Introduction to Fluid Mechanics
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Lecture – 20
Fluid under rigid body motion

Now, that we have seen the stability criteria and so on, we will come to our final topic in fluid statics which ironically is not fluid statics, but fluid with rigid body motion. And as we discussed earlier that we are going to address this issue within a purview of fluid statics for a very simple reason that when you have fluid on the rigid body motion, it is still fluid element without any shear. So, when it is without any shear it is just the normal force which is acting on the surface. So, the distribution of force in terms of pressure on the surface remains unaltered, no matter whether it is at rest or under rigid body motion when the fluid elements are deforming then only you have the shear. So, let us take a simple example for fluid under rigid body motion to begin with.

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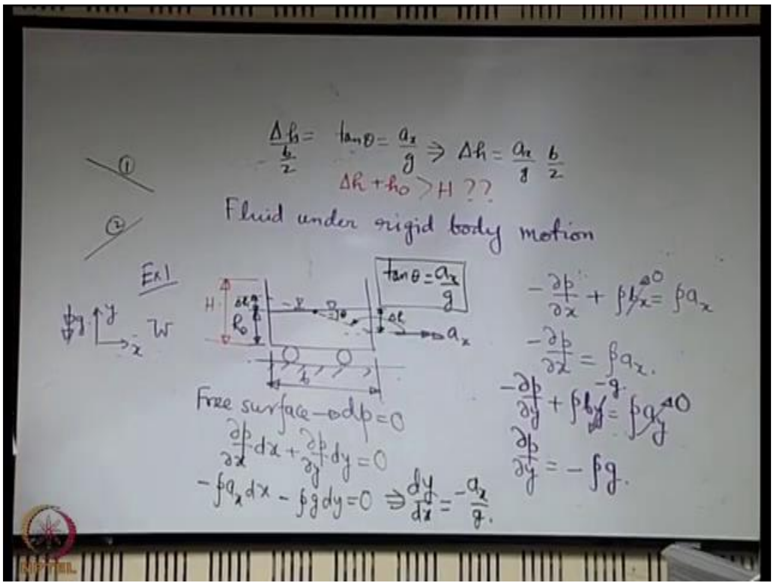

Let us say that we have a tank partly filled up with water and the tank is accelerating towards the right with an acceleration of a x along x, fixed acceleration and neglect the deformation of the water that is there in a tank. In reality that deformation is there so whatever analysis that we are presenting here is not perfectly correct because in reality there will be shear and deformation and so on, but just as an idealization just like many times we discuss about frictionless surfaces not that they are through this idealization we learn certain concepts. So, let us say that there is no deformation. So, the water which is there within the tank just gets deflected in its configuration in terms of the free surface like a rigid body.

So, how we calculate that what should be the location new location of the free surface because of this acceleration that is what we want to see. Say initially the height of water in a tank was h 0; maybe let us specify the breadth of the tank as b and maybe with perpendicular to the plane of the board as w. Let us say; that means, it is a rectangular tank.

Now recall that we derived certain expressions with regard to distribution of pressure in presence of body force. So, so far as I remember this is the expression that we derived sometime back when we were actually starting with our discussion on fluid statics. So, we will try to use this expression here. So, along x if we have minus because of there is no body force acting along x. Let us say that the y direction, so we have x and y as our chosen direction let us say that the y direction is the direction opposite to the acceleration due to gravity. So, for y direction what you can write. So, what is a y? a y is 0 it is accelerating along x, so a y is 0. What is b y? Minus g, so you can get minus rho g.

Now when you have a free surface, the free surface is characterized by what? The free surface has the same pressure throughout because it is exposed to the atmosphere which are same pressure throughout. So, there is a pressure equilibrium between that and the atmosphere. So, for the free surface you must have dp equal to 0 that should be the governing parameter for locating the free surface. So, when you have dp equals 0 remember that now p is a function of both x and y. So, you can write dp as this one.

So, you can substitute in place of the partial derivative with respect to x as minus rho a x d x and this 1 is minus rho g d y is equal to 0; that means, what is d y d x. That is equal to minus a x by g, what does this d y d x represent? It represents the slope of the free surface. So, can you tell now whether the slope of the surface will be like this or like this

1 or 2? 1, because you can clearly see that this will represent a kind of tilt like this, so this will become the new free surface and the angle that you are considering for the slope is basically this one, because this is negative a x is positive and g is positive. So, this is negative, so it must be an obtuse angle. So, in the direction in which the tank is accelerating the liquid will be more down and in the other direction it will be more up right and if you specify this angle as say theta then that theta is nothing, but 180 degree minus this slope angle. So, you can say that tan theta is nothing, but equal to a x by g because theta is nothing, but 180 degree minus this angle.

So, you can find out that what is the extent to which the water level will rise on one side and maybe fall on the other side. Let us say that this raise is delta h on one side because of symmetry it will be a fall of equivalent delta h on the other side. So, it will be as if swelling with respect to the centre. So, we can calculate what is delta h. So, if you calculate delta h what will be that it is nothing but delta h divided by b by 2 is tan theta that is x by g. So, from here you can calculate what is delta h x by g in to b by 2. Now let us come to a critical condition if we are happy with this sometimes we are deceived how let us say that the total height of the tank is h, if it so happens that let us say for from calculation we get delta h plus h 0 it is greater than h.

Practically that is not possible right, because the liquid cannot occupy a height which is greater than that is provided by the tank. So, what this will mean? This will win some water has filled out. So, this is a condition from which you can say that it has actually spilled out when it has spilled out it is no more this configuration. So, when it is spilled out what will happen what type of configuration we expect? So, before spilling out it tried its best to climb up to the top most level and then it is spilled out. So, this will be one end of the surface and maybe the other end is like this somewhere. So, with spill out maybe this is the, this is with spill out.

Irrespective of whether it has spilled out or not you still have this expression applicable. So, now, this will be the angle theta you can find out that what should be this length let us say b 1 what should be this length b 1, because you can say capital H by b 1 equal to tan theta which is a x by g capital H being the height of the tank known. So, from there you can find out what is b 1 and therefore, you can calculate what is the volume which is there now within the tank and the difference between the original volume and that volume will give you the what is the volume that has got spilled.

So, you see that it is not just like your solution should not be driven by a magic formula, but based on the numerical data given you have to come to a decision whether the water is there inside or it has got spilled and so on. Let us take a variant of this example.

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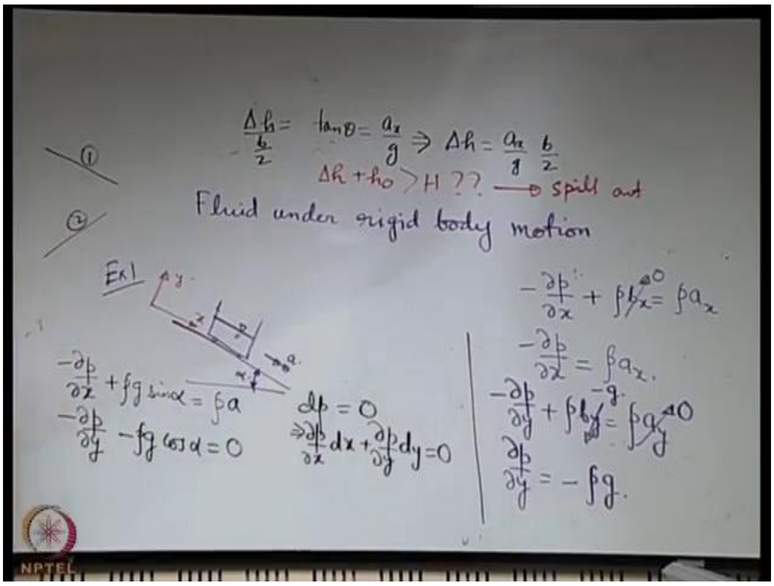

So, the variant of the example is that now we have the tank located on an incline plane with an angle of inclination alpha a tank is there on the plane and it is accelerating say downwards with an acceleration of a, which is a uniform acceleration because of a resultant force which acts along that direction. So, in this case it may be more convenient if you fix up your coordinate axis relative to the inclined plane say x and y. So, the similar equations will be applied and let us just do it very quickly because it is very straight forward.

So, you will have minus partial derivative of p with respect to x plus, what will be b x now? b x is g sine theta. So, plus rho g sine theta is equal to rho here it is not theta we have given a name alpha. So, rho g sine alpha is equal to rho a, a x is a, what about y? So, minus this, y will be minus g cos theta, so minus rho g cos alpha is equal to 0. So, from here you can find out when you have dp equal to 0; that means, you can substitute the expression.

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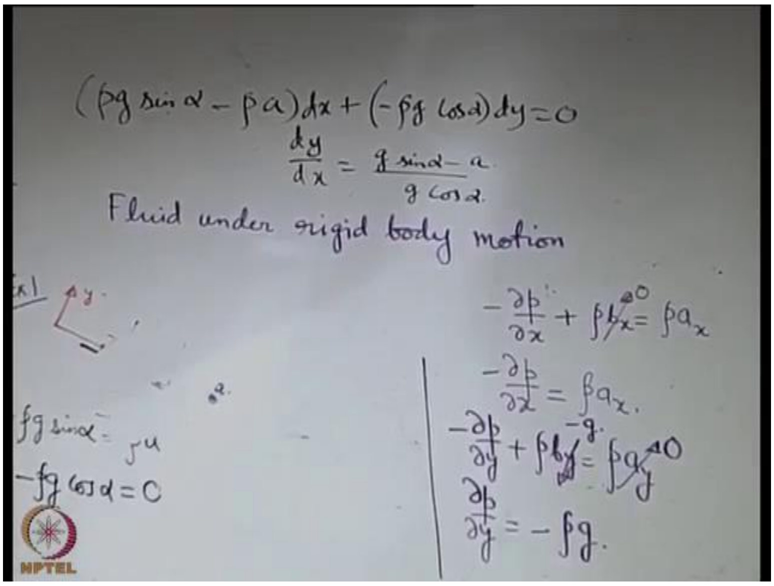

So, it will be let us write it at the top the expression now becomes in place of partial derivative with respect to x you can substitute rho g sine alpha minus rho a d x plus minus rho g cos alpha d y equal to 0. So, d y d x will be g sine alpha minus a divided by g cos alpha right. And this is now equal to the tan theta where theta is the angle relative to the original location of the free surface or like the assumed x direction.

So, you can see now that there is depending on the magnitude of a, this may be positive or negative right. So, you may have a case when g sine alpha is greater than a or g sine alpha is less than a. So, here you cannot trivially say that whether it is 1 or 2, case 1 or case 2. Again you see that it is it is not a magic formula that should drive your decision it depends on what is the physical situation that is prevailing. Let us consider a third example.

Student: (Refer Time: 14:32).

Yes.

Student: Sir, cannot we use the fact that the surface should always be normal to the resultant force acting on the bottom of it.

Surface should always be normal on the resultant.

Student: But we cannot have any therefore, on the surface.

It see here what we are doing here we are implicitly applying the same condition; it is you see that we are having one very important assumption; the assumption is that you do not have an oscillation in the surface. So, sometimes because of this displacement the surface oscillates it becomes like a wavy situation and that is known as sloshing of tanks. So, we are not going into that detail. So, we are assuming that the surface remains flat and when a surface remains flat and under these conditions when you have dp equal to 0 that is exactly the same condition.

Student: In that case we can directly analyse from the vector analysis instead of doing the mathematical derivations.

See vector analysis is not it is also mathematics right when you say that when you are dealing with mathematical analysis and feel this is too mathematical, I mean I do not see any difference like I mean if you have vector analysis this is also vector analysis just dealing with scalar components. So, it is better to be habituated with this because again I am telling you there are situations when it will not be as straightforward as this. So, you have to use this fundamental equation. So, whenever you are solving a problem try to adhere to the fundamental situations I am getting your point why you are telling this because you have been habituated in solving problems in that way through your entrance exams, but we will be encountering more challenging problems than what we have solved earlier through that type of magic situation and we will try to avoid that. So, our basic intention will be that we have this basic equation this should solely guide us for solving whatever complicated problems that we are having of these type.

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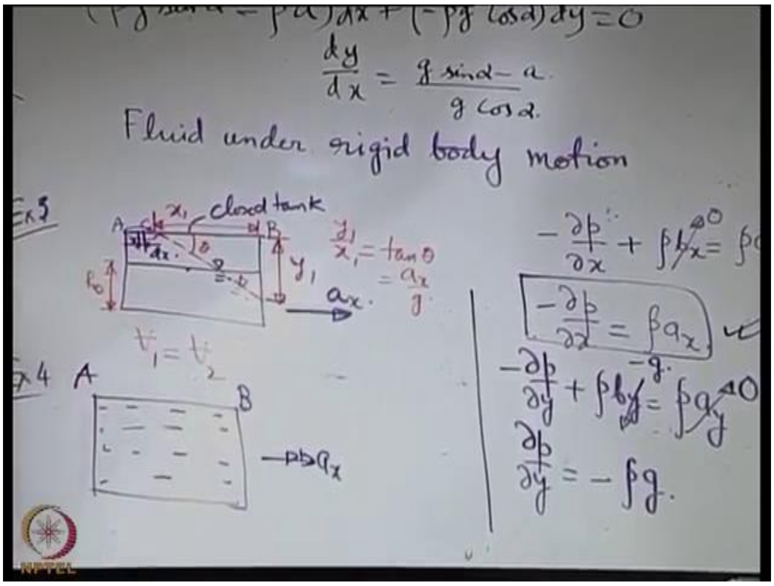

Let us consider another one. Say example 3: now let us say we have a closed tank, closed tank partially filled with water this is closed, again doing the same thing accelerating it along x. What will happen to the free surface? Now there cannot be any spilling. So, when there cannot be any spilling there is even a chance that the free surface is like this. So, the symmetry with respect to the central line which was there for the previous case without spilling is now broken that symmetry is not guaranteed because it may try to escape and since it is not finding an escape route it will break the symmetry and get distributed, in what way? In such a way that the volume of the liquid now is conserved because it cannot go out of the tank, so if you say that these dimension is say y 1 and this is x 1 then you have y 1 by x 1 as tan theta and that is equal to a x by g and you can calculate y 1 and x 1 by considering that the volume which was there originally is same as the volume of water after this tilting of the interface.

So, this is the new interface. So, this will give you another relationship involving x 1 and y1 and you can solve for x 1 y 1. Now let us say that we are interested to find out what is the total force acting on the top surface or the lead of the container, how will you find it out? So, you let us say that this point is c. So, I have to keep in mind that only up to a c it is in contact with the fluid and you can use this one. So, you can find out the pressure distribution of the function of x from a to c, take a small element at a distance x from a of with d x, so the force acting on that is p in to d x in to the third dimension. Integrate it from a to c and a c you can find out from this geometrical considerations so that will give you the total force. Let us consider maybe a fourth variant which I will not solve, but just tell you that such a variant is also possible - say you have a tank now it is completely filled with water and it is closed.

Student: Sir, in the previous example the p will be p naught.

Which p will be p naught?

Student: Sir, when you were calculating the force of on that relative (Refer Time: 20:13).

You assume any one of the points say a as a reference pressure, say p some reference p because always when you have pressure it is relative to some point and so you can express the pressure distribution in terms of the pressure at the point a and then integrate it over the element.

So, next example is you have this tank accelerating towards the right, but completely filled with water, but closed so water cannot escape. So, if I tell you find out the total force on the lead A B how will you do it? I will not do it myself leave it on you as an exercise and I can only tell you that this is the simplest of all the cases that we have considered, but you have to keep in mind that same consideration as this also should work. So, that will give you a natural pressure distribution from a to b and similar to this you can just integrate from a to b to find out, here you do not have the botheration of finding out what is the portion that is exposed with the liquid because since it is completely filled it should be completely exposed.

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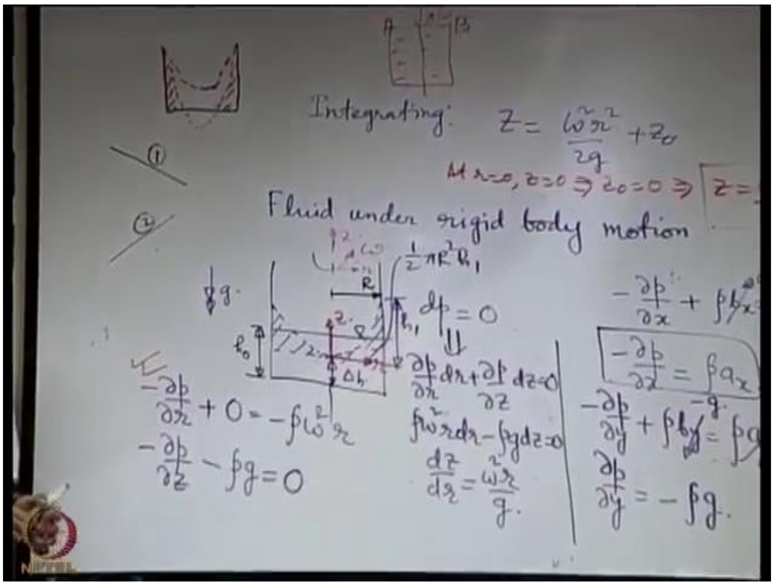

Now, when we considered the rigid body motion it is not always just translation it may also be rotation. So, let us consider a rigid body rotation example. This type of example you have seen earlier that you have a tank filled up with water to some height h naught, it is a cylindrical tank of radius R. So, we are using R z coordinate system axis symmetry coordinate system, say this is R coordinate and along this there is z coordinate. So, this tank is rotated with respect to its axis with an angular velocity omega. So, when it is rotated you already know it that it will come to its free surface will come to deform or deflected shape like this which is a paraboloid of revolution.

Let us quickly see that how we can derive that it should be a parabola revolution from this fundamental principles not from any magic formula. So, let us see you start with here. So, there are 2 directions r and z. So, you have dp dr minus that plus, what is the body force along r?

Student: (Refer Time: 23:02).

Minus omega square r, see this we have writing in the original form of the Newton’s second law of motion not with respect to an accelerating reference frame, so no question of centrifugal. So, you have, now you tell when it is no question of centrifugal whether the body force is there or not, yes or no?

Student: Yes, (Refer Time: 23:32).

You have say you are standing on a platform and you are looking into it from an inertial reference frame this is not an external force that is applied that you can see only when you have a rotating reference frame that is attached to the platform that is having angular motion. So obviously, there will be no b r that you have to keep in mind. This is the form of the Newton's second law in an inertial reference frame. With a non inertial reference frame you have to use a pseudo force for the inertia force, but that then you do not write equation of dynamics, but equivalent to equation of static equilibrium, you convert that into an equivalent static equation through the D'Alembert's principle, but here we are not talking about that we are talking about the proper acceleration.

So, you have no body force, but you have acceleration what is the acceleration? It is nothing, but the centripetal acceleration so, minus rho omega square r. See if you had considered it in a rotating reference frame the right hand side would be 0 because you are writing static equilibrium this would be represented by the pseudo force. So, the final equation would eventually be the same, it is just a matter of the reference frame with respect to which you are writing, but this is written with respect to the inertial reference frame. Then when you come to the z direction minus d p, d z, then let us say this is the acceleration due to gravity direction minus rho g is equal to there is no acceleration it is not vertically moving, again like whenever it is vertically moving and so on you can substitute, so start within this basic equation depending on whatever information is given in the problem you try to use it.

So, for the free surface you have dp equal to 0. So, when you have dp equal to 0 you have this equals to 0. So, this is rho omega square r this is minus rho g. So, you have d z dr is equal to omega square r by g. You can now integrate this with respect to r, so, on integration what follows let us just complete that. Integrating it follows z equals to omega square r square by 2 g plus a constant of integration z naught. You can set up the concept of integration z naught by choosing a reference such that when r equal to 0 z equal to 0. So, if you choose these as our origin of the coordinate. So, this is r and this is

z. So, if you have at small r equal to 0 z equal to 0 then z not equal to 0. So, z will be omega square r square by 2 g this formula you have encountered earlier. So, we can clearly see that it is a equation like a of a parabola. So, it is a paraboloid of revolution because it is a 3 dimensional situation.

And you can calculate the other things just similar to what we did in the previous case with an understanding of what that again if it does not spill whatever was the volume that should be conserved. So, how can you calculate the volume? So, initial volume we can calculate initial volume is pi capital R square into h naught, what is the final volume? Final volume is whatever is the depression say delta h plus the volume of the shaded paraboloid of revolution.

And I leave it on you as an exercise if you calculate the shaded volume you will see that it will just be half of the volume of the circumscribing cylinder; that means, if this height is h 1 then the shaded volumes will be half of pi R square h 1 by simple integration you can find out this volume. So, by equating the initial and final value can find out totally the deflected configuration. Again you may have to check that h 1 plus delta h if it becomes greater than the height of the original tank, then it will spill and spilling again may have 2 different cases. You may have in one case the cylinder is rotating in such a way that you have spilling, but still it is having an interface shape like this. Again it may so happen that it is rotating so fast that is spilling, but only the part of the parabola revolution is within the cylinder. So, then is an imaginary parabola revolution even outside that you have to consider to find out what is the volume that is there inside.

So, it all depends on the rotational speed the given dimensions and so on. So, it is not just like a fixed formula, but you know what is the basic principle we have discussed enough numbers of examples to see that what is the basic principle and that basic principle should guide you to find out that what is the case. Now if it is totally closed cylinder as a final examples say we have a cylinder that is totally closed and filled up with liquid and rotated with respect to its axis, what is the total force on the top lead A B? Again the basic principle is the same, you just use this dp dr formula to find out how pressure varies with r of course, with a reference say at r equal to 0 p equal to p 0 with the reference because pressure you always calculate with a reference. So, you know how p varies with r by integrating this with respect to r. When you integrate with respect to r z is fixed. So, then you can find out the total force by taking an element here an element will be 2 pi R dr. So, integrating over that you can find out the total force on the top surface.

Total force on the bottom surface will be that plus the weight of the fluid which is there right. So, we will close this discussion by seeing just one maybe one example where we will see that how this type of vortex motion is generated in practice.

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You see that it is, you see the this type of a vortex motion that is there I mean it is not exactly a parabola revolution, but it is by rotating the fluid in a container. Why it is not exactly a paraboloid of revolution is because we have neglected here the viscous effects, the shear between various fluid layers we have assumed that the fluid rotates like a rigid body, in reality that is not the case and will look into these situations more emphatically whenever we are discussing with viscous flows. So, we close our discussion on fluid statics with this.