Vibration of Structures

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Lecture – 09
Modal Analysis: Approximate Methods – I

Last three lectures, we have been discussing about Modal Analysis of continuous systems. Now we have solved the problem of modal analysis which is nothing but solving an Eigen value problem as we have seen analytically which means that we are exactly solving the natural frequencies and the modes of vibrations which are characterized by the Eigen functions. Now, doing an analytical solution is always preferable.

Because you can find out the affects of various parameters of the system on the modes of vibration and the modal frequency etcetera. So analytical solution is always preferable. However, we have seen that even in very simple systems the solution of the modal analysis problem requires solving transcendental equations possibly which are, which might be quite cumbersome even numerically.

So, in general analytical solution though preferable are sometimes cumbersome and computation intensive. So it is of interest to know if a numerical method of modal analysis or approximate methods of modal analysis are possible. So in this lecture, in the next lecture, we are going to look at few techniques for solving the model analysis problem or the Eigen value problem approximately, numerically. So what is our motivation for studying the approximate solutions?

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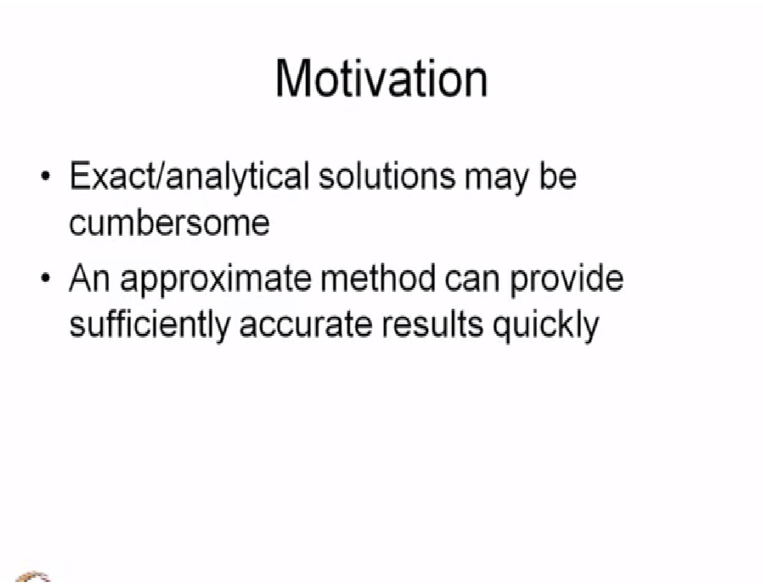

So the first motivation is that the analytical solution may be cumbersome. The other things is an approximate method can provide a quick solution to the modal analysis problem which may be sufficiently accurate for our purposes.

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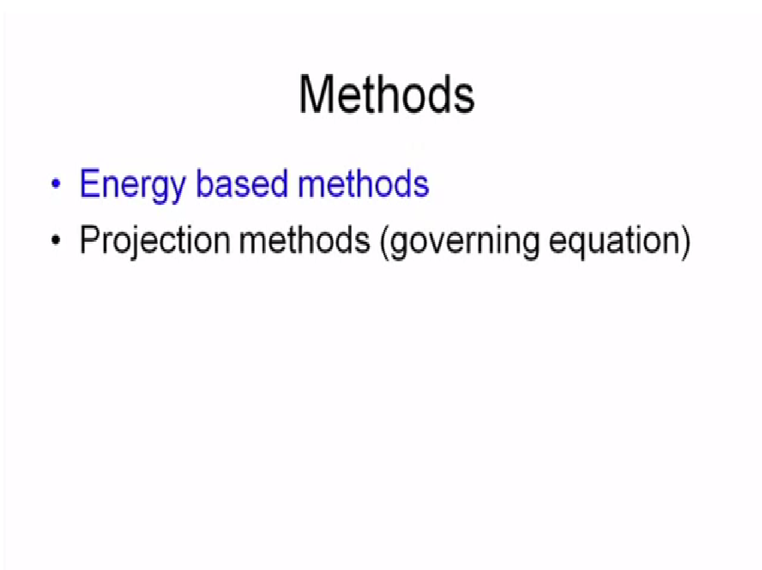

So let us look at the methods that are available to us for approximate modal analysis. So we are going to discuss in this course two methods and two broad methods which the Energy based methods which will be the topic of discussion in today's lecture and in the next lecture we are going to look at Projection methods.

So as the name suggests this Energy based methods will use the kinetic and potential energy of the system to determine the modes of vibration or the natural frequencies where the Projection methods as we will see very soon they use the governing equation of motion directly to solve the modal analysis problem.

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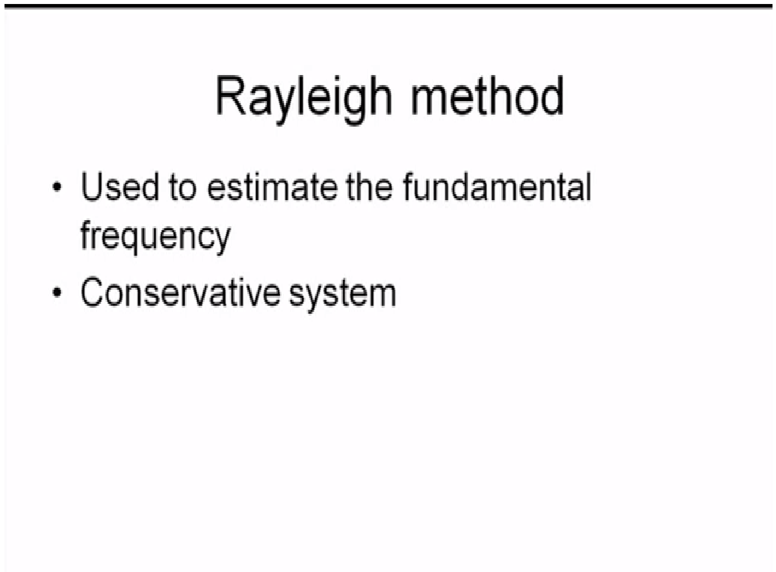

So the first method that we are going to look at is the Rayleigh method. So this Rayleigh method is typically used to determine the fundamental frequency of a continuous system. And this method is use for Conservative systems. So we use Rayleigh method for conservative systems.

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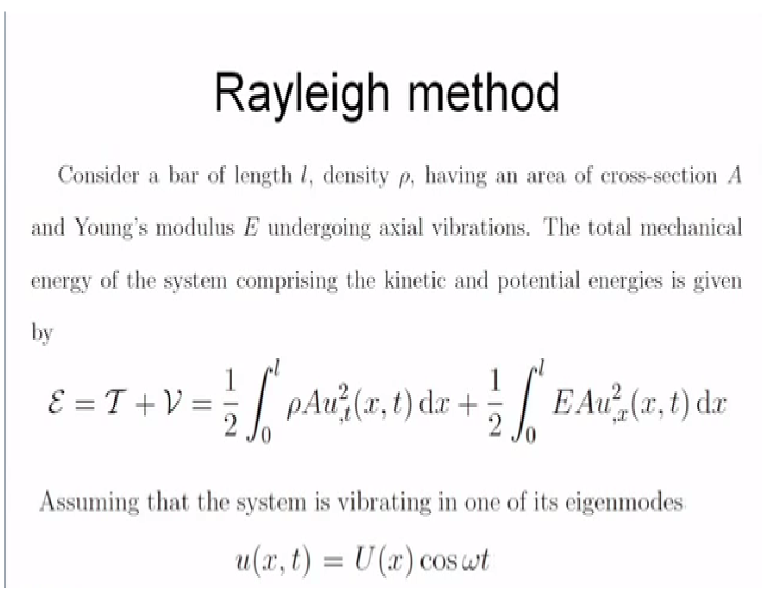

So how does this method work?

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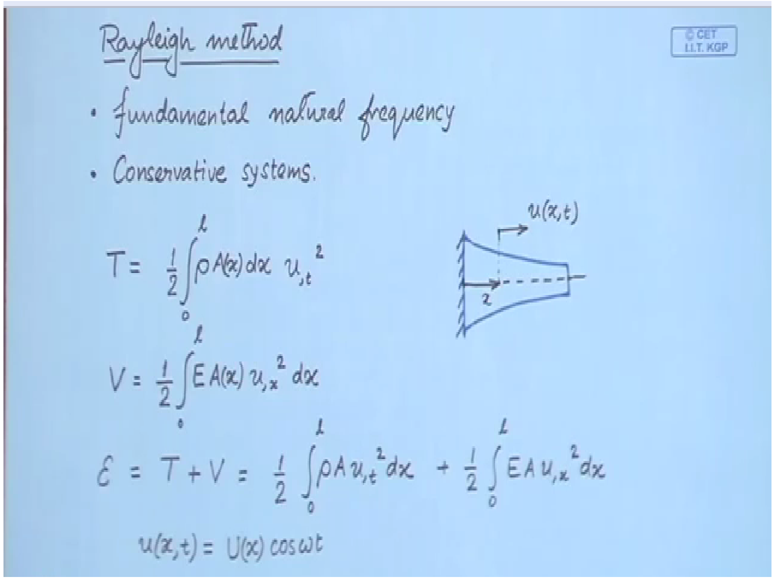

So let us understand this with the example of a bar. So let us consider a tapered bar in axial vibration. So we write the kinetic energy is one half, the density of the material times the area of cross-section times and elemental length and the velocity square, now this integrated over the domain of the bar. So that is the kinetic energy. The potential energy is one half the Young's modulus times spatial derivative of the field variable's square dx and integrated over the domain of the bar.

Now this system this bar a natural vibration as know is a conservative system which means that the total energy of this bar is a constant. So the total mechanical energy-- this is a constant. Now let us suppose that this system is in is vibrating in one of its modes. So as we have already discussed when a system is vibrating when in one its modes the field variable can be written as a separable function and space and time in this form.

So this special solution form is valid for a system for this bar vibrating in one of its modes. Now we will substitute this expression in the total energy of the bar and one we do that.

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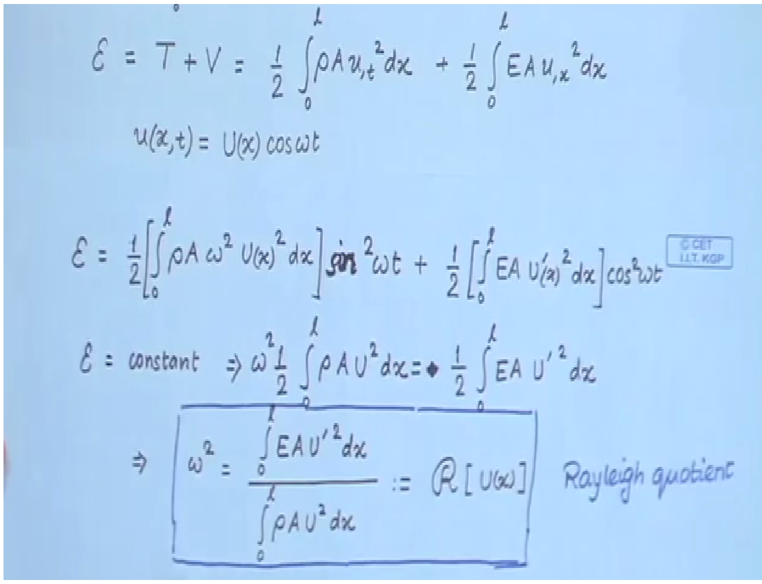

What we obtain – so this is the expression of the total energy after we substitute this solution structure in the total energy expression. Now if this total mechanical energy is to be a constant then what it would require is that the independent of time. And this is possible only when the coefficient of sin square omega T and cos square T they are the same which means – so energy is constant would imply—these two the coefficient of the sin square omega T and cos square omega T they must be equal.

So therefore we obtain this ratio which is defined as the Rayleigh quotient. And this Rayleigh quotient is the key concept in Rayleigh methods.

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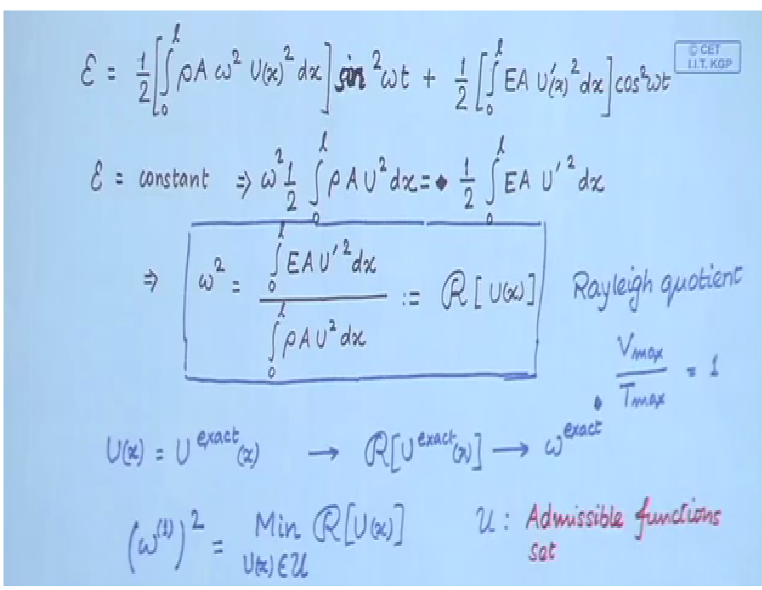

So what we have done is we have considered the modal solution substituted it in the total energy expression total mechanical energy expression and finally we force that the energy to be independent at time by matching the two coefficient of the Sin square and the Cos square omega T terms to obtained omega square as a ratio which is known as the Rayleigh quotient.

Now sometimes this Rayleigh quotient is also expressed as maximum potential energy divided by the maximum kinetic energy. So the amplitude so the maximum potential energy would be the amplitude of-- the maximum potential energy would be amplitude of the sin square omega T term where that the maximum kinetic energy would be the coefficient of the sin square omega T term and of course this omega is solved in terms of this.

So we-- so this is one so from here we solve for omega square as we have done here. Now this-- in this Rayleigh quotient if you know the exact Eigen function, so if you substitute where I am using this exact superscript to indicate that U exact is the exact Eigen function for a particular mode. In that case, if you put this in the Rayleigh quotient what we will get is the exact circular Eigen frequency corresponding to that mode.

But the problem comes because we do not know the exact Eigen function. So in that case how do you use this Rayleigh quotient? So usually what is done is we try to minimize this Rayleigh quotient. So and by minimizing the Rayleigh quotient we obtain the fundamental frequency. So the fundamental frequency square is obtained by minimizing the Rayleigh quotient over a space of possible Eigen function U (x).

Now there is a restriction on how you can use U (x) in this minimization problem? The restriction is U must be a member of the set of what are known as Admissible functions. So this U is a set of Admissible function. So you must choose the possible Eigen function from the set of this Admissible functions. Now what are Admissible functions? This we will come to very soon.

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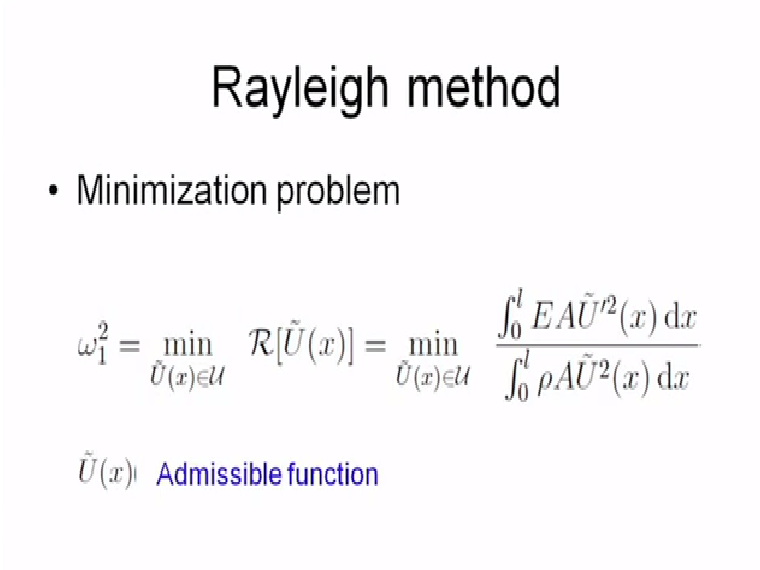

So let me write for this problem.

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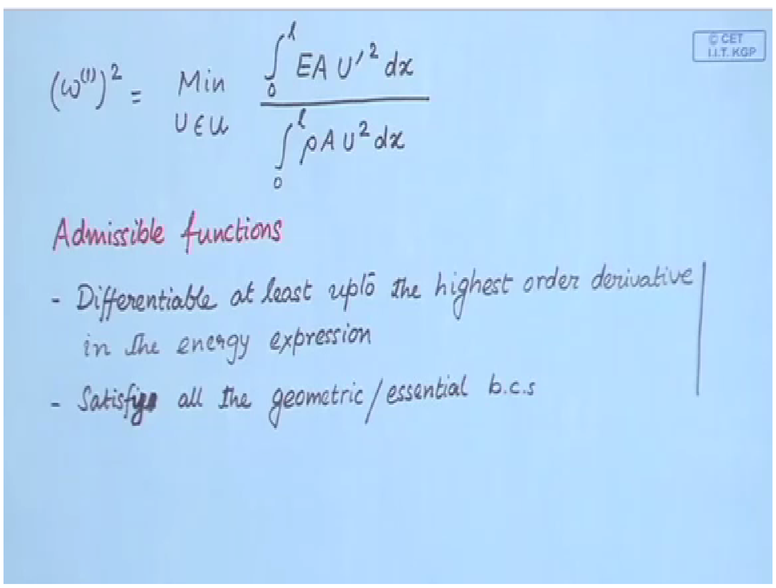

So the fundamental frequency square is minimization over the set of admissible functions of the Rayleigh quotient. Now we come to this admissible function. So what are admissible functions? These are functions which satisfy the following two properties. The first property is it is Differentiable at least up to—so these function are differentiable at least up to the highest order of spatial derivative in the energy expression.

So of course these functions are special function and so they must be differentiable at least up to the highest order of space derivative or spatial derivative in the energy expression. So in this example that we are considering the highest order of space derivative is 1. So the set of admissible function should be differentiable up to first order. The second important property that it should satisfy is that it should satisfy all the geometric boundary conditions of the problem.

So admissible functions must satisfy all the geometric, or essential boundary conditions of the problem. So they satisfy all the geometric or essential boundary conditions of the problem. So these two properties the functions that satisfy these properties are known as Admissible functions. Now such functions can be constructed using Polynomial, trigonometric function and other such elementary functions.

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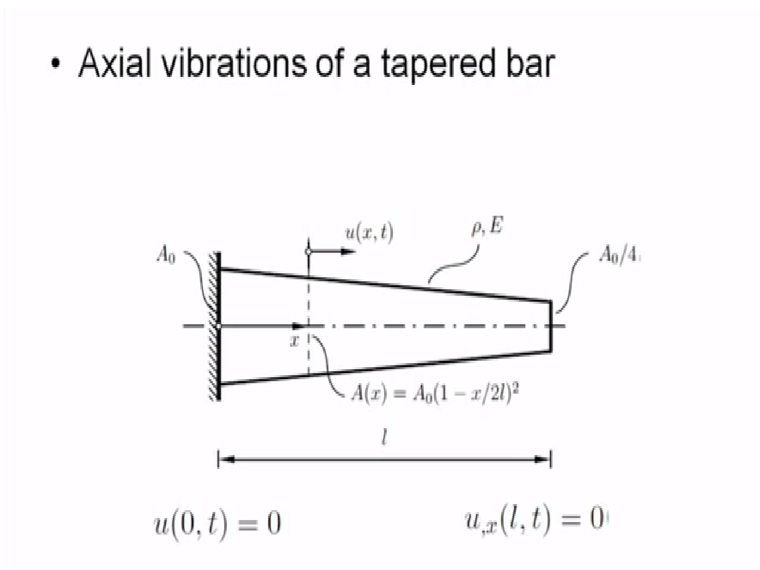

Now let us look at this example of tapered bar. So as you can see it is fixed at X = 0 and its free at X = 2l. So the geometrical essential boundary condition is on the left boundary where the bar is fixed. So we must choose functions admissible functions which satisfy the boundary condition at the left. So let us look at such functions.

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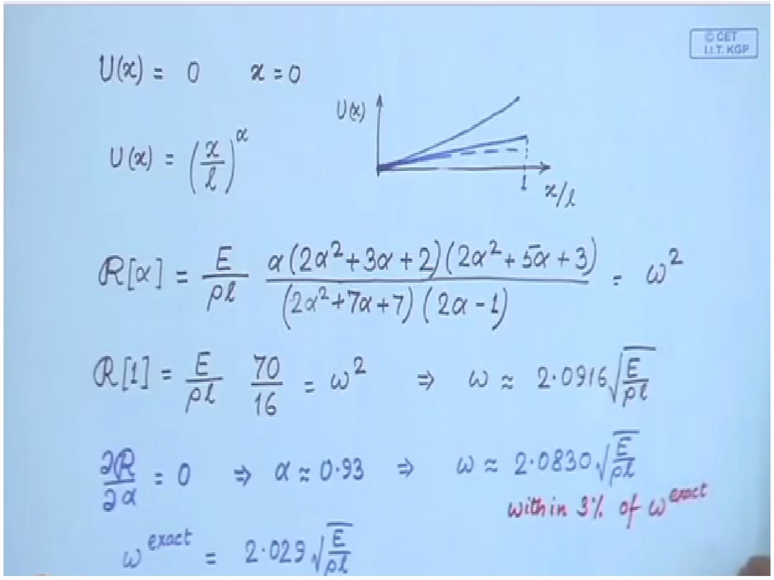

So we must have so we have to choose admissible functions for this problem such that it is 0 and X = 0; this is the minimum condition that is required. So a possible choice we may write it like this x over l but we can have a class of these functions by raising it to the power alpha. So we actually have functions which depending on alpha for example if alpha is one then this is linear for other powers, it can go like this or you can go like this.

Now, so this alpha we will initially keep it arbitrary and we are going to substitute this in the Rayleigh quotient and we are going to look at what is the Rayleigh quotient with this expression of admissible function. Now one thing that you can see that we now have not one function but we have a class of function and we can adjust alpha and see which function minimizes the Rayleigh quotient.

So this alpha provides us with a angle to solve a minimization problem as we have formulated it. So if you calculate the Rayleigh quotient with this admissible function then the Rayleigh quotient turns out to be a function of alpha which is unknown a Z and this expression turns out to be this can be solved very easily and what we obtained is this expression at the Rayleigh quotient.

Now, so this term is the-- I mean has a properties of the bar the geometric as well as material properties of the bar where this coefficient which is a function of alpha determine the Rayleigh quotient, now this alpha is unknown. So we can put various values of alpha or we can minimize the Rayleigh quotient with respect to alpha and determine alpha. So let us see what happens if alpha is 1. And this remember is omega square. So this turns out to be 70 over 16.

So this is an estimate of omega really the first the fundamental circular frequency and this turns out to be-- Now let us see what happens if we minimize with respect to alpha. So which means—so if you do this minimization this gives and corresponding to this—so you see that this value is lower than this. So this is a better estimate of the circular natural frequency. Now we have solved this problem of the tapered bar in a previous lecture analytically and the exact circular natural frequency that we determine was this.

So this is still lower as you can see that this is higher than the exact but they are quite close this is within 3% of the exact. So we have fairly accurately estimated the fundamental frequency of a tapered bar using very simple method. But this of course this gives the best estimate based on the found of the structure of admissible function that we have chosen. But remember that when we go onto calculate stress since this alpha is less than one for our best estimate the stress that we will calculate at X = 0 will be infinite.

So it will have some-- I mean it will give some unrealistic estimates of stress in the bar. However, the frequency estimate is fairly accurate. Now here using Rayleigh method we have estimated the fundamental frequency. Now can we now go on to find out the higher frequencies?

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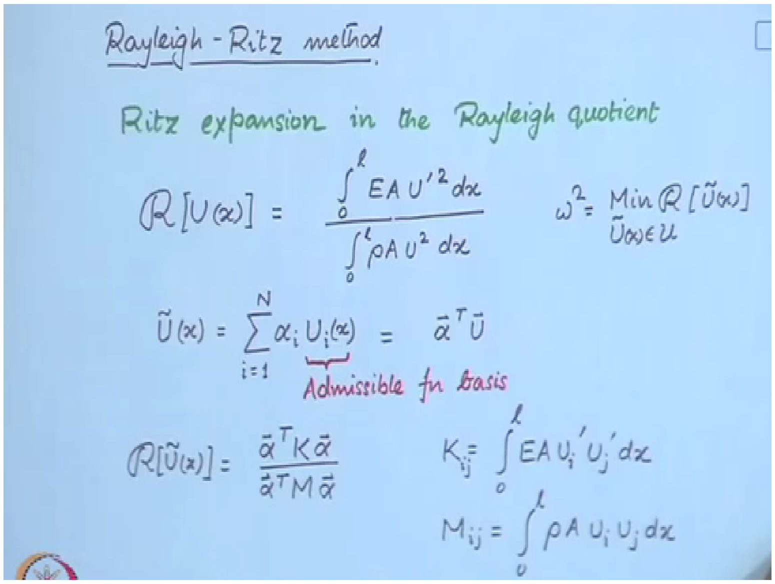

$$R[U(x)] = \frac{\int_0^l EA\,U'^2\,dx}{\int_0^l \rho A\,U^2\,dx} \qquad \omega^2 = \min_{\tilde{U}(x)\in \mathcal{U}} R[\tilde{U}(x)]$$

$$\tilde{U}(x) = \sum_{i=1}^{N} \alpha_i U_i(x) = \vec{\alpha}^T \vec{U}$$

$$R[\tilde{U}(x)] = \frac{\vec{\alpha}^T K \vec{\alpha}}{\vec{\alpha}^T M \vec{\alpha}} \qquad K_{ij} = \int_0^l EA\,U_i' U_j'\,dx \qquad M_{ij} = \int_0^l \rho A\,U_i U_j\,dx$$

Now that is possible using what is known as the Rayleigh-Ritz method. So what we additional have in this method is the Ritz expansion. So we make use of the Ritz expansion of the amplitude function in the Rayleigh quotient. So what I mean, so if you see with this – this Rayleigh quotient let us say for our problem of the tapered bar. So we will minimize this and U has to be chosen from the set of admissible functions.

Now this U the amplitude function we will expand and put this symbol to till this distinguish this function from the basis function that we are using. So like in the previous example in the Rayleigh method we had kept an unknown parameter alpha. Here we will expand the amplitude function in terms of admissible basis function and set an unknown coefficient alpha; so this is a linear combination of this basis functions admissible functions.

So these are all admissible functions. And we can take any number of turns. Now when we substitute this – this kind of an expansion the Rayleigh quotient can be written as so this you can write as a vector multiplication. So alpha is column vector and U is also column vector, so the dot product will represent this scalar function U tilde there. And when I substitute this expression in the Rayleigh quotient, I can write in this form where K the matrix K-- so the ijth element is expressed in this form and similarly for the ijth element of this matrix M is expressed in this form.

Now we-- this remember this alpha this alpha vector is unknown. So we have to minimize with respect to this vector alpha.

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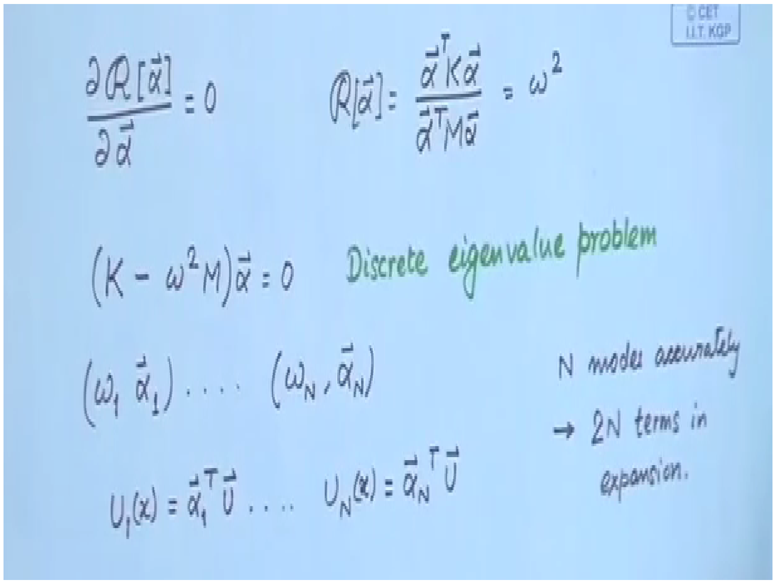

So which implies that- now this Rayleigh quotient is a function of this vector alpha and this must be put to 0 so the derivative must vanish for extremization. Now this derivative implies that the derivative has to be taken with each element with this vector alpha. So we have-- if there are capital N elements in this alpha then the N equations in N unknowns. So if you consider this expression of the Rayleigh quotient and if you perform this derivative which is straight forward finally what you will arrive at.

Now here how will omega enter? W have used this expression here. So if you perform the derivative this coefficient actually turns out to be this ratio which I am replacing by omega square. Now this is a discrete Eigenvalue problem which can be solved very easily to determine omega and this vector alpha the Eigen vector alpha and finally once we have the circular Eigen frequency and the corresponding vector alpha I.

These can be used to determine the corresponding Eigen functions using those basis functions—basis function vector, vector U. So using Rayleigh-Ritz method we can find out not only the fundamental but hard modes of vibration. Now as a thumb rule if we want because the accuracy of these various modes will be different. So as a thumb—rule of thumb we if we want N modes accurately we must state 2N terms in the expansion.

And this is rough estimate I mean rough way of estimating how many turns you must have in the expansion. And this may or may not work always but this is a good way to start. So if you want N modes accurately, reasonably accurately then you must have double the number of terms in your expansion.

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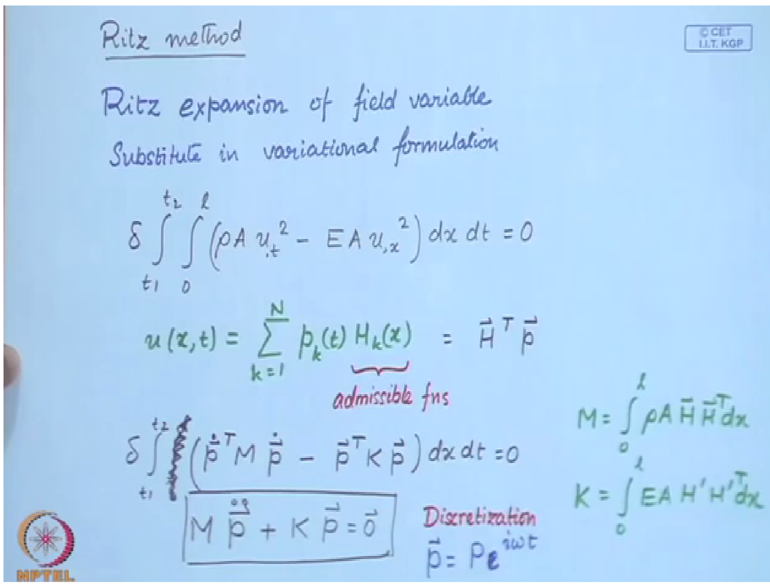

Now there exists another method which is quite powerful in this class of methods which is known as a Ritz method. Now in the Ritz method we use the idea of Ritz expansion of the field variable and substitute this expansion directly in the variational formulation. So this method works with a variational formulation of dynamics. So let us see for once again for the tapered bar we known that the variational formulation for this tapered bar means.

Now here we use this expansion again in terms of admissible function. Now once you substitute in the variational form in the variational formulation and simplify we obtained this where – well this integration over the space as already been performed so we substitute this here and since this function admissible function known is basis to us we can perform this integration and obtain this variation of this discrete problem this metrics M and K a given by this expression.

And we know that and this is now the lagrangian of a discrete system and the equations of motion can be immediately written. So in this Ritz method you have essentially discretized our problem. Now once we have discretized we can search for solutions as we do for discrete systems. So we search for modal solutions of this form and we solve Eigenvalue problem. So after these things are very standard.

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Axial vibration of a tapered bar

$$H_j(x) = \frac{x}{l}\left(1 - \frac{x}{2l}\right)^{j-1} \qquad j = 1,2$$

$$\rho A_0 l \begin{bmatrix} \frac{2}{15} & \frac{7}{80} \\ \frac{7}{80} & \frac{33}{560} \end{bmatrix} \begin{Bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{Bmatrix} + \frac{EA_0}{l} \begin{bmatrix} \frac{7}{12} & \frac{17}{48} \\ \frac{17}{48} & \frac{31}{120} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = 0$$

So let us look at the axial vibrations of the tapered bar of once again. So here I have written out the admissible function that we have chosen so H of x; for J 1 and 2 we have- so we have taken two functions and discretized using these two functions, the equation the discretized equation of motion is shown below. So once you have the discrete equations of motion.

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Axial vibration of a tapered bar

Modal solution $\mathbf{p}(t) = \mathbf{k}e^{i\omega t}$

Eigenvalue problem $[-\omega^2\mathbf{M} + \mathbf{K}]\mathbf{k} = 0$

Characteristic equation

$$\frac{81}{7}\omega^4 - 394\frac{c^2}{l^2}\omega^2 + 1455\frac{c^2}{l^2} = 0$$

Then the standard procedure follows which means that you assume a solution structure as shown here. You come to the Eigenvalue problem and finally the characteristic equation. Now if you solve this characteristic equation you will obtain the Eigen frequencies of the system.

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Axial vibration of a tapered bar

$$\frac{81}{7}\omega^4 - 394\frac{c^2}{l^2}\omega^2 + 1455\frac{c^2}{l^2} = 0$$

$$\omega_1^R = 2.053c/l \qquad \omega_2^R = 5.462c/l$$

$$\omega_1^{exact} = 2.029c/l \qquad \omega_2^{exact} = 4.913c/l$$

Upper-bound property

Now solving this we obtained the circular Eigen frequencies as you can see omega 1 superscript capital R calculated using the Ritz method and similarly omega 2 superscript capital R and they are compared with the exact circular Eigen frequencies. And you can see that the fundamental-- we have taken two turns and the fundamental is fundamental frequencies, circular frequency is matches quite well with the exact while there is some error in the—the second circular natural frequency. Now this Ritz method and even Rayleigh method this has an Upper-bound property.

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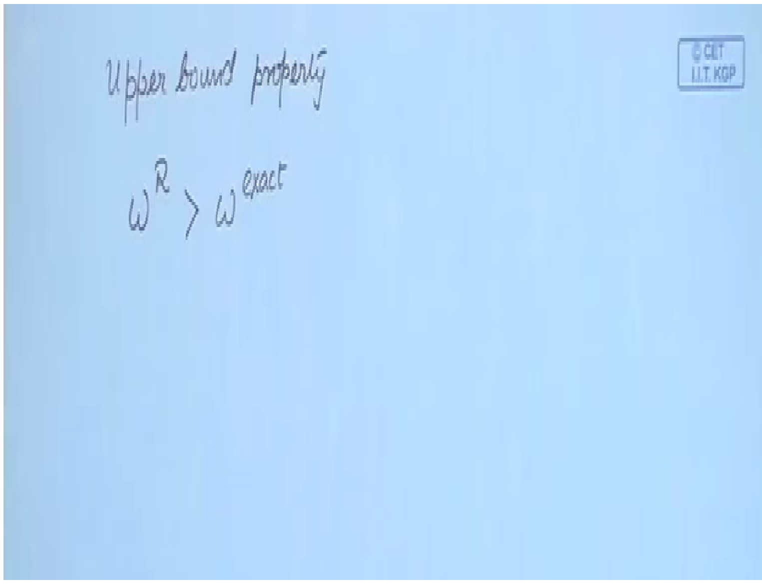

Which means that the natural frequency calculated from this approximate method is always greater than the exact which means that this gives an upper bound the actual natural frequency of the system is lower than what you calculate using the using these approximate methods.

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Axial vibration of a tapered bar

$$\mathbf{k}_1 = \begin{Bmatrix} 1.0 \\ 1.475 \end{Bmatrix} \quad \mathbf{k}_2 = \begin{Bmatrix} 1.0 \\ -1.505 \end{Bmatrix}$$

$$U_1(x) = \mathbf{H}^T\mathbf{k}_1 = \frac{x}{l} + 1.457\frac{x}{l}\left(1 - \frac{x}{2l}\right)$$

$$U_2(x) = \mathbf{H}^T\mathbf{k}_2 = \frac{x}{l} - 1.505\frac{x}{l}\left(1 - \frac{x}{2l}\right)$$

Now here, you can see the Eigen vectors K1 and K2 and the corresponding Eigen frequencies that have been determined by using the Ritz method. And once you plot these Eigen functions,

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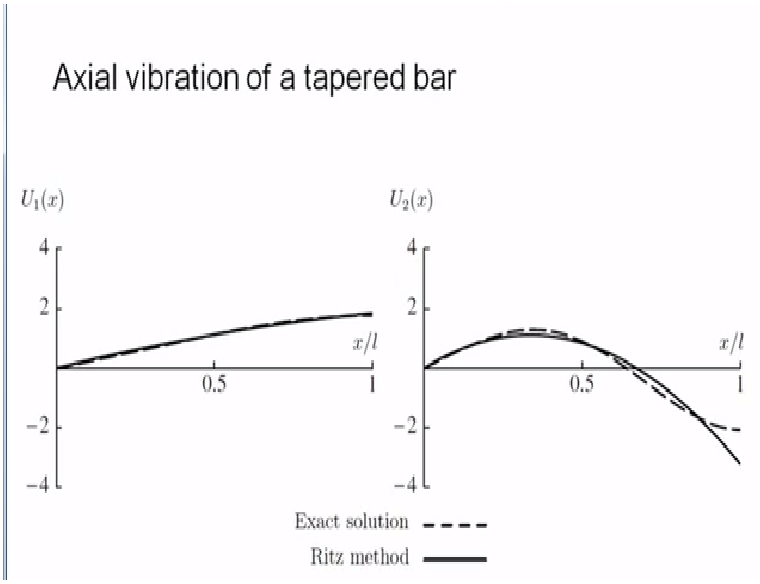

So this shows the comparison of the Eigen function calculated by the Ritz method and those obtained from the exact solution that we had discussed previously. So again you see that the fundamental Eigen frequency matches quite well with the exact while that of the second mode is an error especially at x over l equal to one.

Since we are considering only admissible functions which satisfy the geometric boundary condition which is at x equal to zero while the natural boundary condition is not satisfied with two terms you have to take more and more terms and then there is a conversion to the exact solution.

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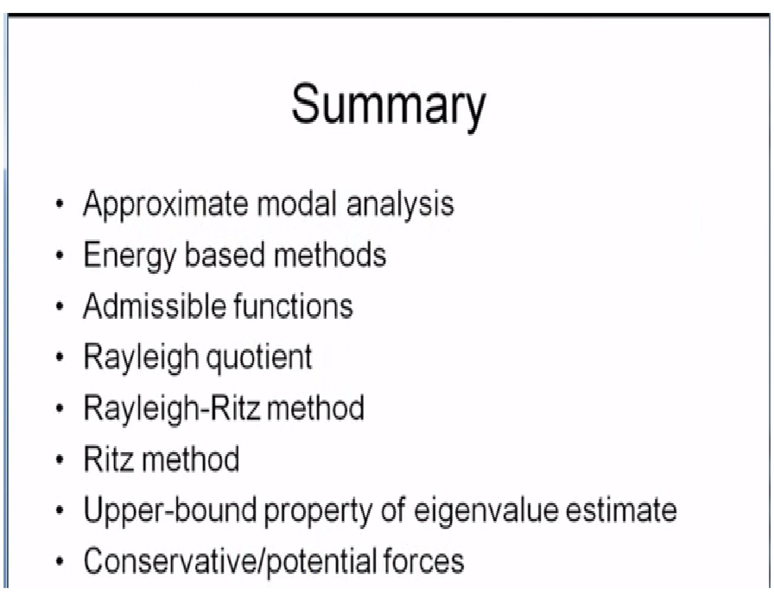

Now will summarize this lecture, so we have considered approximate modal analysis based on energy methods which uses admissible functions. We have looked at three methods Rayleigh quotient, Rayleigh-Ritz method and Ritz method and these methods give have an Upper-bound property of the Eigenvalue estimate. And these methods work for conservative systems with potential forces with that we conclude this lecture.