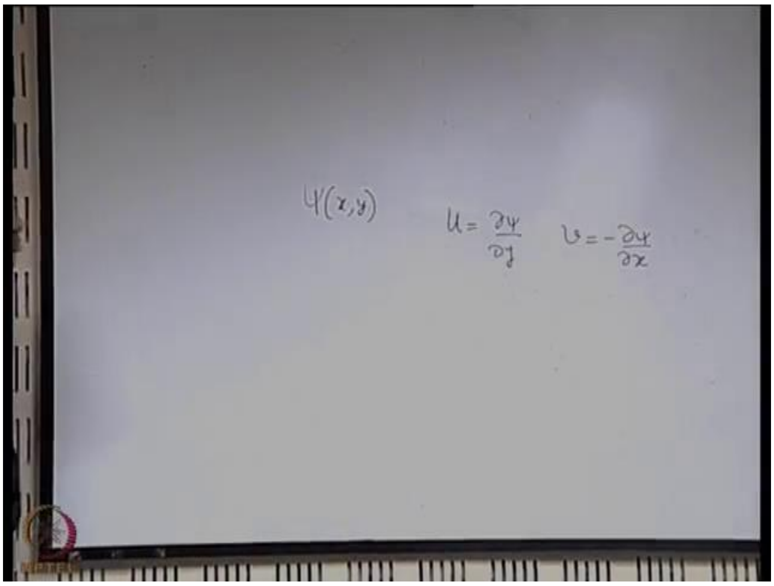
Introduction to Fluid Mechanics

Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 29
Stream Function

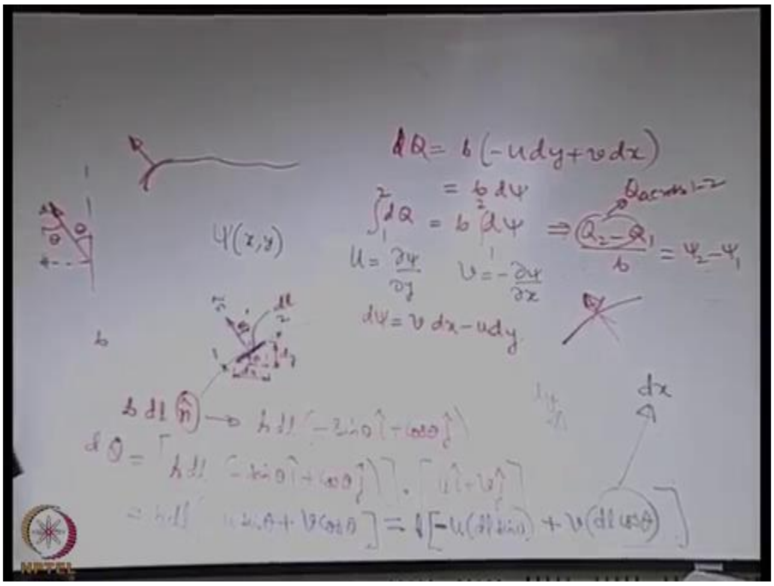
(Refer Slide Time: 00:23)



We continue with our discussion on stream functions. So, the stream function we defined as a function of x and y with an understanding that it should satisfy the requirement of mass conservation constrained by the continuity equation, and special case is giving that parametric form 2 dimensional incompressible flow. So, the important thing to keep in mind that is defined for 2 dimensional incompressible flow in the way which we are defining it. For other types of flows special cases again it may be defined with the slightly adjusted manner.

Now, say we are interested to define it for a 2 dimensional compressible flow, but or 2 dimensional steady flow may be compressible may not be compressible, but 2 dimensional steady flow. When you say a 2 dimensional steady flow, your continuity equation becomes; whether rho is constant or not based on that it will come out of derivative or not. So, this is a bit more general case than the case we had considered, and for these also it may be defined just replace u with rho u and v with rho v then the similar parametric form may be defined. So, it is not that this is a hard and fast rigid definition, but this is like a definition for the most common case that we are talking about.

(Refer Slide Time: 02:04)



Now, the stream function also has some relationship with the rate of flow. To understand that let us say that we have some line elements, and we are marking 2 points 1 and 2. We are interested to find out what is the total rate of flow between these points 1 and 2, how we can do that. To do that let us say that points 1 and 2 are located so closely that they are or may be let us consider a small element within that domain bounded by points 1 and 2, that small element is given by a length of dl which is almost like a cord of this curve.

Now, let us say that the length of this is dl. So, if you consider it like a vector sense it has outward normal vector like this say n cap, and it may be expressed as a function of the important angles. So, if you mark this angle as theta, then you can write dl or decomposed dl as dx dy, length of this is dl. Let us say that b is the width perpendicular to the plane of the figure. So, it is possible to write this dl and subsequently d a which dl into b, as a function of like dx dy and the orientation given by the angle theta. So, dl in the vector form is like dl n cap, it is just like area in a vector form is the magnitude of the area times the unit vector normal to the area and away outward to the area.

So, here the area is represented by dl, b is the width which is. So, it basically b into dl that is a da times the unit vector. So, the unit vector in terms of theta you can write; how you can write say that tangent to the curve makes an angle theta with x. So, the normal to the curve makes an angle theta with y. So, how can you write n cap in terms of theta. So, you have to resolve the magnitude this vector with direction n cap and magnitude 1 because this is a unit vector in x and y components. So, the n cap is like this, it will have x components like this and y component like this. So, x component will be what?

So, remember that this angle is theta that means this angle is theta. So, x component is minus sin theta and y component is cos theta. So, b dl minus sin theta i plus cos theta j; So this is basically the area. Now if you want to find out the volume flow rates, a volume flow rate is nothing but what? The dot product of the velocity with the area because if you have an arbitrary area what will give rise to a volume for it only that component of velocity which is perpendicular to the area, that gives an net flow rate right. So, if this is an area and the velocity vector may be arbitrarily oriented, but its components normal to the direction of area is what is only important; that means, dot product will give the component along that direction.

So, the dot product of the velocity with the area vector will give the component of velocity along the area vector, area vector means area normal. So, that will be the product of that will give the volume flow rate. So, the volume flow rate Q will be the this one b dl minus sin theta i plus cos theta j, this dot product with the velocity vector. What is a velocity vector ui plus vj. It is a; we are assuming it is a 2 dimensional flow, because we are discussing in it in the context of stream function. So, everything is the 2 dimensional concepts that we are discussing.

So, what it will become? So, let us call it may be dQ because we are talking about small length, and the flow rate across this small length is expected to be small. So, the next step of simplification it will b dl times minus u sin theta plus v cos theta. So, you can also write it as b u into minus u into dl sin theta, plus v into dl cos theta. From the figure you can clearly see that dl sin theta is dy and dl cos theta is dx. So, you can substitute dl sin theta with what we can substitute this with dy, and you can substitute dl cos theta with dx. So, dQ is b dl sorry b into minus u dy plus v dx. Try to find out its relationship with the expression for d psi that we derived just in the previous lecture what was d psi.

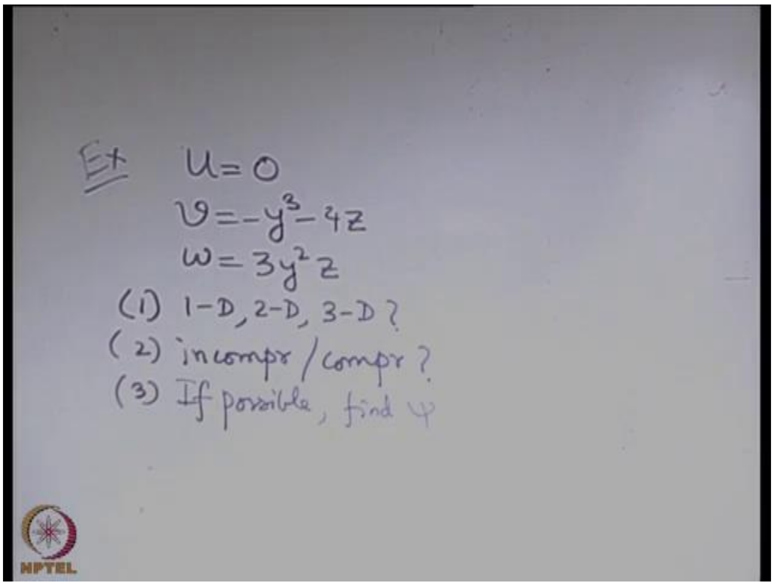
Student: (Refer Time: 09:04).

So, it was v dx minus u dy. So, this is nothing, but b into d psi; that means, if you want to find out the net flow rate between difference in flow rate between 1 to 2; so dQ from 1 to 2 that is b d psi from 1 to 2. So, the difference in flow rate between the points 1 and 2 Q 2 minus Q 1 what you need it is given by difference in stream function; that means, what is this Q 2 minus Q 1? Q 2 minus Q 1 is the net flow rate that is taking place through the element 1 to 2; because it is basically sum of half of the flow rates so for such differentially small elements like dl. So, Q 2 minus Q 1 is the net Q across 1 2; you can clearly see that if psi 2 equals to psi 1 that is if the values of the stream function are same then there is no flow, and it is quite logical because we have defined stream lines in such a way that streamlines are such that the velocity vectors are tangential to the stream line.

So, a very important corollary of that is there cannot be any flow across the stream line. So, if you have a flow if you have a streamlined like this you cannot have any flow across this because all the velocity vectors are tangential to it, there is no normal component of velocity and normal component of velocity can only give a cross flow. So, because it is so, you clearly see from definition of stream function also it is reflected in that way, because streamlines are characterized by constant stream function. So, you do not have change in stream function along a stream function and therefore, no flow across a streamline.

So, the difference in stream function between 2 points, gives a quantitative indication of what is the flow rate across the line element that joints those 2 points per unit width. Per unit width per unit width is important because it is just a 2 dimensional concept. So, with this background on stream function let us workout maybe 1 or 2 problems from your textbook related to stream functions.

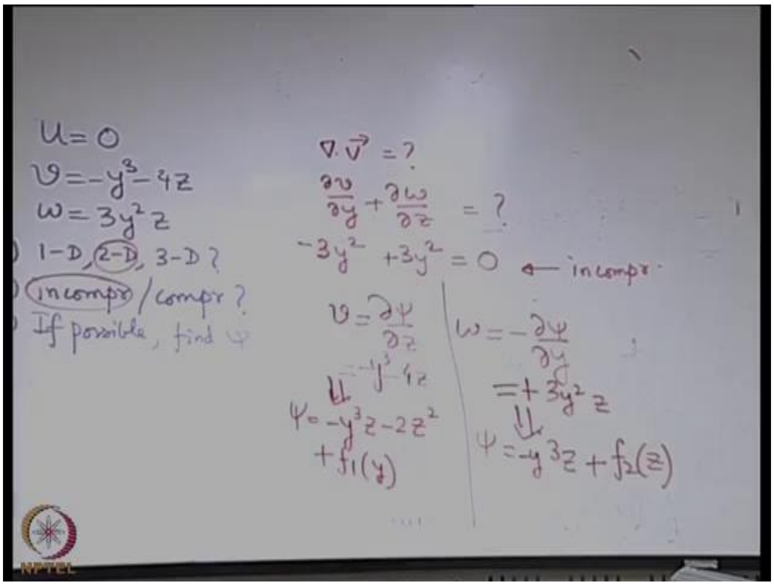
(Refer Slide Time: 11:47)



So, let us workout may be one simple problem to begin with; say you have flow field given by the following components u equal to 0, v equal to minus y cube minus 4 z and w equal to 3 y square z these are velocity components which are given. Now the questions are as follows number 1, if the flow 1-D, 2-D or 3-D perhaps we cannot have a more simple question than these, the next one whether it is incompressible or compressible flow, and third part is that if possible define a stream function for the flow.

So, if possible is very important because the stream function is not defined for all types of flow. So, if that stream function is defined then only we can find out stream function that is the whole idea of the problem. So, let us try to work out this problem. So, the first part it is the 2 dimensional flow because it has 2 velocity components fine. Incompressible or compressible flow, so how will you test? The divergence of the velocity vector you should find out because that gives the rate of volumetric strain. So, if the rate of volumetric strain is 0 that means, it is incompressible flow.

(Refer Slide Time: 13:27)



So, we have check what is the divergence of the velocity vector. So, here it will boil down to the checking of what is this. So, the first one is minus 3 y square, and the second one is plus 3 y square. So, this identically becomes equal to 0; that means it is incompressible flow. So, now, you have a 2-D flow from first part, you have an incompressible flow from the second part since its 2-D and incompressible the stream function is itself defined. So, we can now attempt to find out the stream function. How do you find out the stream function? So, now, it is like it is in a yz plane that we are talking about the flow. So, we have to define in some way v and w in terms of the stream function.

So, how do you define v? May be again plus minus sign is not that important what is most important is it should be of opposite sign, to make this equation trivially satisfied equals 0. So, let us write the corresponding components, yes.

Student: this the different way we are having 1 means (Refer Time: 15:04).

Yes this is a v component and this is w component. So, these are written in such a way that this equation is satisfied, that is what I am always saying that do not write take it as a formula. Now see there with the slight change the components of velocity given as y and z components instead of the previous one, you see that you are facing a dilemma that should not be there. Again the objective will be given a form of continuity equation just write a parametric form which satisfies this, and that will automatically give you the definition of the stream function for that case. So, do not confuse with the case with the case which we were discussing for developing the theory in a x flow in xy plane, now the flow is in yz plane that definition does not work here.

So, this is given by minus y cube minus 4 z. So, if you integrate this with respect to z what will be psi is minus y cube z minus 2 z square then.

Student: (Refer Time: 16:14).

Plus some function of y, because again repeating this is partial integration with respect to z when you are integrating with respect to z as the variable even y is a constant with respect to that.

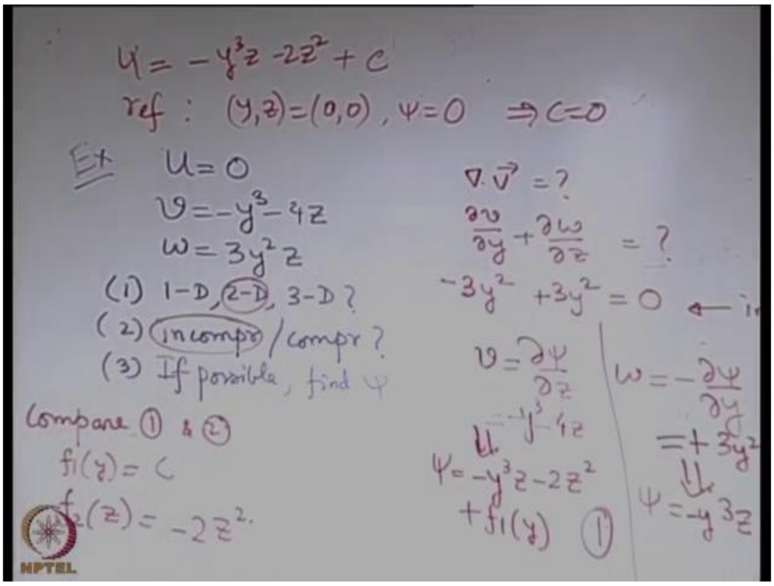
Student: (Refer Time: 16:35).

It cannot be function of x here because there is no dependence of x; I mean x is just like it is flow taking place in yz plane. So, where does the dependence of x comes? So, you have to understand it physically then next here. So, that is minus 3 y square z. So, if you partially integrate with respect to y. So, psi will be y cube yes.

Students: minus y cube minus y square.

Sorry this is plus right psi equal to minus y cube z plus function of z. So, you have say this as equation 1 and equation number 2, both are representing the same psi. So, you have to compare 1 and 2 to get f 1 and f 2. So, 1 and 2 are the same.

(Refer Slide Time: 17:51)



So, if you compare 1 and 2 what follows, what is f 1 y it may be some constant at the most, and what is or you may choose your reference in such a way that constant itself is 0, or let us just keep it as a constant and see that how you can choose it to make it 0, and what is f 2? F 2 z is minus 2 z square.

So, the expression for the stream function becomes minus y cube z, minus 2 z square plus some constant. Now these constant you may arbitrarily choose because there is no absolute value of stream function so to say, you have a reference with respect to that you find out the change because if the velocity component is defined as the partial derivative of the stream functions. So, it is not defined on absolute sense with respect to the value of the stream function. So, if you have the value of the stream function defined arbitrarily; say you define that your reference or origin is such that at y z equal to 0, psi equal to 0. Remember it is not a must it is just your choice, and always such choices are convenient because then you come up with the expression which will have C equal to 0.

It does not matter whether C is any arbitrary function or not because that then difference in stream function is what is important; that C will get cancelled if you find out the difference in stream function. But for working convenience you may set your references in that way, in most practical purposes you have flows on solid boundaries usually the solid boundaries are considered to be. So, solid boundaries itself is a stream lines, because there is no flow across it. So, by physical sense any shape solid boundaries itself a streamline.

So, you can give it a value of a particular stream function if it is a 2 dimensional incompressible flow. So, classically just as a matter of convention we give you psi equal to 0 as a reference, again there is no sensitivity you might give 1, 2, 4, 5, whatever, but it is just a reference a data with respect to which you calculate other stream function. So, solid boundaries are classically referenced as 0 stream function. So, let us may be move on to a next problem which may be similar to this, but let us just workout another problem with respect to the concept of the stream function. So, to do that again we will be solving a problem from your textbook.