Vibrations of Structures
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Lecture - 07
Modal Analysis - II

So let us continue our discussions on Modal Analysis that we had started in the previous lecture.

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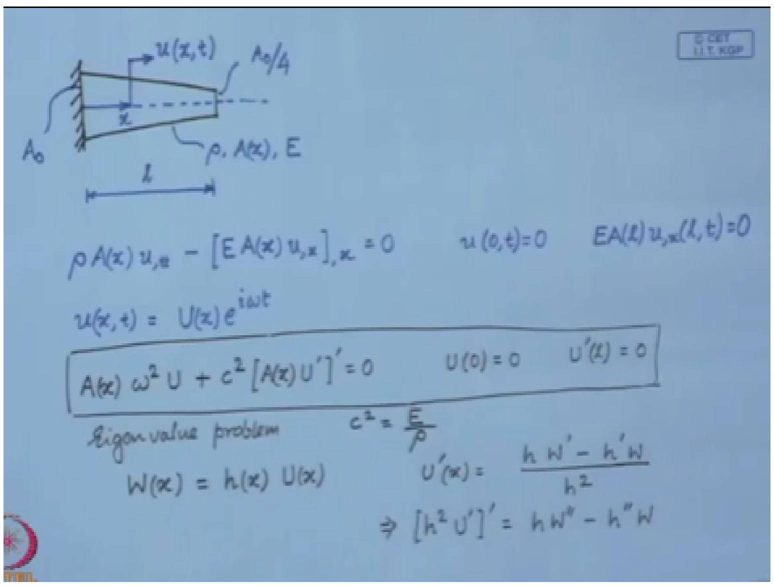

So, today we are going to take yet another example of a system consisting of a bar of varying cross section. So, this is a bar of varying cross section. Consider that A 0 is the area of cross section at the fixed end and at the free end, it is something like A 0 / 4. The field variable is represented by u (x,t), which represents the actual displacement at any point x at any time t. We assume that Rho is the density of the material of the bar.

And of course A as a function of x is the area of cross section and Young's modulus is E. And the length of the bar is l. So, the equation of motion of actual vibrations of a bar of variable cross section, may be written as this. The boundary conditions for the system that we have considered are given by the displacement is 0 at z = 0 and at the free end we have a dynamic boundary condition, which is a no force boundary condition, which can be written as this.

Now, once again we assume solution for as we have discussed in the previous lecture, so you consider the solution of this special structure. And if you introduce the solution in the equation of motion, then, after some simplification we arrive at the differential equation. So, this is an ordinary differential equation obtained by substituting this solution form in the equation of motion and correspondingly the boundary conditions for this differential equation are given by, so this is the Eigen value problem for our system.

So, we have to solve this Eigen value problem in order to find out the circular Eigen frequencies and circular characteristic frequencies Omega and the corresponding modes of vibration which are given by the Eigen functions of this Eigen value problem. So here of course, c is E, the Young's modulus divided by Rho that is c Square. Now, for a general variation of the area of cross section, this may not be solvable analytically.

So, what we are going to attempt here today is to try to find a class of systems or class of variation of cross sectional area for which this problem might be solvable analytically. So to see that how or to find that class let us make variable transformation, let us consider a new variable W (x), which is expressed as some unknown function h of x into our amplitude function U (x).

Now if you differentiate, so U Prime of x can be written as. So this implies, now if you identify this h square, this quantity h square with the variation, the area then you can eliminate or replace this term with this expression and if you make the substitution in this Eigen value problem, the differential equation of the Eigen value problem then you can very easily see that this will turn out to be.

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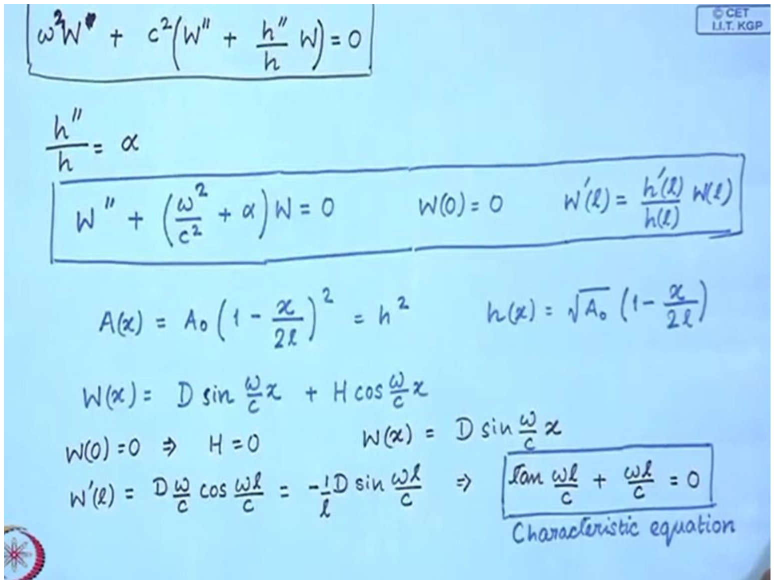

So if you substitute this U in terms of W in the differential equation and make some re arrangements then you can write the differential equation of the Eigen value problem in terms of, in this form and the corresponding, so here this is the differential equation in terms of the new variable W. Now for a special choice of this function h, this differential equation can be written in or can be expressed in a very familiar form or very simple form.

If h double Prime over h is a constant let us say Alpha, where alpha could be a positive or a negative constant. So for such a class of systems our differential equation of the Eigen value problem can be rewritten as. Now the corresponding boundary conditions can be obtained similarly and which turn out to be. So this is our new Eigen value problem in terms of the variable W.

So we are looking at a class of systems for which this function h double Prime over h is a constant, which is alpha, which were positive or negative so this class of system is characterized by variation of h which maybe or which is hyperbolic for Alpha greater than 0, it is harmonic for Alpha less than 0 and its quadratic, h is a quadratic function of x if Alpha is 0. So let us consider a particular case that I had shown in this figure. Here the radius is reducing linearly and the area goes from A0 to A0 over 4.

So in that case, the variation of the cross-sectional area maybe expressed as A0, remember that this is h Square, so if this is h Square then h, so h is linear in x and for this situation if you substitute this expression here then you will find that Alpha for this special case is 0. So if that is 0 then this simplifies further, the differential equation simplifies further and the solution can be written as. So the general solution of this differential equation is given here.

Now when you use the boundary conditions, so W (0) is 0 would imply H is 0 and the second boundary condition at the free end gives us the characteristic equation so which means here, so if H is 0 then W reduces to, now if you substitute this expression here at x equal to l. So, Omega Prime is given by minus of 1 over 2 l under root A 0 into minus of 1 over 2 l and so this expression becomes just -1.

So from here we obtain, here of course 1 over l will remains so what we obtain by applying these two boundary conditions is the characteristic equation of our system. So this is the characteristic equation for a fixed free bar with cross sectional area varying in this form. Now this is a transcendental equation which has to be solved numerically.

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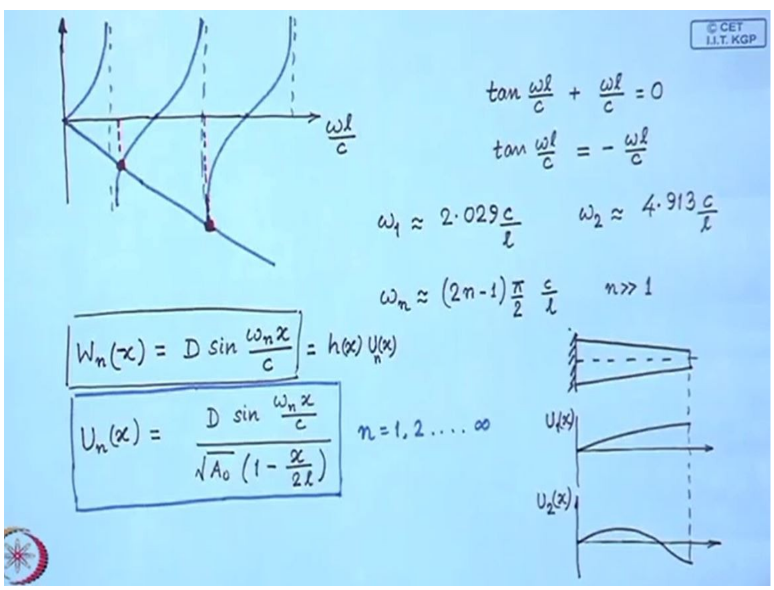

Now a good way to visualize the solution of this transcendental equation is to make a graphical plot. So in the x-axis I have Omega l over c so our characteristic equation is, so I will plot, so I will rewrite this as, so the tangent of Omega l over c looks roughly like this and minus of Omega l over c is a 45 degree line, - 45 degree line.

So these two functions are equal at these points which represent the solutions of the transcendental equation. So these solutions are obtained, so this point the first intersection gives us Omega 1 is 2.029 c over l, similarly Omega 2 and so on. So you can realize that there will be infinitely many intersections which are discrete so they will be countably infinite solutions of this transcendental equation and for higher intersections you have an approximate solution for n, for high values of n.

So once we have these Eigen values are the circular natural frequencies of the system, we can find out the corresponding Eigen functions, which describe the modes of vibration of the system. So these are also now indexed and are given by. So these are in terms of the new variable W. Now we can go back to our original variable U and write the Eigen functions. So this is from the structure of W that we had selected so W was nothing but.

So for our original problem the Eigen functions turn out to be these corresponding to the Eigen values of the circular natural frequencies given here. Now these Eigen functions may be drawn approximately. Here the amplitude function is or it represents the actual displacement of the bar. So this is the first mode of vibration with the circular natural frequency given here. The second mode looks something like this.

These things can be very easily plotted on the computer and visualize. So here we find an antinode, the node, this is the node at which the solution or the bar, so this is the point in the second mode, this point does not move in the, it always remains in its equilibrium position. So this is the node for the second mode, there is one node in a second mode and no node in the fundamental for the Eigen function U 1.

So this node as we discussed in the previous lecture is the point on the bar, which remains stationary at all times.

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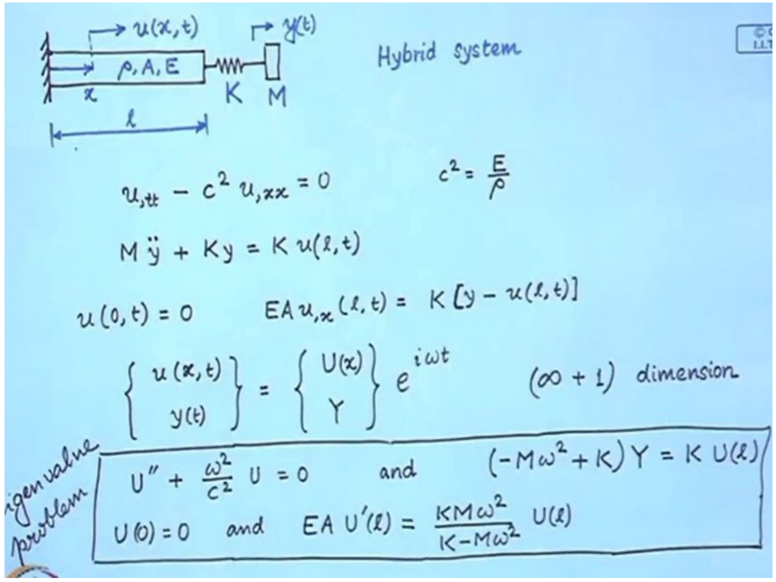

Next let us consider a system, a continuous system which is interacting with a discrete system. So as an example, we consider a uniform bar fixed at one end and attached to a simple harmonic oscillator in this manner. So, here we have a discrete mass represented by capital M and a spring of stiffness capital K, which is attached to a bar of length l. This kind of systems are quite common, when we have to put absorbers, for example, on a vibrating continuous system or a vibrating structure.

So this example is one such system in which we have a continuous system, which is a bar in actual vibration with a discrete oscillator attached. So, we will call them as hybrid systems because we have both continuous as well as discrete systems in this example, so the equation of motion are now the equations of motion because we have a bar and an oscillator. So, we have two equations of motion, for the bar, the equation of motion can be written directly in this form, where c square is E over Rho.

For the oscillator the equation of motion can be easily written. So, y measures the displacement of the mass M from its equilibrium position. So, as you can realize we have two dependent variables, one is the field variable U, function of x and time and the co-ordinate of the discrete mass M given by y. Now the boundary conditions for this bar can be easily written, so u at 0 for all times must be 0 is a fixed end, on the right end of the bar we have this oscillator.

So, we have a dynamic boundary condition, so this must be the force exerted by the spring at this end. So, these are the two boundary conditions for the bar. Now as I mentioned, this system now has a field variable for the bar and a co-ordinate of this and the co-ordinate of this discrete mass M. So we can represent these variables as a vector and search for solutions of the form, this as we had done before.

Now it may be mentioned that this vector that we are representing, it represents the configuration of the system in a dimension, which is infinity + 1, infinity because of this bar as we already know and + 1 because of this discrete system. So the modal space is of dimension infinity + 1, so if you consider a solution structure like this and substitute in the equations of motion, then you can immediately obtain, so as with the structure of solutions that we have been assuming.

We are searching for solutions of this form, we have synchronous motion of the bar and the discrete mass, well all points of the bar and the discrete mass. So this, these are the equations that we obtained after substituting the solution in the differential equations and the boundary conditions tell us U at 0, capital U the amplitude function at 0 must be 0 and if you substitute this structure here and simplify, we obtain the condition at the right boundary in this form.

So, here I have used this equation to simplify the structure of the boundary condition at the right end of the bar. So, our Eigen value problem now is described completely by these equations and the boundary conditions. So this is what we have to now solve.

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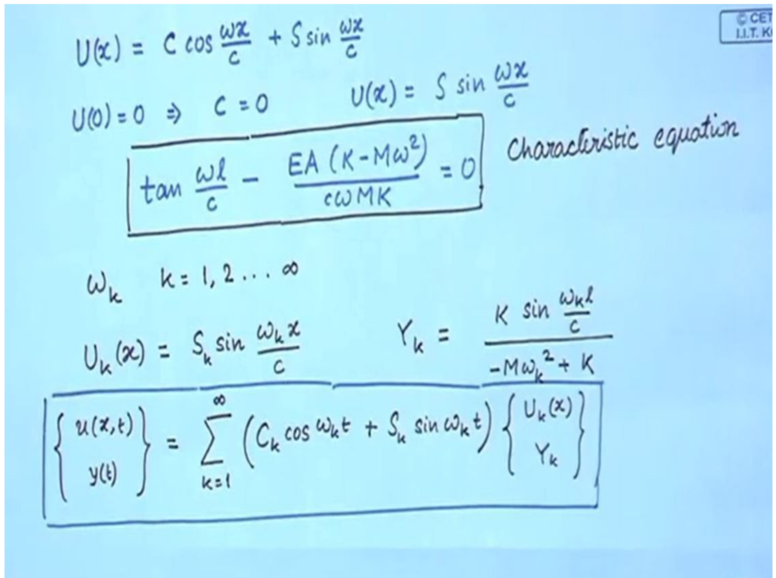

Now the solution of this differential equation may be represented in this form and if you use the boundary conditions, so the first boundary condition for example directly implies C, capital C is equal to 0. If you now, this therefore becomes simply x time Sin of Omega x over c, now if you substitute this in the second boundary condition and simplify, then can be checked that we obtained this condition, which is the characteristic equation of our system.

Now again, this is the transcendental equation which has to be solved numerically for the Eigen values Omega, so you will have discrete solutions of this transcendental equation but infinitely many solutions exist. So, you have countably infinitely many circular natural frequencies of the system obtained by solving this transcendental equation and corresponding to these Eigen values or circular natural frequencies.

You have the Eigen functions, the corresponding Eigen functions for the bar and corresponding to these Eigen functions, you can now find the amplitude function for or amplitude of the discrete mass. So, this is the amplitude function for the bar and this is the amplitude, the corresponding amplitude at the kth mode for the discrete mass. Therefore, the general solution may be represented by super posing all these solution in this form.

So, this is the general solution for the system. So you can see that the motion of the system is taking place in a modal space, which is of dimension infinity + 1 and as we had visualized in the case of string for example, this the motion of this bar with discrete oscillator is nothing but motion of a point in this infinity + 1 dimensional modal space or configuration space of the system. Now we can have two special cases, which fall immediately from the analysis that we performed.

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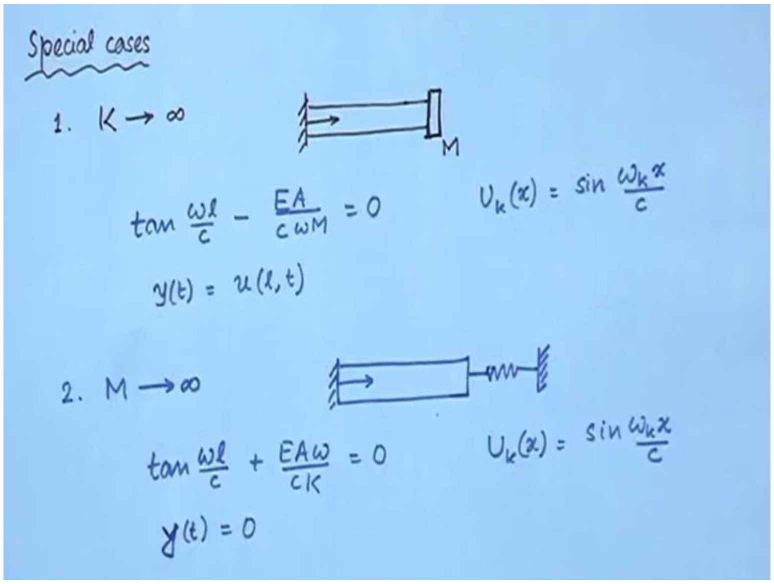

One is when the stiffness of the spring connecting the bar and the discrete mass tends to infinity, which means that the discrete mass is rigidly attached to the bar. In this case, so it immediately follows from this characteristic equation by taking K turning to infinity, the characteristic equation simplifies to this and of course, so you can find out the circular natural frequencies from this characteristic equation.

And the corresponding Eigen functions now only of the bar is given by this, so the discrete co-ordinate, the co-ordinate of this discrete mass become same as, so y is nothing but u at l. The second special case is when this mass becomes infinity M goes to infinity. So in that case, the system simplifies to this, so this is the end of this bar is connected to a spring, which is attached to a rigid wall.

So in this case, the characteristic equation simplifies to this form and the Eigen functions, corresponding Eigen functions are again of the same form. In this case, of course this, since the motion of the mass vanishes, so y (t) becomes 0. Now, if you look back in this example what we have discussed and if you see this Eigen value problem, you see this boundary condition here is dependent on the circular natural frequency or the Eigen value itself.

So this system, in this system the boundary condition is a function of the Eigen value. So to summarize, we have discussed today two further examples, for which we have performed the model analysis by solving the Eigen value problems and we have considered a bar with varying cross section and we have solved a class of problems for which we have obtained analytical solutions, which we will compare against solutions obtained by other methods later in this course.

The other thing that we have discussed today is continuous system interacting with the discrete vibrating system. So, we will continue this discussion further in the next lecture. So this completes today's lecture.