Introduction to Fluid Mechanics
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Lecture - 28
Deformation of fluid elements-Part-III

Now, when you want to define the angular velocity of the fluid element, it is not such a straight forward picture as that of rate of strain or rate of angular deformation. Why it is not so straight forward, let us try to see angular velocity.

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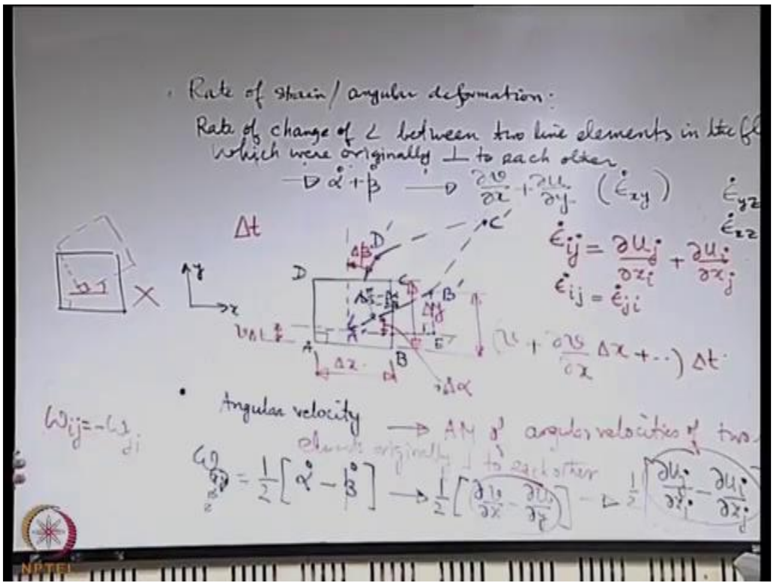

See whenever you are thinking of angular velocity we were discussing about the rigid body motion, but that is the special case. Usually fluid is not under a rigid body motion just like this case. So, when it is not in rigid body motion you cannot really have a unique angular velocity of all the line elements in the flow.

So, the line element say, AB it has some angular velocity, the line element AD it has different angular velocity; that is why this angles alpha dot this angle delta alpha and delta beta they are different. If it was rotating like a rigid body then what would have been the case? Then the case would have been like this. So, you have original fluid element, now you have the fluid element may be like this, so that this angle is preserved that is the rigid body rotation. But that is not happening here, at least in the example we have drawn in the figure. So, if that was the case, we could clearly say that what is the rate of change of this angle, and time rate of change of angle would have given angular velocity very straight forward.

Here those angles are different and therefore, we have to come to our angular velocity definition with a compromise, not exactly same as we do for rigid body mechanics, but keeping in mind that in the special case that it becomes a rigid body it should follow the rigid body mechanics definition of angular velocity. So, when we say that we may define the angular velocity in this way that it may be thought of as the arithmetic average or arithmetic mean of the angular velocities of 2 line elements which were originally perpendicular to each other.

Now if you see the; what is consequence the consequence is as follows. So, if you have a case where you have the fluid element having the angular deformation. So, that it is not a rigid body rotation. So, arithmetic mean of angular velocity of 2 line element; that means, half of angular velocity is of AB and AD which were perpendicular to each other. So, what is the angular velocity of AB that is alpha dot; what is angular velocity of AD?

According to this figure it is minus beta dot, because alpha dot in this figure is in the anticlockwise direction if we take that as positive, beta dot is in clockwise direction we should take it as negative; because we are already putting the magnitudes of alpha dot and beta dot in terms of u and v. So, this will become half of; so, let us call that this is angular velocity in x y plane or sometimes people give index instead of z, because this is also a angular velocity with respect to z axis. Consider the special case when the alpha dot to minus beta dot; when the alpha dot equals to minus beta dot that is the case which is represented in this figure, because when alpha dot is equal to minus beta dot that is given in terms of total deformation delta alpha is equal to minus delta beta, and then delta alpha plus delta beta that becomes 0. So, the angle which was originally pi by 2 remains same as pi by 2, and that is the rigid body deformation case that we are considering.

So, in that case alpha dot is equal to minus beta dot it becomes as good as either alpha dot or beta dot. So, in that limit it corresponds to definition of angular velocity in rigid body mechanics. So, whenever we are defining something we have to keep in mind that in special limit which is already known, it should be consistent with that special limit the definition should not violate that special case. Now what we can see is this also we can write in terms of index notation, and it is; how is it possible? You can write this as half of this one. You can write similar components of rotation in the other plane. So, this is rotation with respect to z axis you can have rotation with respect to x axis y axis and similar such terms will be there.

This half factor is put as matter of definition to ensure that in the limits when it is like a rotating like a rigid body, this angular velocity is same as definition of angular velocity of the rigid body. That is why that adjustment factor comes importantly that if you forget about that adjustment factor; it is the term that is there in square bracket that is actually dictating what should be the angular velocity in term of velocity component. How you can just take as a scale factor or adjustment factor for merging the definition in limiting cases. Now if we see clearly we have come up with 2 different types of angular motion representations: one is to this epsilon i j another is the omega the angular velocity, and it is possible to see that when they combine with each other what do they actually represent?

If you look it into this expression like for the omega, you can clearly see that if you write it in a matrix form, now what is the relation between say omega i j and omega j i. So, you will be replacing j with i and i with j. So, it will be minus of. So, it is if you write in the matrix form it is the skew symmetric matrix. Now it is possible to combine these 2 deformation and write it in some way which is again a straight forward thing that is if you write the partial derivative of u i with respect to say x j, you can write it as this one plus of; why it is important because this is a general velocity gradient, and a general velocity gradient is what is related to what is expected to be related to deformation.

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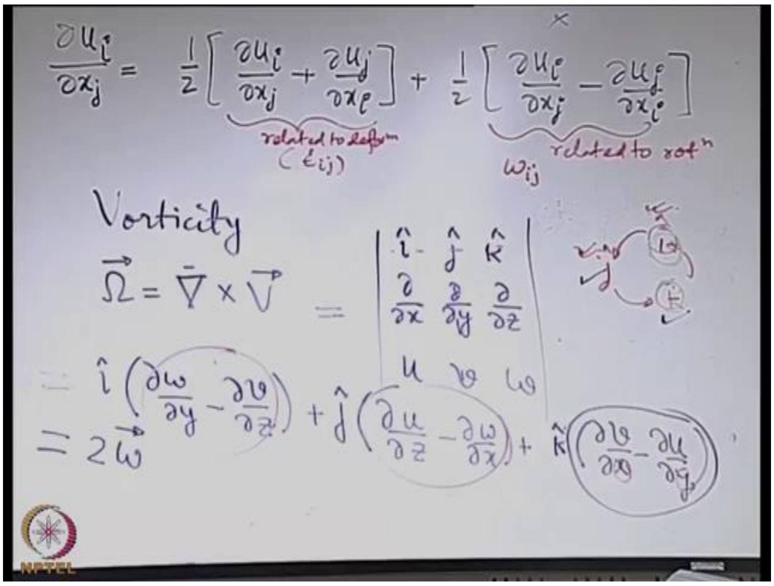

Now, you see that when you have a general velocity gradient, out of that only one part is related to deformation and another part is not related to deformation, it is related to just angular motion like rigid body. So, this id related to the deformation and this related to the rotation. You have to clearly understand the demarcation between the rotation and deformation, and this is a like epsilon dot i j sometimes when half is considerate with a epsilon dot i j, it is given as different name say gamma dot i j. Just a matter of writing the symbols there is no great entity and this one of course; we know that with the half it is omega i j.

So, what we can see from here is a very important thing. So, when we write the general velocity gradient, which should be in general of function or parameterization of the deformation, out of that clearly we distinguished that one part is not related to deformation. So, only this part is related to the deformation and whenever you relate the shear stress with the velocity gradient, this part is what is important; because this part is giving rise to angular deformation the other part is the rigid body motion. So, it should not be relating the constitution of the shear stress in a material in a fluid. So, whenever we have discussed about the Newton law of viscosity, you clearly see that this is the term that we had actually taken. So, if you write if you look it into terms of say u and v it was what was the quantification of this deformation, we wrote partially derivative what we wrote we wrote tau equal to mu times this one; actually we should have written it this plus this one.

Because we were considering a unidirectional flow v was 0, and that why this terms was not considered; we will look into it more that is. So, whatever forms of Newton’s law of viscosity we have discussed in fundamental way it is correct that is the shear stress is linearly proportional to the rate of the angular deformation, but the quantification of the angular deformation that we made earlier was based on a very simple case and assumption? So, as we advance and proceed more and more we will come into more and more rigorous ways of writing the Newton’s law of viscosity, and we will again take it up later on in one of our chapter we will not take it up in this chapter because these just bothers about the kinematics. So, this does not bother about force.

So, just because it has come in this context, and I am just reminding you that it is not that the Newton law of viscosity is like mu into du dy it is like the rate of deformation may be expressed by other terms also, in that special example which we took up in our earlier lectures we consider there is no other velocity component therefore, the other terms were not appearing. The other important observation is that, although it looks something which is nontrivial, but actually there cannot be the more trivial expression than this. Because it is just like you are writing a form every metrics can be written as the sum of symmetric and the skew symmetric metrics that is what we are trying to do.

So, eventually that symmetric matrix is being represented by the components of the angular deformation, and the skew symmetric matrix is represented by the components of the rotation. So, in a more formal way since we are dealing with the tensors; so matrices are some of the ways by which we may represent say a second order tensor in a notational form. So, in general these rules are for tensors we can say that any tensor may be decomposed as the sum of symmetric and skew symmetric tensors. So, metric is the special example illustration way of writing it. Next what we will do we will see that there are interesting quantities of parameter which are related to the angular velocity of the fluid, and based on that we will define a term called as Vorticity.

So, what is the Vorticity? See the entire subject of fluid mechanics in its fundamentals was developed by the mathematicians. So, we are trying to give a physical basis or physical insight to whatever concepts that we are discussing, because we have to keep in mind that after all we are trying to learn it in the engineering context, and we have to understand that what physics goes behind this mathematical derivations. At the same time when the subject was first developed, it was developed in a sort of the true mathematical way, and in certain cases quite a bit abstractive from any physical reality; and in that perspective the term Vortcity which was defined as the vector say it was defined as the chord of the velocity vector.

Now, when it was defined as the chord of the velocity vector, it was just defined as the mathematical quantity, but important thing is that such quantities are always defined in a general mathematical theory known as field theory. See field theory is something which is something which is so general that that it is applicable equally in the electromagnetics than in fluid mechanics and so, filed theory talks about vector or may be the scalar field, but in general in vector field and what are the different rules that governed the different rules that governed the behavior of the change of the vector in the vector field. So, when you are talking about electromagnetics, you are having certain parameters, when we are talking about fluid mechanics you are having different sets of parameters, these are physical sets of the parameters, but when you look it into the mathematical field theory there is hardly any distinguish between fluid mechanism and electromagnetics, in terms of the basic mathematical theory that goes behind.

We will see later on a couple of examples where you will find it very much analogous to as if there is an electrical field and something is happening with the electrical field. Now this Vorticity when it is defined in this way. So, let us try to expand it in terms of its Cartesian components; and let us see that will give us fair idea of what physically it tries to represent, we will write it in a determinant form that is the Vorticity vector. So, let us write its components, what are the components plus. So, we are now expanding with respect to these.

Alright now think of the special case of rotation in the xy plane that we are talking about. So, if you have rotation in the xy plane, you can clearly find out that this is one term that came it into the picture when we define the angular velocity, infact that this is double of that. If you are interested about rotation in the other planes, you will easily find out that those are given by half of these components. So, this is the vector which physically is nothing, but 2 times the angular velocity vector.

So, Vorticity is the mathematical definition, what it gives physically a sense of rotationality in a flow. So, if the rotationality in the flow is very strong we say a vortex is created in the flow, and the strength of that is given by the Vorticity vector. So, that is the physical meaning of Vorticity vector. So, if you have say a line element say this type of the symbol symbolic line element you put it in the fluid, if the fluid has an element of rotationally it will change its orientation otherwise it will move parallel to itself in the flow. So, that will give a visual understanding of the whether the flow has an element of the rotationality or not, and that is mathematically is quantified by the Vorticity vector. If you see it is if you know one of the components Vorticity vector, it is possible to generate other one intuitively with going into all cross products.

So, many times the common mistake is like in the sign. So, which term will come it as positive and which term will come it as negative; for that you have to just keep in mind that you maintain the different cyclic order for the x, y, z components. So, like if you follow this cyclic order, see when you are writing the z component first comes the x and then comes the y, first comes the x derivative and then the y derivative. Similarly when you are writing the x components, first comes the y derivative. So, when you are writing this component first comes the y derivative and then the z derivative. When you are writing the j that is the y component first you are coming with this one and then with this one. So, if you maintain this cyclic order it is possible to reproduce that without doing the cross product each and every time.

In summary we have discussed the quantification of linear and angular motion of the fluid elements. Now based on these we will come up with a few important conceptual understandings. So, those conceptual understandings are based on the 2 important terminologies in fluid mechanics: one is stream function and velocity potential let us see what is stream function.

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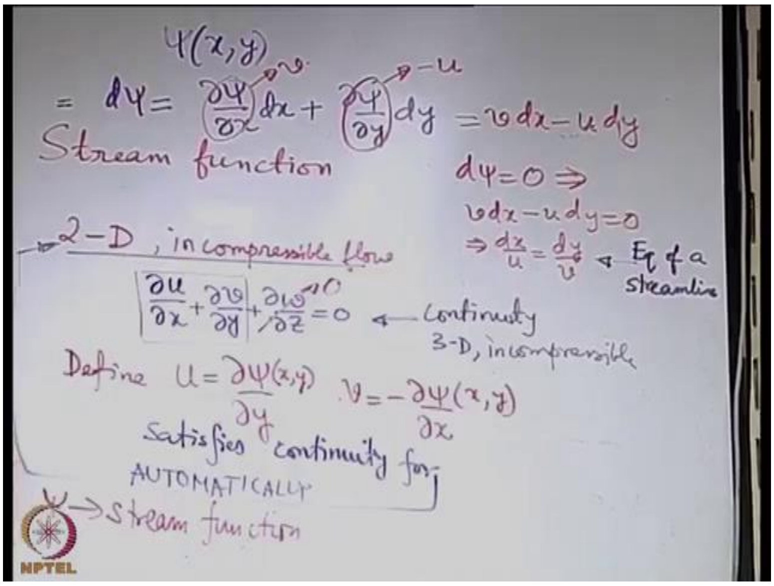

So, stream function is defined as follows; to look it into the definition let us first figure out that what is the motivation behind defining such a function.

Let us take an example of 2 dimensional incomprehensible flow and write form of the continuity equation for that. What is the form of the continuity equation for 2 dimensional incomprehensible flow? If it was the general three dimensional flow then it would have been this equal to 0 for the general 3-D what incomprehensible flow, because the condition for incomprehensibility 0 volumetric strength. So, this is for continuity for 3-D and incomprehensible flow.

Now, when it is 2 dimensional; that means, the third velocity component is not important, so you are left with these 2 terms. Now let us say that you interested to find out the velocity components u and v, we will later on see that this is not the only equation governs the change in u and v, there are other differential equations which need to be coupled. But just in a notional or mathematical form if instead of the 2 variables u and v, you could transform into a single parameter that is express u and v in terms of the single parameter that satisfies automatically this form of the continuity equation then u and v may be parameterized with respect to that new function. So, what type of parameter that we may choose? Let us say we define u equal to partial derivative of psi with respect to y, and v is equal to minus partial derivative of psi with respect to x where psi is the function of x and y. If you do that then what is its effect on this continuity equation? You see that this definition satisfies the continuity equation automatically; provided it is continuous differentiable up to the second order. So, this definition satisfies the continuity equation for 2-D incomprehensible flow so continuity for this special case which special case this special case automatically.

So, the objective is that we are trying to define the velocity components in form of mathematics in terms of a mathematical function, which ensures the satisfaction of the continuity equation automatically. Because not matter how complex or how simple the flow it should satisfy the continuity equation. So, this definition cannot highlight that see why we are restricted to such a case because for a more general case it is not easy to find such parameter.

So, if you had a 3 dimensional case, there you have not been possible to find out such a parameter automatically that satisfies this general equation. So, this definition is restricted for a 2 dimensional incomprehensible flow, this function psi is known as stream function. One could give many names to this function psi, but why the name stream function, what is the specialty about the stream function. The specialty is that it has some relationship with the concept of the stream line that we have learnt, what is that relationship to figure out that let us write psi mathematically as a function of x and y.

So, when we want to write d psi, how we can write d psi, psi is the function of x and y. So, what should be the d psi? Plus right now let us substitute the u and v, remember one thing one could also write u as the minus of this and v as plus of this. So, there is no entity with this plus or minus only thing they have to be of opposite sign to satisfy the continuity equation. So, different books will take either this as plus this as minus or may be this as minus and this as plus nothing, but this wrong because your objective is to satisfy the continuity equation nothing more than that.

So, if you take this particular definition then in place of this like the first term you can write v and term as minus u. So, this will be become v dx minus u dy. So, what is represented by d psi equal to 0 is represented by v dx minus u dy equal to 0; that means, dx by u is equal to dy by v, what is this? This is nothing, but equation of the stream line, so, what is the very important conclusion? Important conclusion is along a stream line there is no variation in stream function; that means, one stream line represent the particular constant stream function that is the stream function equal to constant along a given stream line.

So, the conclusion is psi equal to constant along a stream line, and that is why the name stream functions. Another important thing you have to keep in mind, what is that is that important thing when you say psi is constant along the stream line you should not be confused that always psi is defined; when psi is defined it defined for the 2 dimensional incompressible flow, stream line is defined for all types of flow. So, this relationship is not for all type of flow, is only for that case when both are there stream line is always there, but always psi equal constant along the stream line is not relevant because all psi is not defined, only for 2 dimensional incomprehensible flow this definition works and only for that case we may say that t is constant along the stream line.

It does not mean that the stream line is the not there, if it is not a 2 dimensional incomprehensible flow it is very much there, but the stream function is not defined in this way. So, you cannot have analogy or relationship between those 2. So, we stop here and we will continue again in the next class.