Vibrations of Structures
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Lecture - 05
Variational Formulations II

Let us resume our discussions on the variational formulation that we have started in our previous lecture. So today what we are going to look at is the transverse vibrations of a hanging chain. So this is a chain as we can see this is an inextensible continuum. So we are going to look at the equation of motion of such as chain.

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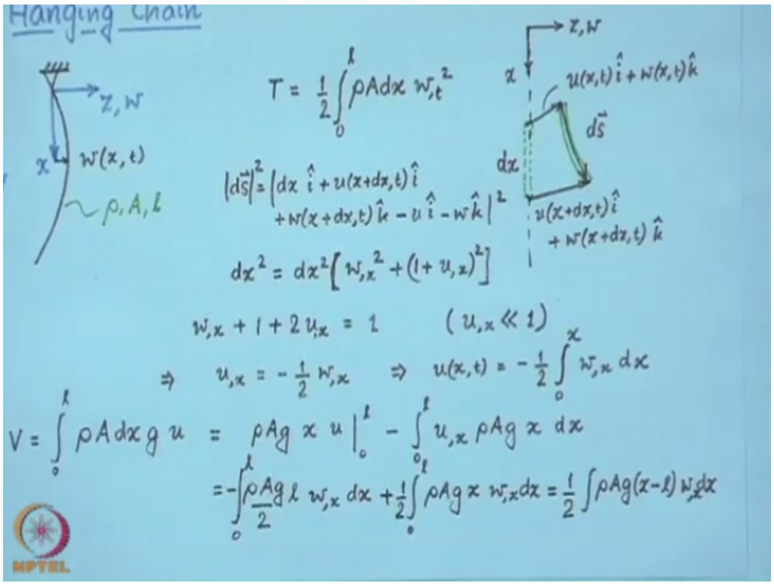

So let us consider this chain hanging in a unform gravitational field. So we assume that this chain is made of material of density rho has a constant area of cross section A has a length L. Now this is an inextensible chain. So though it does not resemble a string in that sense which is an elastic continuum one dimensional elastic continuum this is inextensible. Yet as we will see very soon the equation of motion at least up to the linear order is similar to that of a string.

So we are going to write down the energy expression. So the kinetic energy so at any location X our field variable is WX, T. So the kinetic energy of infinitesimal element can be written as half rho ADX is the mask of the infinitesimal element and velocity square. So if I integrate this over the length of the string then what I have this is the kinetic energy. Now the potential energy.

Now the potential energy of this chain. Now why chain restores back to its vertical position? It is because as it deflects or as it is disturbed the potential energy of the chain changes or increases. Now to calculate that let us look at a small element of this chain. So this is the undeformed configuration of the chain and this is the deflected configuration. So this infinitesimal element had a length DX.

Now since chain is inexpensive as we have assumed so the length of this infinitesimal element in the undeflected or in the equilibrium configuration is same as in the deflected configuration. So this is also the DX, but now this is deflected so I will represent it as a vector whose length or its magnitude is DX and these vectors I will represent as this way and this vector at X plus DX is given by this vector.

So if you write out this vector equation. So we can represent this DS vector in terms of the deflection of the chain. Now if you take the magnitude of the vectors on both sides then as I mentioned that the magnitude of this is still DX because the chain is inextensible. So what I will do is I will take the magnitude square let say and if you simplify the right hand side we have this expression from which we can write.

Now here I have made this approximation so in this step while coming from this step from this step I have made this approximation that Del U, Del X is much, much smaller than one. So the expansion of this I have in this expansion I have dropped Del U, Del X whole square. So this implies or so I have represented the actual deflection or actual motion of the chain. Now this is very different from that in strings.

So in strings we have neglected the actual motion, but in a hanging chain we must consider this actual emotion because that is the reason why the potential energy of the chain is changing as the chain deflects, from its equilibrium position. Now once we have this expression of the actual deflection we can write down the potential energy of the chain. So rho into A is mass per unit length. DX that gives the mass of a little as an infinitesimal element of the chain into the acceleration due to gravity, into U.

So that would be the potential energy and if you integrate over the length of the chain that will give you the net potential energy assuming that the potential energy is 0 in its equilibrium configuration. Now what I can do is I can integrate this expression by parts. I can write this as taking U as the first function and rho AG as the second function. So the first functions integral the second function minus integral of derivative of the first function of the integral of the second function.

Now this simplifies to this boundary term can be written as U evaluated at L so that would mean this integral with X replaced by L and one minus from here when I replace Del U, Del X here. So we have the kinetic and the potential energy expressions of the chain.

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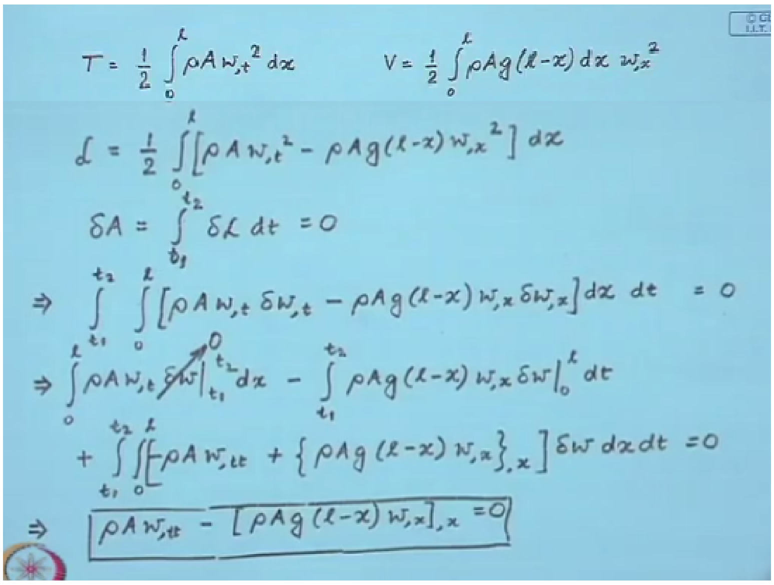

So let me write them again. So if you look at these expressions this kinetic energy expression this resembles the kinetic energy of the normal elastic string and this potential energy expression also resembles that as a string except for this term. Now in a normal elastic string here this term is the tension which is constant, but in the case of a hanging string or handing chain as we know the tension varies with the location.

So our Lagrangian now may be expressed like this and from Hamilton's principles which says the variation of the action is 0. We can write now here once again as we had done before will integrate by parts this term with respect to time and this term with respect to space to obtain so this is what we obtain finally. Now here as we had stated before since we know the configuration of the chain at the time instance of T1 and T2 so there cannot be any variation of the configuration.

So this at both times T1, T2 this variation must vanish. Now here is a term which is evaluated only at the boundaries and here it is evaluated or it is over the full domain. It is integrated over the complete domain 0 to L. Now I can always the boundary variation fix and change the inner portion arbitrarily. So therefore if this sum has to vanish they must vanish separately.

So for arbitrary variation therefore if this integral has to vanish then we can write this must be 0 which is our equation of motion of the hanging chain. Now this boundary terms must also vanish. So this will give us the boundary condition. So, let us have a look at the boundary conditions.

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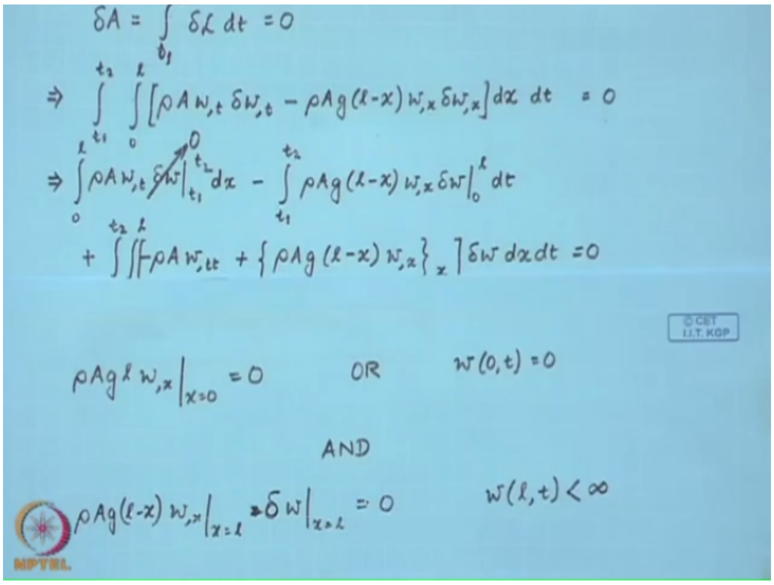

Now at X equal to 0 so at this boundary either this must be equal to 0 or now in our case if you consider that the chain is fixed at the top end then this is the boundary condition. If it is free then this is going to be boundary condition. So these are the possible boundary conditions at X equal 0 and we can have so here as you can see that at X equal to L this term vanishes.

Now this remember was a tension in the string and tension at the free end of hanging chain is definitely 0. So this of course there is a product now here in this case of a chain with a free end this vanishes. Now in order to have 0 contribution from the boundary at X equal to L therefore you must have this variation to be finite because this part of the boundary term is 0 ay X equal to L.

So therefore we usually write we say that the deflection of the string of the chain at the free end must be finite and the implications of this boundary condition will be seen when we discuss this solution procedure for the string they are a hanging chain.

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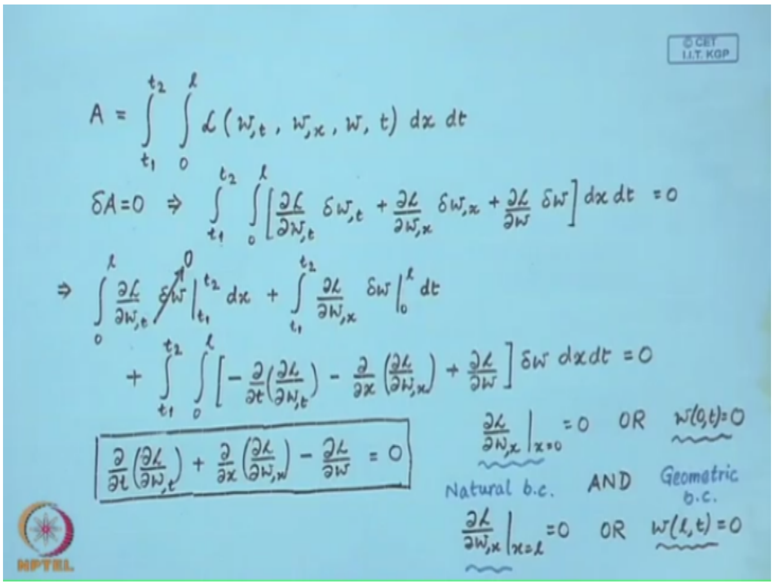

Next, we are going to slightly generalize the procedure so we have been looking at actual integral in which the Lagrangian is a function of the velocity, the slope. It may be also function of the field variable W itself and it may be a function of time. So let us slightly generalize what we have been doing for a Lagrangian which has this form which is a function of the velocity, the special derivative of the field variable, the field variable itself and maybe time.

So if you say that the variation is 0 that would imply so delta L maybe written as in this form. Now I will integrate by parts the first term in the integrant, the first two terms. The first term with respect to time and the second term with respect to space and of course this must be 0. So here we have minus of derivative with respect to time minus derivative with respect to X delta W comes out from here and that must be 0.

So as you know that at these two time instants I definitely know the configuration of the system which means I completely know my field variable. So there cannot be any variation on the field variable at these two time instants. So if the rest of the terms must vanish for arbitrary variation delta W you see this term which is evaluated at the boundaries and this is depended on the total domain.

Now I can fix the variation at the boundary and change the variation over the domain. So these two terms must independently vanish for this sum to vanish. So for arbitrary variation if this integral is to vanish then it must be true and this is our equation of motion and what we have from this term we have the boundary conditions. So the boundaries conditions of possible boundary conditions could be so Del L, Del W, Del X at X equal 0 is 0 or W itself at 0 must be 0.

And Del L, Del W, X at X equal to L must be 0 or W at L must vanish. So these are the possible boundary conditions for the problem. Now it happens that boundary conditions of this type they are the geometric boundary conditions while these boundary conditions are the natural boundary conditions of a problem.

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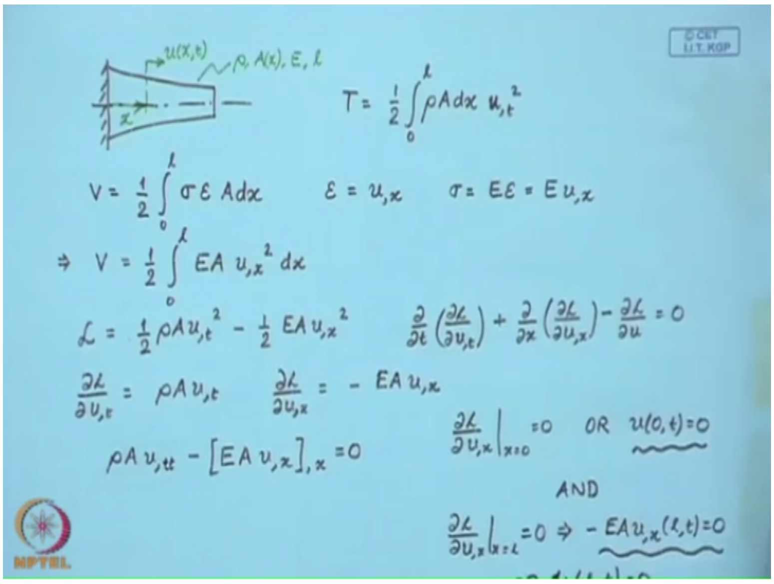

So let us look at some applications of this generalized formulation for vibrations of bars. So we consider first actual vibrations of a bar. So the field variable is U, X, T. So first we write down the kinetic energy. So if rho is a density A is the cross section then this is the mass of infinitesimal element into velocity square and this integrated over 0 to L it will give you the kinetic energy of the bar.

Now the potential energy of the bar maybe written from elasticity theory as half times the stress times the string and this is third unit volume. So I must integrate this over the volume. So if A is eta cross section A, D, X is this little volume element and this integrated from 0 to L will give us the total potential energy of the bar. Now string as we derived before is Del U, Del X and sigma is so therefore this is the expression of the potential energy of the bar.

Now what we have been describing as the Lagrangian or what is more appropriately called the Lagrangian density is given by now as derived the equation of motion they will be obtained from this equation. Now as you see that there is no exclusive dependence of this Lagrangian density on U. So this term here is 0 so what we have is we have the only these two term. So Del L Del U, T is and Del L and Del U, X.

So therefore the equation of motion is obtained in this form which we have derived before as well and the boundary conditions are obtained the possible boundary conditions would be so in this case of course the boundary is fixed at X equal 0. So this is our boundary conditions at X equal to 0 and X equal to L this the boundary condition at X equal to L. There is another possibility which does not apply here.

So if the bar is fixed at the other end as well then we have this boundary condition so that does not apply here. So in our case for this bar a fixed free bar these are the boundaries conditions of the problem.

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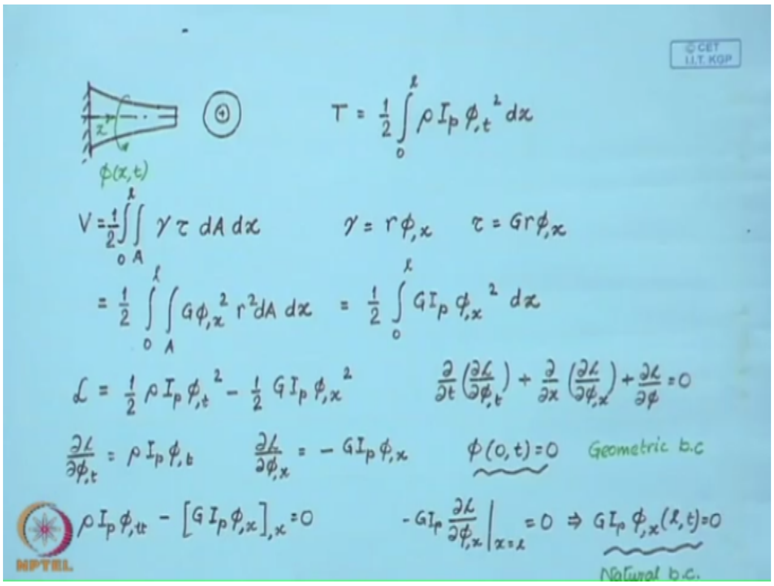

Now let us discuss this for torsional vibration of a circular bar as well. So if phi is the field variable then I can write the kinetic energy in this form. Now to write the potential energy of this bar so again from theory of elasticity now as we know that the stress the shear stress is varying over the cross section. So I must first integrate over the cross section and then finally over the length.

So this is energy per unit volume and that I integrate over the total volume to obtain the total potential energy. And we know that the shear strain is given by r times del phi / del x and shear stress is G times the shear strain. So if you substitute these two expressions in here this will simplify to G times IP. This of course comes with a square so that is the potential energy expression for this bar.

So our Lagrangian density is given by this expression. Now once again we apply our equation of motion so which reads once again we find that this Lagrangian density is independent of phi. So this term the third term is 0. So once I have evaluated these terms I can now write down the equation of motion in this form and the boundary conditions are obtained as X equal to 0 since this bar is fixed to the wall.

At the x equal to 0 there is no twist in the bar and at X equal to L this is a free end. So phi is not fixed so therefore so this term must be 0 which implies so these are our boundaries conditions for the problem. So this is the natural boundary, so this is a geometric boundary condition and this is a natural boundary condition. So what we have looked at in this lecture we started with the vibrations of the dynamics of a hanging chain.

We derive the equation of motion and the boundary conditions then we generalized the variational approach to find the equation of motion and the boundary conditions. And we have derived a general form of the equation of motion and the boundary conditions and we have applied it to 2 examples. So we will carry forward this and see applications in other examples in the subsequent lectures. That is the end of this lecture.