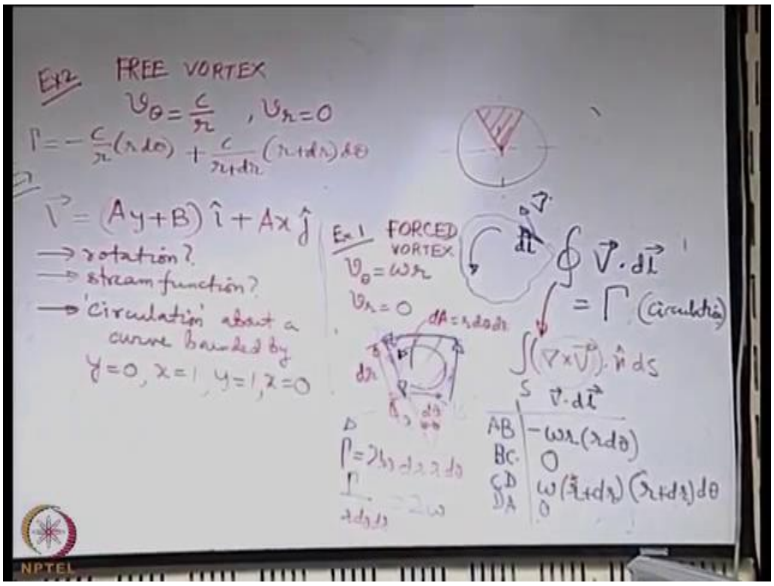

A y plus B, i cap plus A x j cap, where A and B are dimensional constants these are given in terms of some numbers we are not going into that, we have to find out the fluid rotation and the stream function. Again the this is very simple case to talk about, but whenever we are trying to understand the fluid rotation we will now try to understand a very important concept, which is asked in this particular problem that is what is the circulation about the curve we will now learn what is the meaning of circulation about a curve bounded by the following lines. What are the lines? Y equal to 0, x equal to 1, then y equal to 1 and x equal to 0, so we have come across a new terminology in this problem called as circulation.

So, we will keep this problem a bit aside, try to learn; what is the meaning of the terminology circulation, and then we will try to apply it. The terminology circulation is not a very new terminology it is very much related to what we have already learned. So, and we will see that it is very much related to the concept of viscosity, how it is related to that. Let us say that we have a line a closed contour, a contour which is closed by some curve. Now if you take a small line element located on the contour, let us say that we take a line element like this. So, you may have a Vorticity vector say V and you may assign a vector to this line element by giving it directionality. Say you are traversing along a clockwise direction or an anticlockwise direction; here we are traversing along a anticlockwise direction.

So, if you find out the V dot d l the dot product of the Vorticity vector, which this one and find the contour integral, that is the integral of these over this close path by traversing in a particular direction, that is known as circulation along the curve, which is given by a symbol capital gamma this is circulation. Now it is possible to express this in terms of an area integral by using a stoke s theorem in vector calculus. So, you can express this in terms of called f or here the called V where the vector function f is here V, where S is an element of the surface d s is an element of the surface and a S is the total surface that is bounded by the closed curve that is very very important.

So, you can clearly see that if you talk about the value in a plane it is it is possible to make out from this that first observation is this is nothing, but the Vorticity vector. So, this Vorticity vector when you utilize this Vorticity vector here. So, the circulation is nothing, but how you express it in terms of Vorticity, it is just like roughly Vorticity per unit area or it is in the other way; it is like a Vorticity into the area is the circulation. So, it is the other way not Vorticity per unit area, but Vorticity into area is the circulation.

So, let us take an example before going into this particular mathematical form to figure out that how you can calculate what is the circulation. Now to do that let us take an example which is not the same example, but we will utilize what we learnt from the example to solve this problem. Say you have another example where you have the Vorticity components given by polar coordinates; say V theta is equal to omega r, and V r equal to 0 there is no V, V upto dimensional flow. Let us say that we want to find out what is the circulation around the closed contour.

So, a closed contour in an element in a polar coordinate form, we can generate in this way. So, you consider that at a distance r, I am magnifying d r to show that as if d r is greater than r, but it is just for clearly drawing the figure of d r, and this angle say is d theta. So, what is the contour that we have chosen? We have chosen the contour which is formed by this element in this polar coordinate. So, these like A, B, C, D; now we are interested to find out the circulation. So, we are first we start the traversal from the point A. So, what is the V theta defined? V theta is defined as omega r. So, we are going from A to B in this direction, what is V dot d l? Here d l is also along A B because it is a small arc it is almost like tangent to the curve, V theta is also tangent to the curve so, but the thing is here positive theta direction is anticlockwise.

So, V theta and d l, so, V theta is in the opposite direction as that of your d l, if you are trying to traverse the path in this way. So, when you go from A to B, V theta is opposite to that of your d l that is drawn. So, V theta dot d l will give a minus sign what is the magnitude of V theta dot d l? So, for A B, V theta dot d l will be omega r into r d theta with a minus sign. So, what we are writing here is V theta V dot d l, then for B C, V theta is along the tangential direction, B C is along the normal direction. So, their dot product is 0. So, this does not come into the picture it is 0, for C D what is V theta. So, for C D what is d l? The d l is from C to D you are traversing it in a particular direction. So, we are traversing it like this. For C D what is V theta?

Student: omega into r plus d r.

Omega into r plus d r and what is;

Student: r plus d l.

D l that is r plus d r into d theta and for D A it will be 0 just like before. So, when you sum it up this contour integral will be the sum of this 4 line integral. So, sum of this 4 line integrals will give what is the total circulation that is sum of these. So, what will be that? You see the first term minus omega r into r d theta will be cancelled with 1 omega plus omega into r d theta, then there will be terms omega d r r d theta. There will be term omega into d r square d theta then any more term. So, one of the terms this has got cancelled with omega r into r d theta, then omega r into d r d theta we have written, then omega r into d r d theta, so another omega r into d r d theta.

So, 2 omega r d r d theta, so if you find out what is the circulation per unit area, what is the circulation per unit area? This divided by r d theta d r that is d A, the d A of this shaded 1 is r d theta d r. So, that is nothing, but 2 omega; actually this omega is the angular velocity and 2 into that is the Vorticity. So, this is like the Vorticity. So, what we get from here this is an example of illustration of the same concept this is just from vector calculus theorem, this is just detail illustration of that. So, you can conclude from this that in such as case you can find out the circulation per unit area equal to Vorticity. So, from that you can relate Vorticity with circulation. So, if you find out one, you can automatically find out the other. This type of example where you have V theta equal to is proportional to r and V r equal to 0; this is known as a forced vortex.

So, it is something like you are creating a rigid body type of rotation in a flow by having an angular Vorticity to the system. So, it is a an example is like we have discussed in the class earlier that you have a cylindrical tank, and you start rotating the cylindrical tank in the limit as you are neglecting the viscosity, it takes the shape of a parabola that of revolution type that is an example of a forced vortex. So, you can have a forced vortex with this type of velocity. So, a velocity is 0 at r equal to 0, you have to keep in mind 1 very important thing. So, the circulation per unit area is given by the Vorticity.

Let us say that somebody has taken an area like this. So, this is a circle somebody has taken a area like this, this is the centre of the circle and he is trying to find out what is the circulation. We will see that in certain cases this definition will be restricted if we include the origin. So, let us take another example that will illustrate that what is the corresponding problem. So, take an example 2, with understanding of example 1 and example 2 this example problem that we are going to solve it will be easy for us to appreciate. Example 2 is something which we call as free vortex, what is a free vortex? So, free vortex is defined as the V theta component is inversely proportional to the radius, here it is directly proportional to the radius inversely proportional to the radius and V r is equal to 0.

So, this kind of a situation you get in a kitchen sink. So, you open the tap and what you will see that as it comes. So, the sink is first say you fill up the sink first with water, close the valve so that the water cannot go out. Now suddenly open the valve, you will see that as the water comes to the outlet and goes through the pipe when it comes to the centre of the sink it is coming out with a very high velocity. So, there and it has a sense of rotationality in the flow. So, the V theta is inversely proportional to the radius, it is singular at r equal to 0 mathematically. Because at r equal to 0 it is as if V theta tends to infinity, and you cannot have an infinite velocity. So, you have to keep in mind that this is something where this r is defined in the limit as r tends to 0 plus, not r equal to 0 r equal to 0 is a point of singularity for such a case.

So, when you are writing now you try to figure out that what should be the circulation. So, the circulation here just we will have similar element AB, BC, CD, DA we will straightaway write because by now you know how to write it. So, for AB, what will be the circulation? So, minus C by r into r d theta, for BC it is 0, for CD plus c y r plus d r because new r is now r plus d r into r plus d r into d theta and for d A, it is 0.

So, very simple observation is this r f x will get cancelled out so it is 0. And clearly it is so, because you can also find out the Vorticity you will find that the Vorticity is also here 0, these type of cases where the Vorticity is 0 is known as Irrotational flow. So, Irrotational flow is a flow with 0 Vorticity, we will come into the concept of Irrotational flow soon. Irrotational flow means 0 or null Vorticity vector that is the very simple definition, and from the name itself it is quite clear that there is no element of rotationality in the flow, that is why it is called as Irrotational flow.

Now if in this example you take a close contour like this, by definition is a close contour you try to find out what is the circulation. So, for these 2 edges the circulation will not be there, for this edge the circulation will be there. And clearly when you sum it up over the contour it is not 0, if you take such an element. What is wrong with that? A very important conceptual mistake is there by choosing this element; you have taken the element by including the point of singularity. So, you cannot take an element which contains the point of singularity to establish the relationship between the circulation and the Vorticity.

So; obviously, this is not a valid element, yes.

Student: in the (Refer Time: 17:27).

Yes.

Student: So, we cannot (Refer Time: 17:28) Vorticity (Refer Time: 17:28) that is why no rotational.

Yes.

Student: (Refer Time: 17:37).

No, no, no that is I mean that is like it is just coming with it is that is not a circulation, it is like when you will always see that when it comes we will see now that practical example when r equal to 0 it is not defined. So, it is always a forced vortex close to r equal to 0. So, what is the practical case? The practical case is never a purely forced vortex or a purely free vortex.

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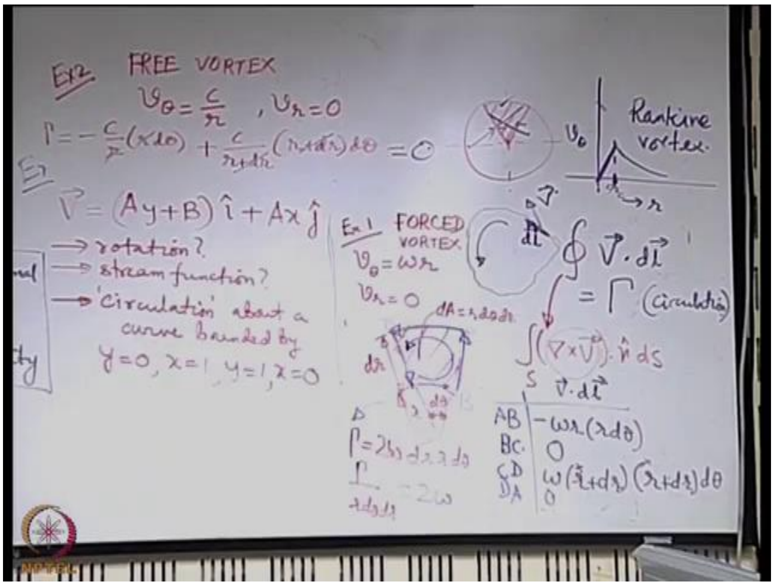

So, the practical case you draw the velocity profile V theta versus r, you know that at r equal to 0 if you use a completely free vortex understanding then V theta will be undefined. So, actually always very close to r equal to 0, it is a forced vortex. So, close to r equal to 0, V theta will be equal to omega r. So, it will be something like this, and away from that it may become a forced vortex a free vortex.

So, it is like a rectangular hyperbola type.

Student: (Refer Time: 18:38) when there is a rotation.

There is a rotation close to this one. So, this is known as this is the practical example; free vortex or forced vortex neither of these are like very practical example, but their combinations are quite practical this is known as Rankin vortex. So, it is a combination of free and forced vortex. So, up to a particular radius say critical radius it is like a forced vortex beyond that it is like a free vortex; very classical example that occurs in nature is a tornado. So, a tornado very close to the eye of the tornado, it is up to that it is a forced vortex. So, it has a strong element of rotationality, beyond that it is like a free vortex. So, a tornado may be very well approximated by a Rankin vortex, which is the combination of free and forced vortex. To keep in mind that if you have such a combination you have to satisfy that at this point at this critical radius I mean where you have a like transition of behavior from free to forced vortex, V theta should be same as given by the considerations of free and forced vortex; because you cannot have a discontinuous velocity field, the velocity field in the physical sense is continuous.

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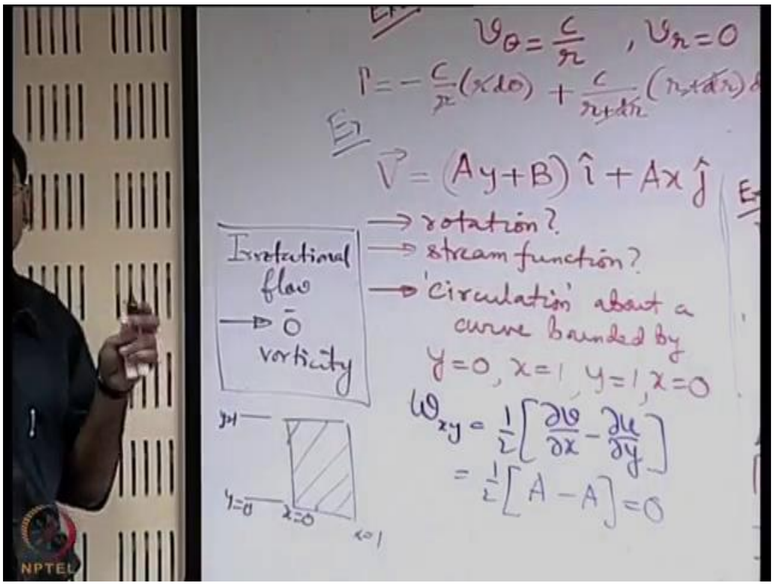

Now, if you come to this example we will try to utilize this example to find out the circulation. So, first what is the rotation? So, the rotation is given by the first we find the guard of the velocity vector. So, it is a 2 dimensional field. So, the guard of the velocity vector eventually will boil down to that you want to find out the angular velocity in the x y plane. So, what is that? Half of this one. So, this is basically if you find out half of the guard of the velocity vector, and take it is component in it is plane wherever it is important it will also boil down to the same expression.

So, that will be A minus A that is 0 the rotation is 0; that means, it is Vorticity is 0 and by the relationship between circulation and Vorticity, now you have a curve bounded by what y equal to 0 and 1 and x equal to 0 and 1. So, it is like a rectangular contour, this is x equal to 0 this is x equal to 1, this is y equal to 0 this is y equal to 1. So, this is the contour this is the area bounding the contour. So, this does not include any point of singularity. So, this is the valid area and therefore, the circulation about this curve is expressed in terms of the Vorticity, since the Vorticity is 0 this is also 0 you need not work it out. But I would suggest that you work it out yourself, check just by following this that you are getting 0 that will be a good exercise for you. And the stream function we already know how to do it I am not going into the details of it, the whole purpose of this problem was to demonstrate the rotation and the circulation process.

Next we will go to concept that is very much related to the rotationality of the flow again, and that concept is given by the name of velocity potential. So, we will see that what is the velocity potential and what are the important considerations that go behind this, so velocity potential.

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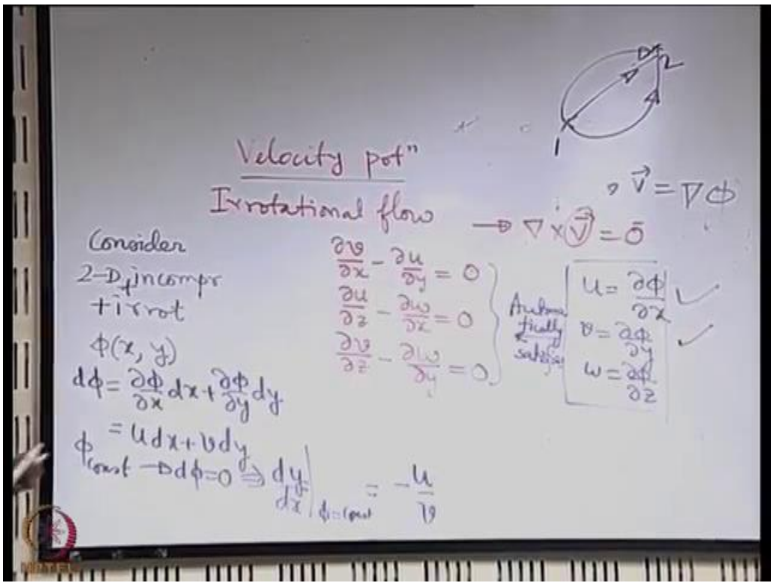

Let us consider the case of Irrotational flow. So, this is defined only for Irrotational flow. So, for nothing else therefore, we must find out the condition for irrotationality. So, what is the condition for irrotationality? The curl of the Vorticity vector is a null vector. So, what it will boil down to. So, if you write the three scalar components of these by expanding in a determinant form, and equate the scalar components to 0 you will get these.

So, from these you can see that if you now parameterize say u equal to, and w equal to del phi del z where phi is a function it is in general a function of x y z, then these definition automatically satisfies these requirement. This is not a magic this is a very much obvious, because from the theory of vector fields you know that if a curl of the vector is a null vector then that vector may be represented as the gradient of a scalar constant. So, if the curl is 0, it is a or a null vector it is possible to express V as gradient of a scalar function phi. So, that is what which is the mathematical basis of such a coincidence it is not a coincidence, but directly it follows from a important vector identity.

Now, such a field where the curl of the field is null vector, the field what we talk about is the general vector field here it is the velocity vector field, we call it as conservative velocity field. So, in general in field theory if the curl of a particular vector field is null vector that that field is a conservative field, it may be expressed in the form of gradient of a scalar potential. Maybe with plus or minus sign, but it is all the same because again with minus or plus this will be 0. So, when you express the vector field as a gradient of a scalar potential, what is the significant? Think about the case of a gravitational field. So, when you say are having a displacement of a particle from 1 to 2 in a gravitational field, it does not matter whether you are going by path this, this or this. So, why there only gravitational only gravity is the important force, then the work done for going from 1 to 2 is independent of the path, and just is dependent on the difference in the potential that is the that is the potential energy in this case. So, that path dependence comes from the conservative nature of the field.

So, in a conservative field, similarly this is talking about a conservative velocity field not a conservative force field, but the concept is very much analogous. So, whenever you have such a conservative field the field is expressible in form of gradient of a potential. In such particle mechanics in a gravitational field, that potential is a potential energy that we know. So, here we are talking about a velocity potential. So, now, we are interested say about a relationship between the velocity potential and the stream function. Now when we are interested about a relationship they must be comparable. So, when under what case, under what circumstances they are comparable. So, what is our objective now? Objective is to find out the relationship between stream function and the velocity potential. So, what type of flow should we consider, consider what? So, we should consider a case when both are defined.

So, when a string function defined? It is defined for 2 dimensional incompressible flow need not be steady, 2 dimensional incompressible flow when the velocity potential is defined for Irrotational flow. So, it has to be 2 dimensional plus incompressible plus Irrotational, so that both phi and psi functions are defined. On this basis we want to compare the 2. So, when you are considering a 2 dimensional Irrotational flow, you have only the relevant components as this u and v. So, now, let us find out what is say how we can express d phi, exactly in the same way in which we expressed d psi. So, what is d phi d phi is this one. So, it is u d x plus v d y, because u is defined as the partial derivative of phi with respect to x and so on.

Now, if you have a line of phi equal to constant, just like we had psi equal to constant then that will mean d phi equal to 0; that means, for that line which is having phi equal to constant you can have d y d x for phi equal to constant is equal to minus u by v. Let us now recall what was the case with the line of psi equal to constant? So, psi equal to constant. So, what was the expression for d psi?

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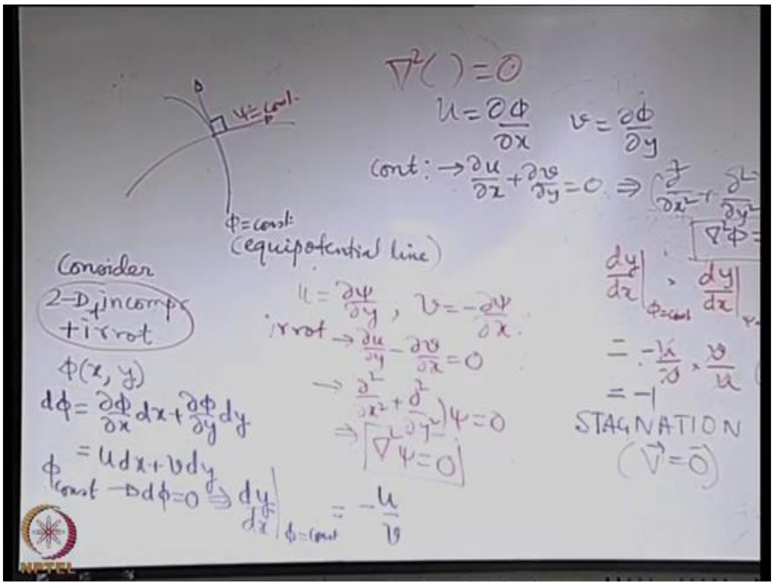

So, what was d psi? V d x minus u d y, so what is d y d x along the line with psi equal to constant? That is d psi equal to 0 that is equal to v by u. Now if at a common x comma y if you want to find out the product of this d y d x or phi equal to constant, and d y d x for psi equal to constant what is that? That is u by v sorry minus u by v into v by u, it is not trivially minus 1 that is a very important thing, because who has guaranteed that u and V are non zero, if at least one of this components you have 0, then you get a division by 0.

So, you have to make sure that these are non zero. So, if these are non zero then only you may cancel out this and when you are cancelling out both you are ensuring that u v both are non zero because both are appearing in the denominator. So, then in that case this is minus 1, and what does this minus 1 signify that if you have a line of constant psi and if you have a line of constant phi, they are orthogonal to each other. So, you have phi equal to constant and psi equal to constant. So, we do not say that they are perpendicular because it is not that the curves are perpendicular, but their common tangents at the point at the common point. So, here you have a tangent like this and here you have a tangent like this. So, these are perpendicular to each other. And therefore, the important conclusion is that phi equal to constant lines which are called as equi-potential lines. So, this is equi-potential line. So, this is very much analogous to the equi-potential line that you get in say electromagnetic field theory.

So, the phi equal to constant and psi equal to constant these lines are orthogonal to each other everywhere in the flow field, except for certain points. Which are the points the points are where the velocities are 0 those points are known as stagnation points. So, in a flow field the points where the velocities are 0 those are called as stagnation points. So, stagnation point is a point where v is 0, and at the stagnation point you do not have such a relationship because at the stagnation point you cannot really workout these. So, it is not true that the stream lines and equi-potential lines are orthogonal everywhere in the flow field; they are orthogonal at each and every point except for the stagnation point that it is not defined in that.

Now, finally, what we will see we will see a relationship or the governing equation for the expressions for phi and psi. So, if you say that you are interested about this case when you have both 2 dimensional incompressible and Irrotational flow. So, when you have a 2 dimensional incompressible flow. So, you have say u Irrotational flow you have u equal to this and v equal to this. Now if you want to satisfy this with the continuity equation continuity also has to be satisfied. So, you have this equal to 0; that means, you can write or laplacian operator of phi equal to 0.

Similarly, if you start with the definition of the stream function, you have u equal to and v equal to minus this one. Now this automatically satisfies continuity, but for the case when you are considering both phi and psi at define it also has to satisfy the irrotationality constant. So, for irrotationality you must have this equal to 0. So, for

Irrotational flow now if you substitute this you will get the same thing just replace phi with psi. So, you get laplacian of psi equal to 0. So, we can see that for 2 dimensional incompressible and Irrotational flow, both phi and psi satisfy the Laplace equation this prototype equation laplacian of phi that is the del square of phi equal to 0 or del square of psi equal to 0, the general prototype is del square of something equal to 0 that is known as Laplace equation.

So, both phi and psi satisfy the Laplace equation it does not mean that their solutions are same because boundary conditions are different. All though governing equations for phi and psi are the same, but their boundary conditions are different and therefore, solutions are different. So, in summary what we can see that we have defined what is the stream function, we have defined what is the velocity potential, we have seen that there is a relationship between these 2 when we have both defined that is 2 dimensional incompressible rotational flow, and both in terms of their governing equations and also in terms of their orthogonality.

Now, what we will do finally, we will look into some visual demonstration of certain types of flows, and end up the discussion today.

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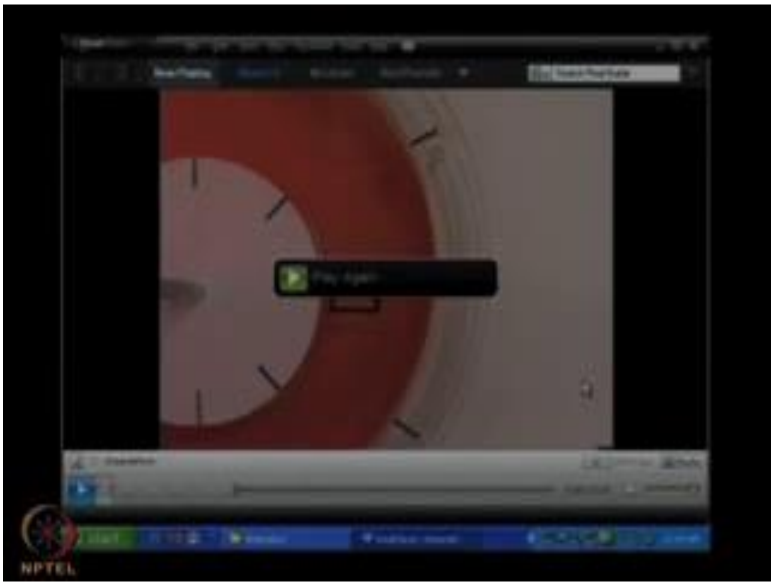

So, in this visual demonstration we will see some examples. So, first you see this is a this is a shear deformation; see there is a line element which is marked and this is the flow between 2 concentric cylinders, we have seen such a case when we were discussing about viscosity. So, you can clearly see that the fluid element is deforming. So, from a rectangular shape the fluid element is coming to the deformed shaped. So, this is the angular deformation that we are talking about we just play it again so that you can see it again. So, originally it was rectangular, but because of the shear you see that how it is getting deformed.

Now, we will look into a second example after this, where we will see that like if you have a general case when a fluid element is moving along a path.

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Then alike it can have a general type of behavior. So, it can have deformation. So, here it is just rotating you see that it is not deforming. So, it is just a pure rotation now let us look into a third example.

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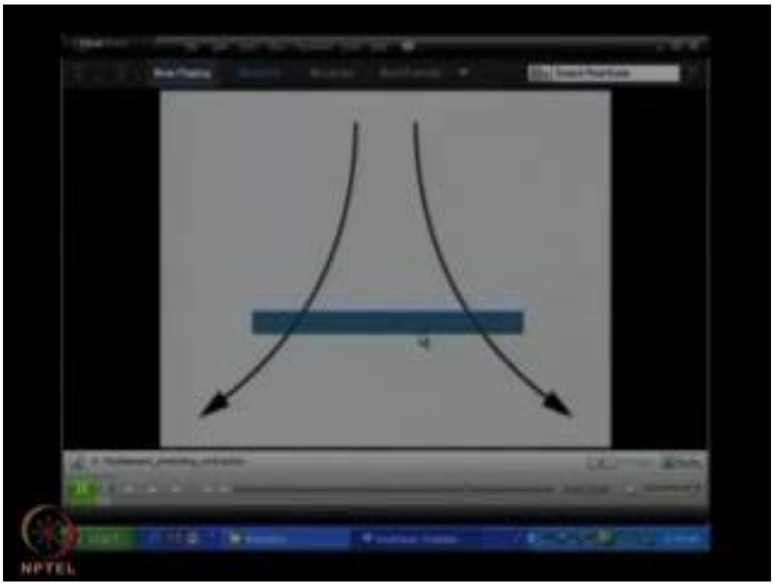

Where we see a flow this is this is a very interesting case. This is a case when it does not have any type of angular motion angular deformation, neither shear deformation nor rotation. So, it is as if it is getting stretched. So, I will leave it you to you on a as an exercise you find out what are the velocity components u and V that will lead to this condition. What are the important restrictions you must have both the partial derivative of u with respect to y, and partial derivative of v with respect to x equal to 0, then both the angular velocity as well as the angular deformation will be equal to 0 and then such a case is visible, that is known as the stagnation point flow.

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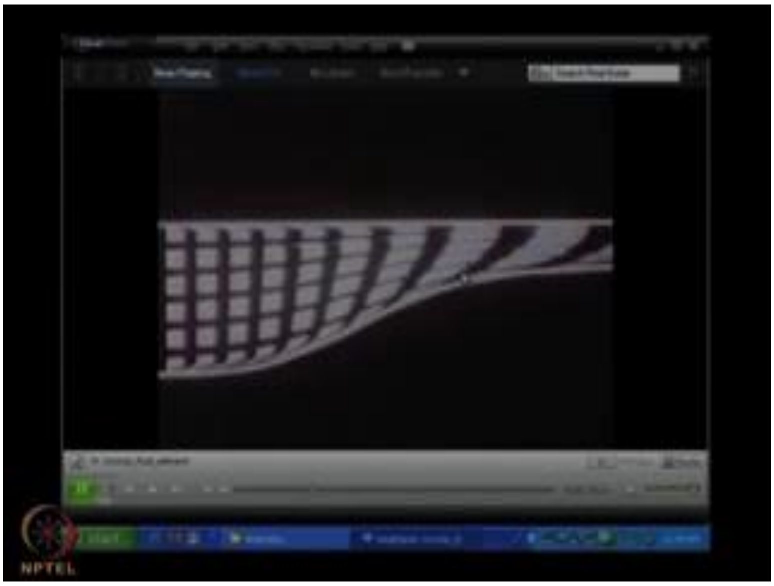

Finally let us look into a general case of an incompressible fluid element how this fluid elements flow. So, this is a visual demonstration of the continuity equation. So, you see that whatever flow is entering the same flow is leaving. So, if you have a constant passage what is happening is that the velocities are increasing to compensate for the decrease in the cross section area. So, this is what like a 1 u 1 equal to a 2 u 2 that type of expression that we have seen. So, we stop here today.

Thank you very much.