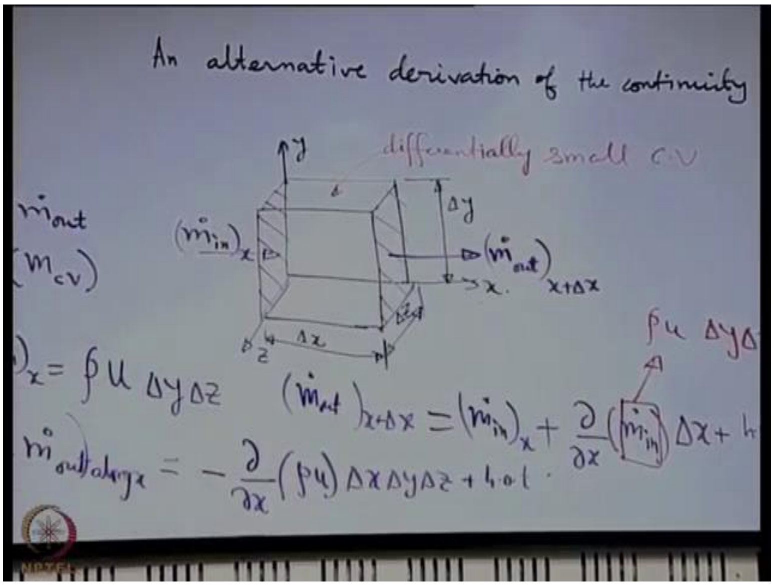
Introduction to Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 25
Derivation of continuity equation

We were discussing about the continuity equation last time and we will see now that there may be a different way of deriving the continuity equation; not only one different way, but there could be many different ways of looking into that we will look into one such alternative of deriving the continuity equation here and in our subsequent chapters, we will look into other possibilities. So, more number of different ways we look into it gives us better and better insight of what is there actually in the continuity equation.

(Refer Slide Time: 00:57)



So, we look into an alternative derivation of the continuity equation. In this alternative derivation, our object will be to look into the entire thing from Eulerian viewpoint; that means, we will identify a specified region in space across which fluid is flowing and that we call as a control volume. So, in a control volume of a particular extent, let us for simplicity in deriving the equation assume that the control volume of rectangular parallel of perfect shape. So, it has its dimensions along x, y and z as; say delta x, delta y and delta z which are small and in the limit we will take all these as tending to 0.

So, this is differentially small control volume that is the entity; differentially small control volume. Now what is happening across this control volume? Some fluid is coming in, some fluid is going out and that is occurring over six different faces and each face has a direction normal and basically the mass flow rate across that face is taking place normal to the direction of that respective face. So, if we consider, the flow rate along x then we should be bothered about which faces? We should be bothered about these 2 faces because these 2 faces have direction normal along x.

Let us see; what is the mass flow rate that gets transported across this faces. So, there is a mass flow rate that enters the control volume along x. Let us symbolize that in this way. Across the opposite face say, there is some mass flow rate that goes out and that occurs at x plus delta x. If this is x, this must be x plus delta x. So, how do we characterize the difference between these 2 because what we have to remember? We have to remember the mass balance, what is the mass balance? What is the rate at which the mass is entering say m dot in, it may be along x, y or z minus m dot out that also maybe along x, y, z. So, what is the difference between these 2?

Say there is some mass flow rate coming at the rate of 10 kg per second and say there is a mass flow rate that leaves the control volume at the rate of 8 kg per second. So, what the remaining 2 kg per second will do? That will increase the mass of the mass within the control volume. See volume has the particular volume, it does not have a fixed mass. So, if it is a compressible flow say it is highly possible that the mass inside these changes. So, that remaining 2 kg per second may contribute to the rate of change of mass within the control volume. So, we can say that mass flow rate in minus out is nothing, but the rate of the time rate of change of mass within the control volume.

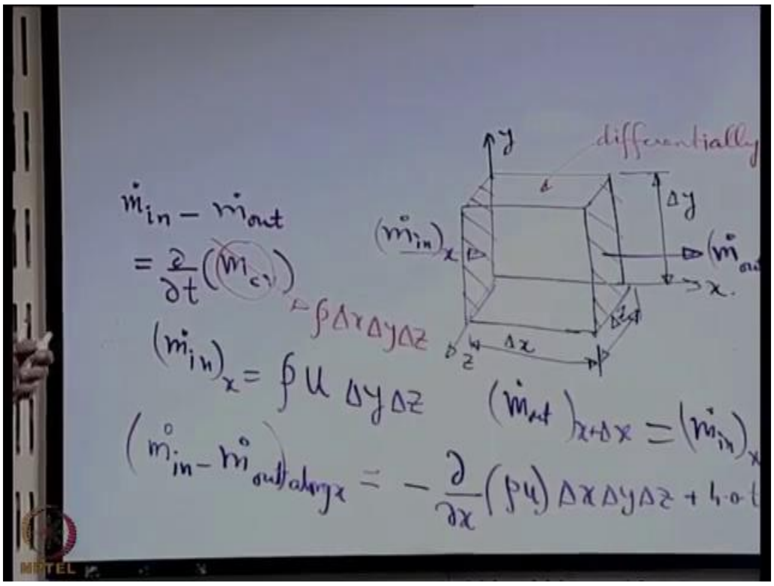
Again why do we use a partial derivative here? Because by specifying the control volume here by some fixed coordinates, we are assuming that we are freezing its locations with respect to position and trying to see, what happens in that frozen position with respect to time? So, that is why a partial derivative with respect to time. So, when you say m dot in minus m dot out, you have to remember that it has like contributions for flow along x, y and z. So, we can try to write what happens along x similar expressions will be along y and z. So, m dot in x. So, what is how do you calculate a mass flow rate given a density? So, to calculate the mass flow rate you requite first to obtain the volume flow rate.

So, what is the volume flow rate volume flow rate is the normal component of velocity perpendicular that is velocity component perpendicular to the area times the area. So, the component of velocity along x is u, what is the area perpendicular to that into delta y into delta z that is the volume flow rate that multiplied by the density is nothing, but the mass flow rate very straight forward. So, when you write m dot out at x plus delta x basically you are looking for the value of this function at x plus delta x.

You know the value of the function say at x, again you can use the Taylor series expansion in the Taylor series expansion when you expand it you keep in mind that delta y into delta z is phi delta z is fixed. So, you what you can do you can also take it out of the derivative and think about the expansion of rho into u. So, this will be m dot in x plus higher order terms, let us substitute what we can write in place of m dot in x so that is rho u delta y delta z.

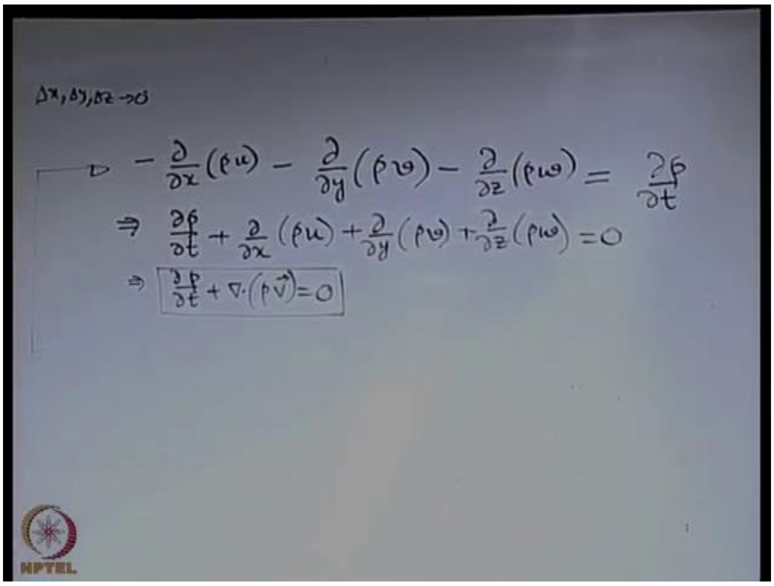
So, clearly if we write, what is m dot in, minus m dot out along x what is that? So, this term will come in the other side, it will be minus of that into delta x, delta y, delta z plus may be higher minus higher order terms, just by simple transformation or taking one side terms in the one side to the other. So, the minus sign appears similarly you get what happens along y and z and you can write therefore, expression of m dot in minus m dot out we will do that, but before that let us see what happens to the right hand side. So, what is the mass within the control volume? So, we were to write basically the time derivative partial time derivative of that.

(Refer Slide Time: 09:16)



So, what is the mass within the control volume let us try to write what is the mass within the control volume what is that yes. So, it is the density times the volume. So, rho if it is the density that times delta x into delta y into delta z that is the mass within the control volume.

(Refer Slide Time: 09:42)

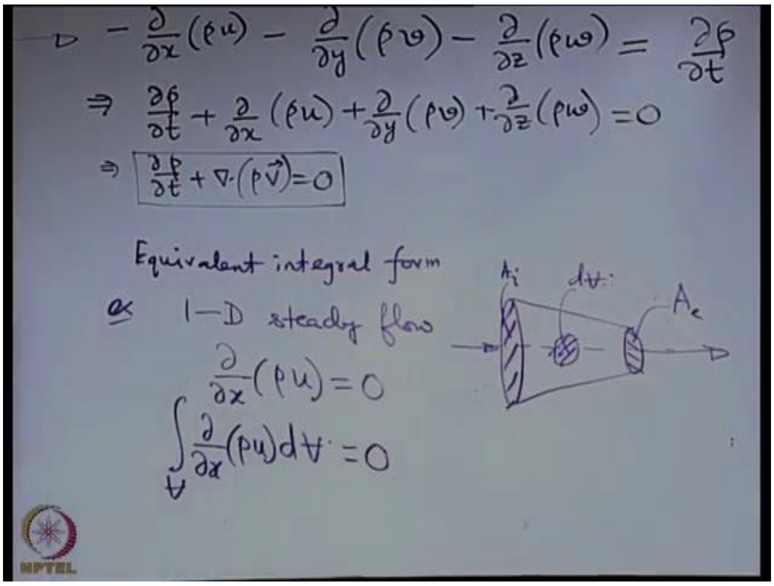


So, what we can do? Now we can write a mathematical expression of this physical balance because we now know how to write all the terms and we take the limit as delta x, delta y delta z tend to 0. So, that delta x delta y, delta z that gets cancel from both sides and in the higher order terms some small terms remain which in the limit as that tends to 0 that terms will be 0.

So, with that limit that is delta x, delta y, delta z, all tending to 0. So, you will have for flow along x that will be the case flow along y what should be the term and flow along z see individual velocity components are responsible for flows along certain directions and that is what we have to keep in mind that is equal to what because delta x delta y delta z gets cancelled. So, you can write this in the well known form that we saw in the previous class or equivalent vector notation which is the continuity equation.

So, we have seen at least 2 different ways of deriving the continuity equation and keep in mind these are not the only 2 ways, but at least these have given us some insight of what this law or what this equation is talking about. We will have a couple of important observations related to this before we go onto a problem where we illustrate how to make use of such equations.

(Refer Slide Time: 12:12)



So, the first point is that you see this is the differential form now let us say that you want to express in a integral form. So, how will you do it? We will later on formally see one methodology by which you can convert easily from differential to an integral form, but without going to that formality let us look into a very simple example by which we see that how to do it.

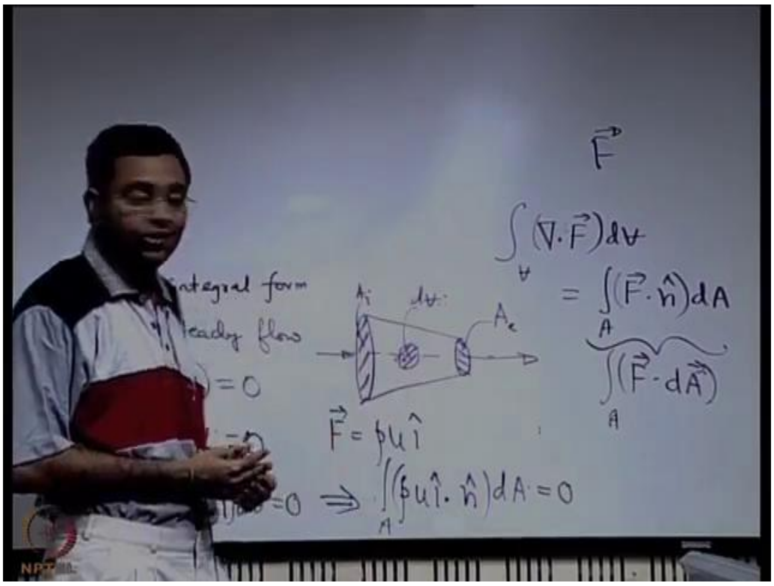
So, we are now interested about equivalent integral form what is that equivalent integral form we will not go for the most general case that we will steady later, but we will considered a very simple case has an example one dimensional steady flow say the flow is taking place along x may be just for your visual understanding let us say that the flow is taking place through a nozzle like this nozzle is something where you have when the flow is entering it is entering with a higher velocity and as it is moving along it the area of cross section gets reduced. So, the velocity of the flow gets increased. So, it is you may assume that it is something like a conical shape maybe frustum of a cone, cone or something like that the shape is not important for us we will just keep in mind that there is an inlet section with area e i, A i and there is an outlet section with or exit section with area a e.

And the flow is taking place along x, 1 dimensional steady flow. So, how do you simplify this differential form when you have a steady flow? The time derivative is not there when you have one dimensional flow it will just boil down to only the x component of velocity, but it may be compressible or I mean otherwise rho maybe a function of position or whatever. So, we are not commuting ourselves to a constant rho and we are just putting the rho inside keeping the rho inside now what we will do is we will try to integrate this over the entire volume of the nozzle so; that means, what we are trying to do we are trying to integrate it. So, we have a small volume element say a small volume element D v, why we require a small volume element to consider because u especially varying.

So, we are taking u at a location where you that u is that u at that particular x and then we are integrating that over the entire volume by considering such elemental volumes. So, that is that integrated over the entire volume that should be equal to 0. It is very straightforward if the function is 0, its integral should be 0. On the other hand, if the integral is 0, function need not always be 0, but we will see later on that there are certain cases when if the integral is 0 we may say that the function itself is 0 under certain important considerations, but not for all considerations, but here it is the other way which is more straightforward now when you have it see our objective is to convert this volume integral to a area integral because we are interested about the areas across with the fluid is flowing.

So, if you recall in vector calculus there is a theorem called divergence theorem gauss divergence theorem which converts the volume integral into a area integral or vice versa. So, what is that theorem?

(Refer Slide Time: 16:22)



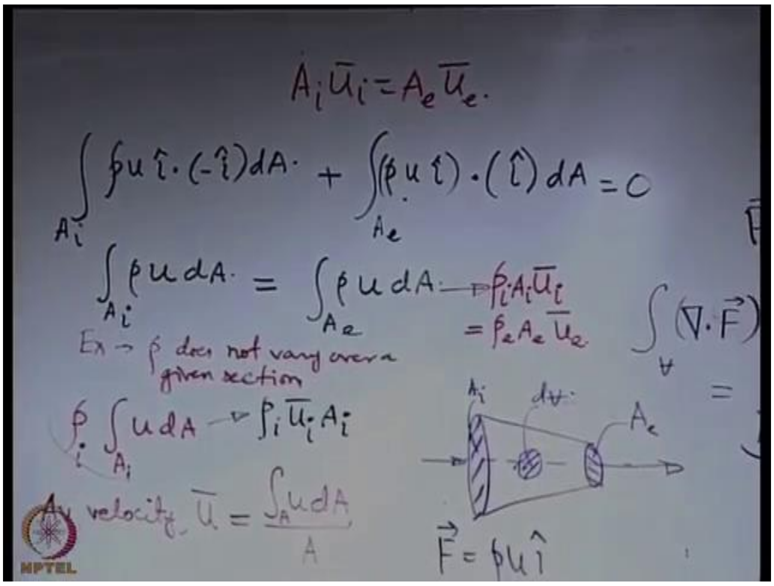
So, if you have if you have say f as a vector function general vector function. So, if you have divergence of f over a volume that is given by area integral sometimes this is also equivalent notation as f dot d a by giving the area directly a vector sense. So, giving an area vector sense is like magnitude of the area times the unit vector normal to the area. So, that anyway takes care of this.

So, the when you write d a as a vector it is n cap d a scalar. So, you have to keep in mind that it is always the outward normal that is considered to be the positive direction of any area. So, if you see this theorem you if you want to convert it to an area integral you have to cast its form in a divergence. So, say we want to cast it in the form of a divergence. So, what should be the vector function here f. So, what f should you choose such that it is in the form of a divergence yes rho u i right? So, the other divergence of that will give you this partial derivative. So, we can write this also as divergence of rho u i d v now if you look at this theorem this is the mathematical statement of the theorem, but it has a very important understanding what is this a and what is this v it is not any arbitrary a and arbitrary v this a is the area of the surface that bounds the volume v closely.

So, when you have the volume v here it is bounded by say lateral surfaces and these cross sections. So, when we are writing this in terms of an area integral that area should consider a i, A e and lateral surfaces also lateral surfaces at then will not be important because there is no flow across those surfaces. So, those are like irrelevant from flow computation considerations, but fundamentally it is the entire surface that is bounding the volume. So, you can write this by using the divergence theorem as how do write this rho u i dot n d A over the area that is equal to 0. Now let us look into this form, so when you consider this d A that area element, now you have as we mention 3 types of like one is inlet another is exit and another we may call as wall across with there is no flow.

So, when you have the wall, it is not necessary to calculate to bother about this integral for the wall because there is no flow there we will therefore, break it up into 2 integrals one for the area A i another for the area A e other areas are not relevant. So, when you consider the area A i what is the n cap for the area A i minus i.

(Refer Slide Time: 20:36)



So, you have rho u i dot minus i d A, this integral over A i plus rho u i now what is for A e n cap, it is i. So, these dot i d A for A e equal to 0. So, you can write integral of rho u d A. So, what does it say physically? It says that at for steady state whatever is the mass flow rate across this section the same must be the mass flow rate out across this section that is what it is math it is physically saying mathematically the statement is straight forward now it is many times convenient to express this in terms of the average velocity because it might. So, happened that u is a function of the transverse coordinates let us say we have transfers coordinate as may be say r or y that type of a coordinate and it is possible and it is almost always likely that you will be the function of a transverse coordinate because you will be 0 at the walls by no slip condition and then you will change may be maximum at the centre line.

So, it is expected that along the transverse direction you will vary. So, it is not that we are talking about a cross sectionally constant u it is rather a cross sectionally variable u now if you assume that rho is not varying across the section as an example. So, let us take an example where rho does not vary over a given section, but it may vary from one section to the other. So, rho does not vary over a given section; that means, you can take that rho out of the integral and your left with these types of terms for example, for the left hand side it will be rho at the inlet surface inlet section times this integral now there is a definition which is called as average velocity.

So, how the average velocity is defined the average velocity is a cross sectionally average velocity. So, for a one dimensional flow it is like u average. So, we call we give a notation u bar to indicate that it is an average. So, it is basically integral of u d a divided by a what is the physical meaning of this physical meaning is see the entire section has a variable velocity with that variable velocity it has a flow rate volume flow rate now if you have the same volume flow rate with an equivalent velocity that would have been uniform throughout then that uniform equivalent velocity is the average velocity. So, what we are basically doing we are equating the volume flow rates in one case its variable velocity the real case in the other case, it is an equivalent idealistic case where it is an uniform velocity what the end effect the mass the volume flow rate is the same and then that equivalent velocity equivalent uniform velocity over that section this is known as the average velocity.

So, we can replace these integral or this term by what we can write this as rho i into u i average into A i therefore, we can clearly say that this equation these boils down to a very simple form rho i A i u i average is equal to rho e A e u e average. If the densities are not varying specially then rho i and rho e get cancelled out. So, you get A i u i average is equal to A e u e average which is like A 1 v 1 equal to a 2 b 2 these types of equations we have used earlier for solving simple problems now what you realize here see whenever you come up with an equation again I am saying you have to keep in mind that what are the assumptions.

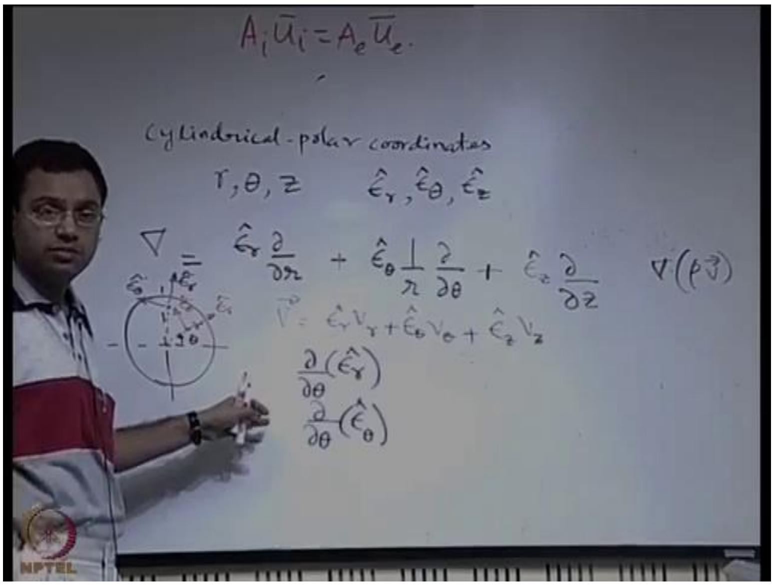
So, if you have say an equation like this A i u i is equal to A e u e. So, what are the assumptions under which it is valid see we have of course, we may go on deeper and deeper into the assumptions, but let us talk about only the major assumptions what are the major assumptions first rho is a constant the density is a constant then we are talking about this velocities not local velocity at a point, but cross sectionally average velocity if it is an ideal fluid flow when it is possible that that local velocity same everywhere because the velocity gradient is created by viscosity. So, if you have no viscous effect then the effect of the wall is not propagated in to the fluid and it is possible that there is a uniform velocity profile.

So, then the average velocity and the local velocity may be the same. So, if you consider say cross section like this now let us identify 3 different point say these 3 different locations and these 3 different locations the velocities are different. So, when you write say A i u i bar may be sometimes in your in your previous studies you have written it as the velocity at this point right fundamentally that is in correct that is wrong when you get read of that wrongness by only one thing either your writing your although your thinking that your writing velocity at that point actually you are writing the average velocity over the section or even if it is velocity at this point that may be if the velocity is not varying over the section that is a uniform velocity profile if it is there across the section.

That means your implicitly treating it as an in viscid flow. So, these are some certain important concepts that go into the equation that is why I always say that try to get read of whatever you have learn for the entrance exam preparations because you know the and formula, but many times you do not know that what are the restrictions under which you are using that and formula and that maybe dangerous that may be more dangerous i would say it is worst than not knowing the formula. So, let us keep that in mind now next let us look into another issue that we have discussed about the continuity equation in a general vector form, but we have not looked into the other coordinate systems we have looked in to the Cartesian coordinate system, but let us say that we are also interested about the cylindrical polar coordinate system that coordinate system many times is important if you have something of say a cylindrical symmetry some body of cylindrical symmetry.

We will not going to the detail derivation of the continuity equation for a cylindrical coordinate, but I will tell you how to do it and I will leave it on you as an exercise to complete it.

(Refer Slide Time: 28:51)



So, if you considered say cylindrical coordinates cylindrical polar coordinates. So, in the cylindrical polar coordinate you have polar nature; that means you have the r theta coordinate system just like the polar coordinate and you also have 3 dimensionalities. So, have the axial coordinate system given by the z. So, r theta z coordinate system and let us say that these have their unit vectors as given by epsilon r epsilon theta and epsilon z just like i j k.

Now, we can. So, we have to see that what are the differences in the Cartesian system and in this system? So, first we have to know what is the del operator in the system. So, first just like the i j and the k. Here also you will have the corresponding epsilon r epsilon theta epsilon z. So, for the first component that is for the component along r it is just like component along x for the Cartesian coordinate for the component along theta it will be this one because you have to keep in mind that the line element along theta is like r d theta that is the length element not only that even if you forget about that fundamental consideration just look into the dimensionality.

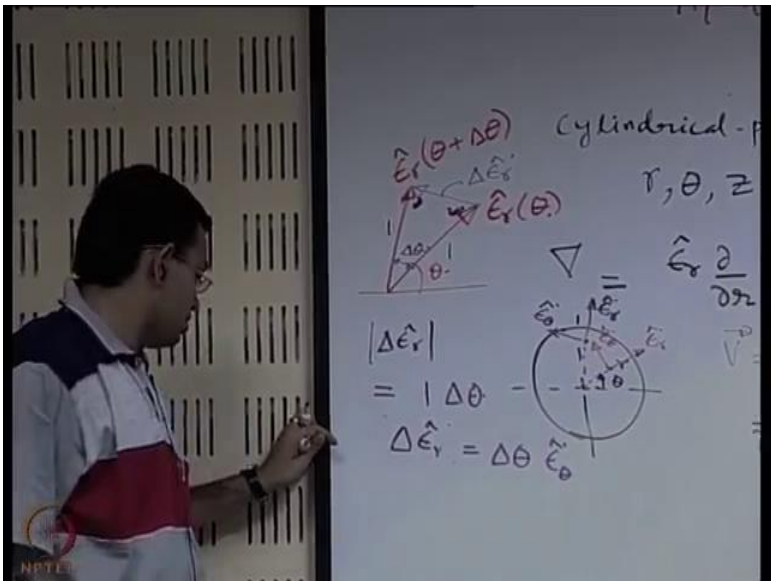
It is 1 by length. So, it has to be there this unit should length. So, when you write the theta derivative theta is like it does not have a dimension. So, you have to adjust with the linear dimension. So, just from the dimensional arguments also like these thing I am telling because if you are confused that you are not being reminded that what should you write? At least this commonsense thing should guide you that what should be the correct way of writing this and then this.

So, in the continuity equation see the first term is the time derivative. So, the time derivative you do not care much you know that like it is not dependent on these operators the time derivative of the density what the next term is you have the divergence of rho v. So, there calculation with a del operator will be important can you tell where does it fundamentally differ from what you do in the Cartesian system yes there is only one fundamental difference and if you keep that difference in mind it is just a straightforward exercise in the Cartesian system when you have i j k, those are invariant in direction whereas, when you have epsilon r epsilon theta these are not invariant in direction.

So, if you consider say point which is located at a position r. So, what how do you write it epsilon r and epsilon theta. So, this is a radial direction. So, this will be the epsilon r and perpendicular to that will be epsilon theta you go to a now a different point let us say you go to this r even if you keep the radial magnitude same now what will be your epsilon r your epsilon r will be this and your epsilon theta will be again perpendicular to that. So, it i and j they are invariant, but epsilon r and epsilon theta they actually vary with theta where theta is the angular coordinate.

So, when you differentiate. So, this del operator is basically for differentiation and when you differentiate you have to keep one thing in mind what you have to keep in mind. So, when you write v you also write v in terms of its r theta and z components. So, when you write v it is epsilon r v r plus epsilon theta v theta plus epsilon z, v, z. So, when your differentiating it is possible that you have to find out these quantities and these ones these kinds of quantities they were not relevant for Cartesian system because invariant direction unit vectors. So, we will find one let us say you want to find out the derivative of epsilon r. So, how do you look into that?

(Refer Slide Time: 34:34)



Let us say that you have epsilon r for a particular angle theta as this one and let us say that epsilon r has changed with the angle theta plus d theta. So, let us say this is epsilon r for theta and let us say this is epsilon r for theta plus we can write d theta straight away or if you want to be more fundamental let us consider it has delta theta in the limit as delta theta tends to 0. So, let us say that this is at theta plus delta theta where theta plus delta theta is the corresponding angular position. So, you can see that in a scalar form actually magnitudes of these are the same, both are unit vectors.

So, length of this is 1, length of this is 1, what has change is the directionality. So, what is the delta epsilon r? What is the change in the epsilon r? That is nothing, but delta epsilon r; the change in epsilon r. What is the magnitude of this delta epsilon r? So, if this angle is delta theta, so magnitude of delta epsilon r is like it is like 1 into delta theta. So, for small delta theta it is just like (Refer Time: 36:12) circles. So, it is one into delta theta what is its direction you have to keep in mind that we are talking about the limit as delta theta tends to 0 some of the 3 angles of this triangles is 180 degree.

This is an isosceles triangle. So, these 2 angles should be equal. So, when this tends to 0 these 2 tend to 90 degree almost; that means, delta epsilon r has a direction epsilon theta yeah that is perpendicular to epsilon r. So, we can say that delta epsilon r is like delta theta now you give the directionality epsilon theta sorry, now you get read of the magnitude. Now on the basis of this, you can find out, what is delta epsilon r delta theta that is you can take limit as delta theta tends to 0, delta epsilon r delta theta that is epsilon theta. So, this is nothing, but ok.

Similarly, if you work this out, it will be minus epsilon r. I mean I need not work it out again because it is just very very similar. So, now, when you plug in those considerations in this del operator in this divergence operation by considering this as the del operator for the cylindrical polar coordinate system you will get form of the continuity equation in r theta z system, I would advise you that you complete that exercise and then in your book you will see that the final form in written in the r theta z system. So, you check or verify your final expression with that.

That will give you a confidence that you have done it correctly in terms of the cylindrical polar coordinate systems. So, we have got some preliminary understanding of the use of the continuity equation whether the understanding is good enough let us try to work out a problem and see.

So, we stop here today we will continue in the next class again.