Introduction to Fluid Mechanics
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Lecture - 27
Deformation of fluid elements –Part-II

We were discussing about the continuity equation last time and let us work out a problem from a text book to go ahead.

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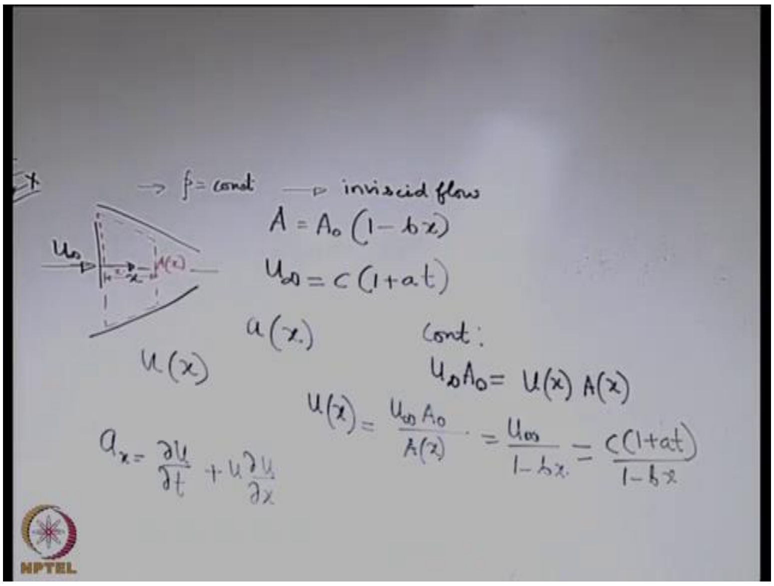

So, we have a nozzle like this and it is carrying a fluid with density equal to constant and some other input data are given for the problem the area of the cross section of the nozzle; it varies with the area of the cross section of the inlet in the following manner where x is the coordinate measure from the entrance.

And other information that is given that the velocity at which the fluid enters the nozzle; let us say that is u infinity that varies with time, it is of the form of c into 1 plus a t. where theses A naught, b, c, a, these are some given numbers; these are dimensional parameters to adjust the dimensions of the expressions. What; we are asked to find out that what is the acceleration as a function of x?

One important assumption is density is equal to constant. The other assumption that may not be stated explicitly in the problem that we will assume to go ahead is that it is an inviscid flow. So, when we assume it as an inviscid flow; that means we are not bothering about the variation of velocity along the transverse direction, we are assuming that at each section that velocity is uniform.

Now, to find out the acceleration, we have to go step by step, we have to first find what is the velocity how do we find out the velocity here? If you are given these information how do you find out the velocity; say u at a given x, how do we find that out?

You know the u at x equal to 0 you know the A at x equal to 0, you know the A at some given x. So, you can relate that with u at given x to the integral form of the continuity equation. So, if you take a control volume like this one maybe where say this distance is x. So, here the area a is a function of x and how do we express u infinity in terms of u at x, see in this control surface the surfaces which are the lateral surfaces across which there is no flow. So, only flow is through the inlet and through the surface at x. So, you can write from the continuity equation, it is density equal to constant. So, it is as good as u infinity into A naught is equal to u at x into A at x.

As we discussed earlier, this u should have ideally been the average velocity at the cross section, but because it is uniform it is same as at the central line, there is no difference. So, you can find out what is u at x that is u infinity A naught by A at x. So, that is u infinity divided by 1 minus b x because A naught by A is 1 by 1 minus b x. You also have to keep in mind that u infinity is not a constant, but it is the function of time. So, it is c into 1 plus a t divided by 1 minus b x.

Once you know that what is you as a function of x, the next thing that you can do you can utilize the expression for acceleration. So, it is not only a function of x, but also a function of time. So, the remaining work is very straight forward that acceleration along x, it is just a 1 dimensional type of flow, but you here, we write it is not just u as the function of x, but it is also a function of time, so here we write it u x t to be more precise because at given x, it will also vary with time and then you can differentiate this expression that is the straight forward exercise to find out the acceleration component along x.

There is no acceleration along any other direction because it is just a 1 dimensional flow. So, here you can see an example where you have both the temporal component as well as the spatially varying component of the acceleration. So, given this acceleration, it is also possible to find out that how much time a fluid particle will take to traverse say from one end of the nozzle to the other or given in fact the velocity components, just here one component of velocity you may work that what should be the time necessary for a fluid particle which is injected here to move along the central line from one end to the other if you know the velocity component. So, it is just like tracing the path line and finding it out when in along that path it moves and comes to the end of the channel. So, that is a straight forward extension of this one. So, we will move one to our next concept and that concept is related to the angular deformation of the fluid elements.

So, till now we have discussed about the linear deformation of the fluid elements and from the linear deformation of the fluid elements we found out that we got as a consequence a very important equation known as the continuity equation. For the angular deformation, we will try to first sketch that how a fluid element when deformed angularly will look and then try to quantify it in terms of the velocity components.

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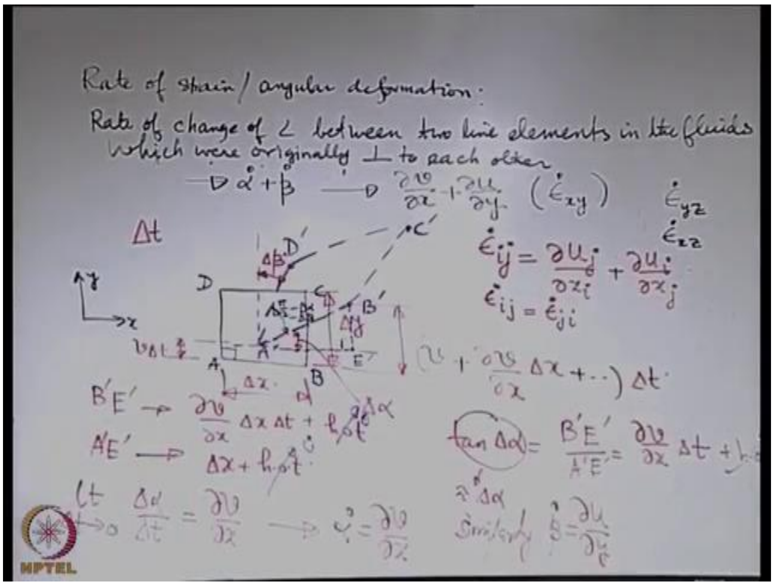

So, angular deformation of fluid elements; let us say that we start with rectangular fluid element like this when we are thinking about angular deformation the deformation is in terms of change of an angle and that angle change may take place in a prime. So, although the general fluid element is 3 dimensional element, but you can always take a 2 dimensional element and consider the deformation in the prime because no matter how complicated the deformation is it may be resolved in different planes. So, let us say that this is one such prime in which the deformation is taking place.

So, what is this special prime? This is a x y prime. So, for example, if there is a rotation with respect to the z axis then that occurs in this prime. So, it does not means that if we are focusing our attention on a prime, we are actually restricted to 2 d we are actually restricted to one component of the deformation and other components will be very very similar. Let us say that the name of this fluid element is A, B, C, D and it has its dimensions say delta x and delta y along x and y.

Now, lets us state our imagination and assume that this fluid element has got deformed with time. When it is got deformed, it is possible that it has got deformed in 2 ways, one is its volume might have got changed which is like an extension of the linear deformation, but its shapes might have also have got distorted which is more common if it is under shear. So, if its shape is distorted, let us say that may be it has come to this shape, this may not be a very regular shape this is just a schematic. So, now, if you see that what are the important parameters that are characterizing this deformation?

If you draw 2 lines through this new through this new location of a say one along x axis and another along y, the first important parameter that will come very much apparent is this angle say delta alpha. Here we are considering a small interval of time delta t within which this deformation has occurred because if we allow large time we will not be able to keep track of the deformation fluid is under continuous deformation. So, we take a small time interval in the small time interval the element A B which was originally oriented along x, now is oriented at an angle delta alpha with x, similarly let us say that this angle is delta beta.

Our objective will be to quantify the time rate of change of these angles say alpha or beta in terms of the velocity component here u and v because it is in a x y prime to do that we may make some simple geometrical constructions not actually a construction, but just to figure out what is happening. So, if you think of right angle triangle like this may be A prime, B prime, C prime.

This right angle triangle is important because in that right angle triangle if you know that what is B prime E prime, then you may possibly be able to express delta alpha or tan of delta alpha in terms of that and A prime E prime. So, let us try figure out what is B prime

E prime. That is the next objective. To do that we first understand or we first try to figure out that what is the vertical displacement of the point A. See to know B prime E prime, our consideration is the vertical component of the displacement. So, first we find out what is the vertical component of the displacement at A, also we find out what is the vertical component of the displacement at B, the net difference between these 2 is this length B prime E prime that we are talking about.

So, what is this displacement? Let us say that the velocity at the point at A is given by the 2 components u and v; u comma v are the components of the velocity vector. So, at A if you allow a time of delta t, what should be the vertical displacement? V into delta t.

Next we try to find out that what is the corresponding vertical displacement for the point B?

So, if this is v, we have find out what is v at B. So, v at A is v, what is v at B? That v we have to write here. So, what is that v? V plus higher order terms. So, here the change is because of change in x. So, that is why partial derivate with respect to x. So, from A to B when you go, it is not change in y, but change in x that is why this term is appearing then that into delta t. So, we can say that B prime E prime that is nothing, but the difference between these 2. So, that is like this.

Similarly, what is A prime E prime? A prime E prime is like delta x plus some higher order term, it is it is not exactly delta x, but it is why it is not exactly delta x because when the fluid element has got deformed. Now this component along x gives the linear deformation along x. So, linear deformation along x is something it is not 0. In general therefore, it is not same as delta x, but it is actually delta x plus the change in delta x. So, delta x plus the strain along x times delta x and that is much smaller than delta x itself. So, and we have figured out earlier that what is that. So, if you go back to that you will see that its delta x plus some higher order term.

That is the new length along x. So, if you find out what is tan of delta alpha then that is equal to B prime E prime by A prime by E prime that will be like this and may be some higher order terms. The delta x from the numerator and denominator it has got cancelled out. It is like this is a negligible term. Remember we are taking the limit as delta t tends to 0. So, when you are talking the limit as delta tends delta t tends to 0, this higher order term is tending to 0, this higher order term is tending to 0, this higher order term is also tending to 0. So, effectively in the denominator you are left with only delta x in the numerator, you are left with only this term, other terms are vanishingly small in comparison to this dominant term. So, in that limit, now as we know that if we are allowing a very small time interval delta t this angle delta alpha also will be also very small. So, in that case this tan delta alpha is as good as delta alpha.

The conclusion from this expression is that we can write these with limit as delta t tends to 0 we have already taken that limit, but now we are just writing it more formally delta alpha divided by delta t is equal to the partial derivative of v with respect to x and this is nothing, but the rate of change of the angle of this line element with the horizon term. So, let us call it alpha dot that is partial derivative of v with respect to x similarly we can say that what is beta dot? So, similarly just by same consideration what will be beta dot it will be the partial derivative of u with respect to y. So, whatever geometrical consideration that you had on this side if will have it on the other side it will exactly lead you to the same thing.

So, we have got a quantification of the rate of change of these angles and these are like rates of deformation. So, to say now, this rate of this deformation we have to quantify in terms of certain parameters. So, when we quantify the rate of deformation; we have to keep certain thing in mind that we have to give a definition to what is rate of deformation these are just changes in angles now how we translate that into a more formal definition. So, to have a more formal definition we may say that we are defining the rate of strain on the rate of deformation in the following way.

Rate of strain or angular deformation in what way see there is a very important quantification of the distortion in the angle what is that if you consider 2 line elements say A B and A D, there were originally at an angle 90 degree with each other now these line elements are no more at an angle 90 degree with each other they at now at an angle 90 degree minus delta alpha minus delta beta. So, this difference between 90 degree and this one gives an indication of the angular deformation because if there was no angular deformation this angle would have remained as 90 degree.

So, what is the change in this angle? So, what is the specialty of this angle we are trying to identify what is the change in angle between 2 line elements in the fluid which were originally perpendicular to each other. So, these are representatives of 2 line elements which were originally perpendicular to each other, but with deformation they are no more perpendicular to each other. So, what we are interested to find out that; what is the rate of change of angle between 2 line elements in the fluid which were originally perpendicular to each other?

So, if we want to quantify that we now know that we can quantify that in terms of delta alpha and delta beta and the rate in terms of alpha dot and beta dot. So, what is the change? The change is pi by 2 minus pi by 2 minus delta alpha minus delta beta. So, it becomes delta alpha plus delta beta that is the total change. So, the total rate of change is alpha dot plus beta dot. It is like delta alpha plus delta beta is the change the rate of change is you divide those by delta t and take the limit as delta t tends to 0. So, this is nothing, but. So, this we may call as say epsilon dot x y as a symbol epsilon for strain epsilon dot for rate of strain x y to indicate that it is actually an angular strain in the xy plane that is just a nomenclature one may use of course, different nomenclature, but this also works.

In this way, it is possible to write epsilon dot y z and epsilon dot x z you see that specification of this epsilon dot requires 2 indices. So, what are the indices the indices are the coordinate directions along which your original line were oriented and now because of deformation those line elements are not oriented any more along those directions. So, if you write if you want to write it in the index notation say if you want to write now it as epsilon dot i j, how will you write it? Just think in place of i and x and y i and j. So, what will be this one? So, this is the component of velocity along j. So, partial derivative of u j with respect xi plus partial derivative of u i with respect to x j the advantage with this notation is now once you write it you do not have to bother whether you are we writing in xy plane y z plane or x z plane it is like you just replace i and j with the particular indices 1, 2, 1, 3, 2, 3, like that.

One important observation that you can see is epsilon dot i j is same as epsilon dot j i. Just replace i with j and j with i. So, it is symmetric. So, if you write it in the form of a matrix where you can there are 2 indices each index varies from 1 to 3; that means, you can write a 3 by 3 matrix using these components again out of that you will have 6 independent components because of the symmetry. So, this is let me now this may be written in the matrix form that is the components of this not only that what is the thing that you get from this is also a second order tenser because it may be shown that it maps a vector on to a vector and not only that it requires 2 indices for its specification in the Cartesian notation.

So, we have seen the rate of angular deformation, but one important thing is we have seen examples earlier that there are cases when the fluid element is not having angular deformation as such like this, but it may be rotating like a rigid body. So, if the fluid element is rotating like a rigid body without any angular deformation then there also will be a change in some angle that will not be an angular deformation, but the change in angle because of rigid body rotation. So, that is also an aspect of angular deformation and in place of deformation we can just call it rotation because it is not actually a deformation it is the change in some angle because of a rigid body type of motion.

So, we will now see that; what is the way in which we may parameterize or we may describe the angular velocity of a fluid element and the angular velocity definition should be such that whenever there is a rigid body motion; the angular velocity will only be there, but the rate of detonation would be 0. So, keeping that generality in mind we have to define the angular velocity. These are all definitions we know that what we want to qualitatively represent and we are now trying to put into some mathematical definitions to quantify those physical features that is what is the exercise we are undergoing at the movement.

So, the next objective therefore, is finding the angular velocity of the fluid element.