Vibrations of Structures
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Lecture - 04
Variational Formulation - I

In this lecture, we are going to initiate some discussions on an alternate formulation of dynamics of continuous systems.

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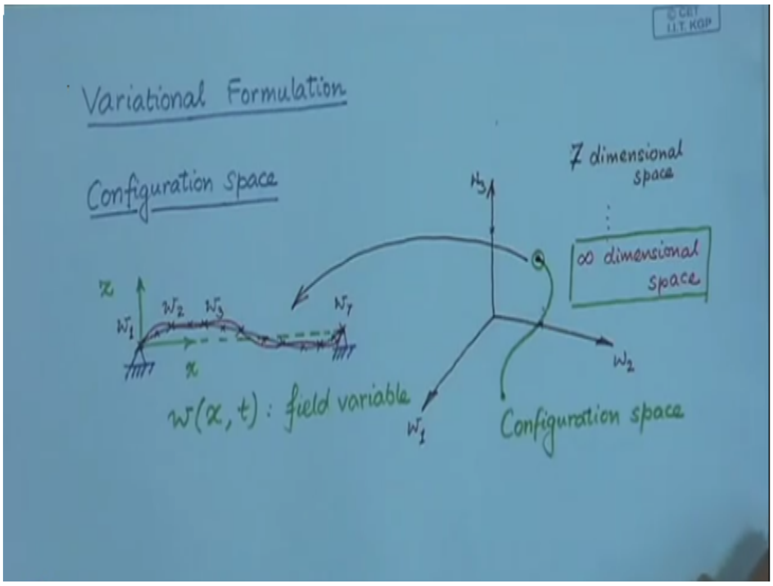

This formulation is known as the variational formulation. Before we get into this variational formulation, we have to understand a few concepts based on which this variational formulation has been made. So this concept is of configuration space. Now what is this configuration space? Let us picture a string, A taut string, which has been displaced from its equilibrium position which is the x axis.

Now suppose I want to represent or track the configuration of this string. So the simplest thing that I can think of is track certain particles on this on material points on this string. So here for example I have 7 points so let me call them as the displacement corresponding to these points as W1, W2, W3 etcetera up to W7. Now to visualize this configuration of the strength at this instance I can think of an equilibrium like space.

Here what I have done is I have taken three acces and that is the best thing I can draw W1, W2, W3 and then there are acces like this up to W7. So here I cannot draw a space like this with 7 access, but I will appeal to your imagination that you think of a space in which there are 7 such access and then mark on this access the displacement of each of these points. Say for example W1 is 0, W2 has a certain displacement, W3 has a certain displacement.

So like this in this space which is now a 7 dimensional space. There is a point which represents this configuration. So in such a space a point represents a configuration so this point, for example, represents this configuration of the string, but suppose now this string is actually like this. Then also I mean these 7 points have the same locations I have drawn this red configuration of the string very carefully.

So that this point in a 7 dimensional space is the configuration of the string, but we can see from this figure that it is not exactly the same as the previous configuration the blue configuration. So what do we do? So we increase the number of points and like this if you go on increasing the number of points to capture more and more configurations or infinitely many possible configurations.

Finally, what you come to is an infinite dimensional space which can track the configuration of the string. So finally we have we require an infinite dimensional space to represent all possible configurations of the string. Then as you can see such infinite possibilities therefore if you want to represent them we require what are known as field variables and that is what we have been using till now.

So W the transverse displacement of the string from the equilibrium position is therefore represented as the field variable which can capture all this infinite possible configurations of the string. So this space in which a point represents any configuration of the string is known as the configuration space of the string. So therefore you can clearly see that for a string and for any continuous system the dimension of the configuration space is infinity.

So such systems continuous systems are represented as points an infinite dimensional configuration space and the number of coordinates of the configuration space represents the degree of freedom of the system. So the degree of freedom of a string or any continuous system is infinity. What is very important now to move on is to remember that in the configuration space any configuration of a continuos system is represented by a single point.

So therefore as the string vibrates as it moves through configurations there are trajectories in this configuration space through which the string is going to move.

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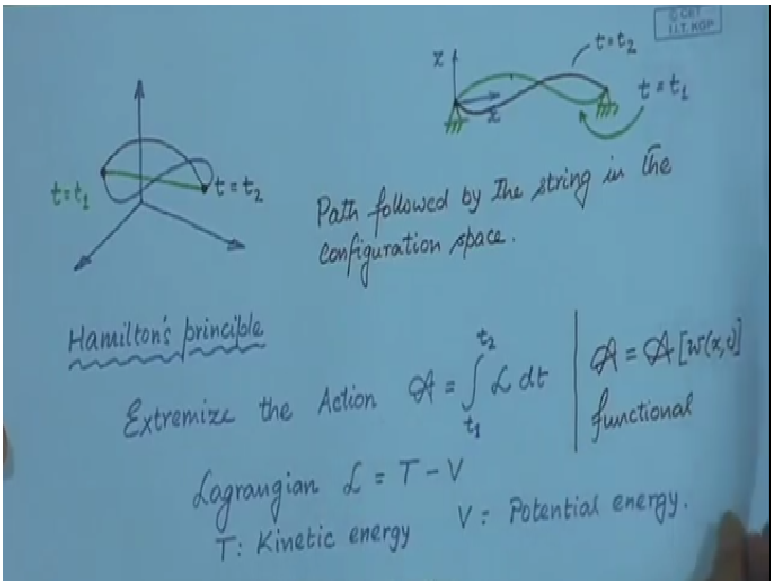

So with this basic definition let us look at the variational formulation of dynamics. So imagine an infinite dimensional configuration space and a time T1 I observe the configuration of the system is represented by this point T equal to T1. So this is this point in the infinite dimensional configuration space represents the configuration of the string let say at time T equal to T1 which may for example look like this.

Now so this is I have observed this is the configuration of the string that I have observed that time T equal to T1. Now suppose I close my eyes and allow this string to move and suppose at time T equal to T2 it attains a configuration like this. So this was the configuration at time T equal to T1 the black configuration is the configuration of the string at time T equal to T2. So at time T equal to T2 I have opened my eyes and I observe the configuration of the string over this which is another point in the configuration space of the string.

Now the question is how did the string move from this configuration at time T time equal to T1 to this configuration at time T equal to T2. Well it could have moved in this manner or it could have moved in this manner or may be this. The question is can I say which path did the string follow in the configuration space so this is our question. So is there is a way of knowing because I have not seen how it has evolved from this state to this state.

Is there a way of knowing which path it followed? So this question is answered by something known as the Hamilton principle. So what is this Hamilton principles? So what Hamilton principle says is that of all these infinitely many available path for the system to move from the configuration at time T equal to T1 to another configuration at time T equal to T2. The one the system follows will extremize the action which is defined as an integral from T1 to T2 of a scalar known as the Lagrangian which is again defined as a difference of the kinetic and potential energies of the system.

So we again to reiterate we have observed two configuration of the string and we have not observed the intermediate configuration how it went from configuration 1 to configuration 2, but this principle says that the path taken by the string of the configuration intermediate configuration attained by the string in moving from configuration 1 to configuration 2 extremizes the action which is defined in this manner.

Now what is this extremization? So extremization has a connotation of minimization or maximization, but in mechanical systems such as a string or a bar this is the connotation of extremization which is actually minimization. Now let us see what is meant by this extremization. Now the thing that has to be understood is this we are talking in terms of paths in the configuration space.

So this action is a function of this path which is a function of time. So this action given the motion of the string or given the path taken by the string from configuration 1 to configuration 2 which is a function of time we have defined this action as an integral over the scaler function which is defined in terms of those paths such a thing is called a functional. So it is a function of a function so this action is a function of a function.

So we have to minimize over functions. We have to find a function which minimizes the action. Now this is slightly different from something we talk about in minimization of only functions.

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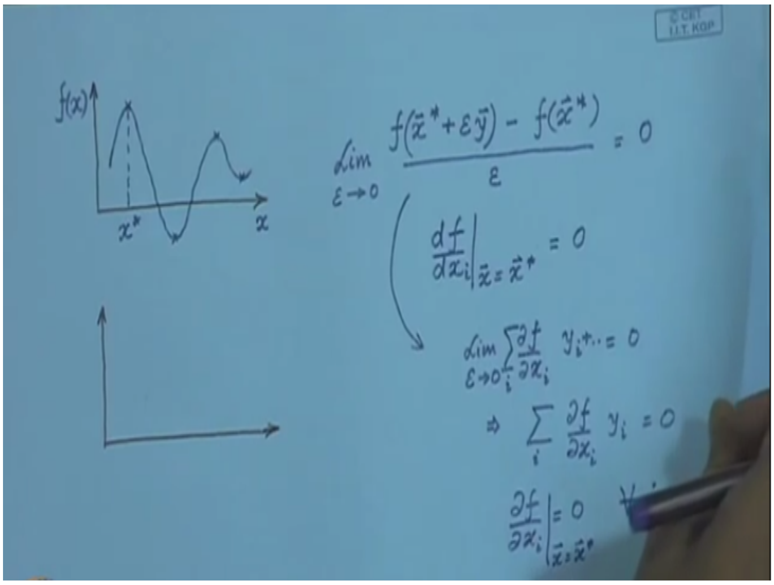

So let us see. So suppose we have a function F of X, how do you find out the extremum points, so these are extremum points. So what we say is that suppose X* is an extremum point in this case the maxima. Then how do we detect that this is a maxima or how do we detect this point an extremum point. So we make a test. So let us suppose that X* is a solution of an extremum point let us make this test.

We disturb or perturb this point this X* by a small amount. Here I have written this small amount as epsilon times Y where epsilon maybe a small quantity. Now if I compare take this difference divided by epsilon the and take the limit epsilon tends to 0. For arbitrary perturbations Y. And if this turns out to be 0 then we say we have found an extremum. So if you I mean this is the standard definition, this is in terms of the derivative.

This is what we do, but remember this X is a variable. Now suppose we want to draw an analogy from here to understand what is Hamilton's principles. So let me draw an analogy. So here by the way this formulation I mean this way of doing things will work even if x is a vector. If x is a vector, then let us see what happens. If you Taylor expand, then the first term.

So this is the first terms there will be further terms. So finally those terms have epsilon so when you take this limit they will vanish. So this is what we will get and now as I said that this should vanish for arbitrary perturbations Y. So if this is to vanish for arbitrary perturbations Y then we must have this should vanish for all.

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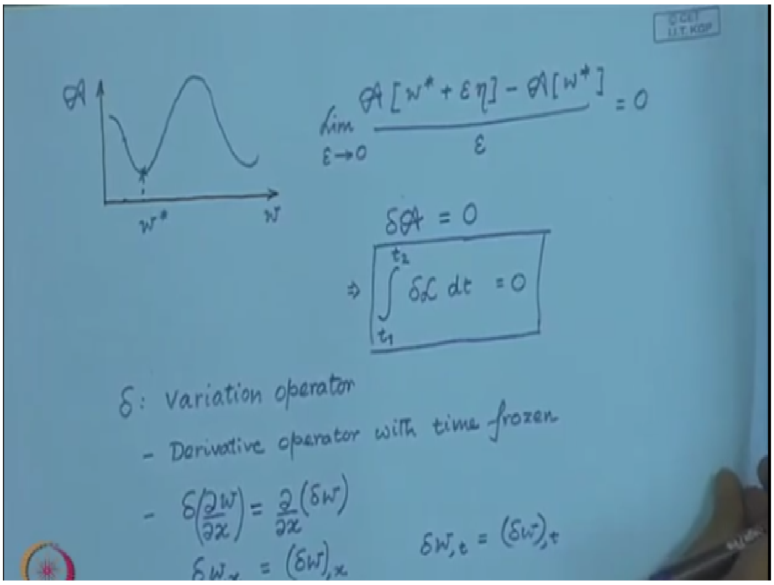

Now let me extend this to understand Hamilton's principles. So let me make a rough analogy I mean this I mean definitely cannot be representation of a Hamilton's principle perse, but with a little abuse of this notations let me imagine that this action has been calculated for different functions. So on this X axis now I have different functions. So a point here is a function and for which I am going to calculate A.

So let us just imagine for a moment that this is what our calculations result. So if I had to find out the extremum points then I must calculate this action and I want to understand in the similar manner what is this extremization. So suppose W* is the actual path taken by the string as it moves from configuration 1 to configuration 2. So this I can perturb with another function eta and small quantity of epsilon and take the difference divided by epsilon take the limit epsilon goes to 0.

And if this turns out to be 0 then I say I have found a path that the string has taken to move from configuration 1 to configuration 2. Remember this action has been calculated as an integral from T1 to T2 of the Lagrangian. Now this is represented in this manner just as in the case of functions we say del F, del X vanishes here we write delta of the action is equal to 0. So this therefore implies this condition.

And this is what we will use in this formulation. Now what is this delta? Delta is known as the variation operator. It is very similar to total derivative operator except that it does not differentiate time so time is frozen. So when this operator is applied it is assumed that time has been frozen. All this thing can be understood if this limit that we have calculated is analyzed.

So what we are looking is paths. So we will be perturb being paths. So time will be help frozen and paths will be perturbed. So this variation operator it is like a total derivative operator with time frozen. The second property of this operator that we will assume is that this operator commutes with partial derivatives which is to say so suppose you have partial derivative of W with respect of X and you operate delta over this.

And this can be written as so now an annotation we will be writing and similarly.

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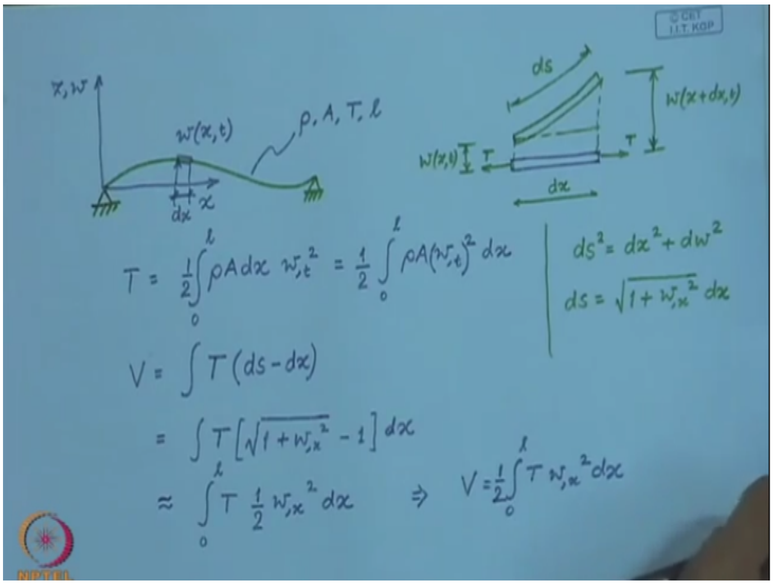

Now let us go over to some examples and see the application of Hamilton's principles. So here we have a string whose transverse displacement is represented by this field variable W. We assume that this string is made of a material of density rho area of cross section A and is under a attention T has a link L. This is the first thing that we need to write is the Lagrangian and construct the action integral and on which we will apply this variational formulation or Hamilton's principles.

So what is the kinetic energy of this string? So if I consider a small element of this string at a certain location X then the mass of this little element rho A, D, X where D, X is the length of this element. Now with this I multiply the velocity square of this little element and if I integrate this over 0 to L and multiply by half then I obtain the kinetic energy of the string. When I write del W, del T. So W, T, whole square.

So this is assumed that it is like this so that is the kinetic energy of the string. Now we have to write the potential energy. Now to write the potential energy let us look at the string and infinite decimal portion of the string as it goes. So this is the equilibrium configuration. So the length at the equilibrium configuration was D, X after it has been displaced its length changes to D, S.

Now this change in length is taking place under a tension T which we have assumed not to change with this placement. This was one of the assumptions of our model. So tension of course actually changes, but then that change is assumed to be negligible. There is no actual force on the string. So what is this length of, this deformed length of the string. So this is the traverse displacement at X and this whole thing is W, X plus D, X at time T.

So this length can be written as which maybe represented in this manner. So let us look at the work done by this tension as the string stretches. So tension which is almost constant times D, S minus D, X. Now I integrate this over the length of the string. So that should give me the work done and which is stored as potential energy in the string. So this now I expand this assuming that Del W, Del X is small which we have assumed.

So then this turns out to be so this will be 1 plus half Del W, Del X whole square so that one cancels off. So here I am left with only this term. So that is my expression of potential energy of the string.

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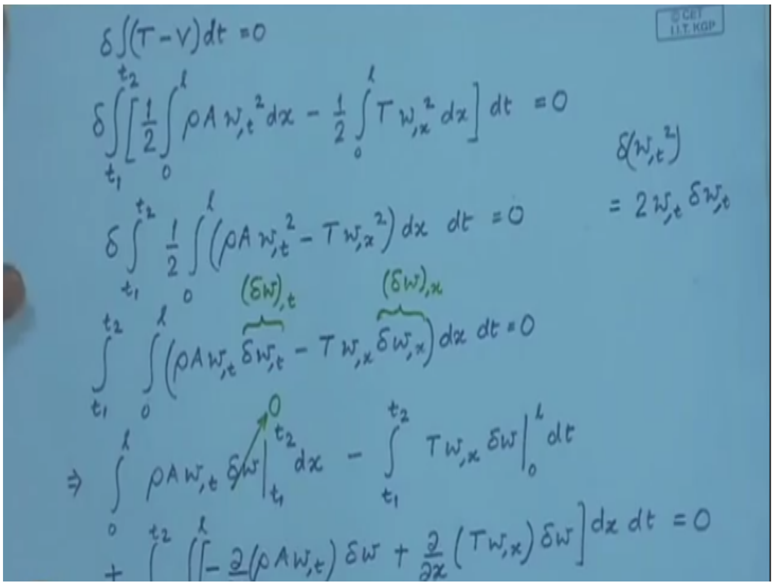

Now let look at the variational principle. So what Hamilton's principle says so let me first write so this is a statement of Hamilton's principle so which now can be written as this and I can simplify this further in this manner. Now I am going to apply this variation operator on the integral to obtain so Del W, Del T whole square when I operate that will give me two times Del W, Del T delta of Del W, Del T. Now this two will cancel with this half this two in the denominator.

So I am writing out the expression after such a simplification. Now this here as I had mentioned that this variation operator can commute with this time derivative and here the space derivative and this delta W is the small perturbation on the function W. Now since we want to separate out this small perturbation arbitrary perturbation, arbitrary variation over W I would like to have something in terms of delta W.

Now to obtain that I integrate by parts these terms this will be integrated by parts with respective to time and this will be integrated by parts with respect to spatial coordinate x. So if I do that and I assume that this integral will commute then I can write so I will take this as the first function and here this is nothing, but these expressions. So first function time integral of the second function.

So this is the first part obtained from this term in the integrant. Similarly, this is going to give me this has to be integrated by parts over space so my time integral will remain. So first function in the special integral of the second function. Now minus the time derivative of this and once again the integral and in the same manner. Now if you look here this variation over the configuration have two-time instance.

This variation has to be calculated at two-time instance. Now if you remember the formulation of the problem in the first place I know the configuration of the string at these two-time instance so there cannot be any variation I am not trying to vary these two configurations which are at time T equal to T1 and time T equal to T2. So I am not looking at variations of the initial and final configurations, but I am looking at variations at the intermediate times. So this variation must vanish. So here I finally have this.

Now the first term in this integral if you see the variation is at the boundaries delta W and all this terms will have to be evaluated at the boundaries, but here they are at the intermediate positions of the string. Now these two can be independently done so you can hold the variation over the string as fix and just vary the boundary or vice versa. So therefore if this has to vanish if this total expression has to vanish for arbitrary variations delta W.

Then I must have this integrant here in this square brackets must vanish. So this must be 0 which gives us the equation of motion as we can easily recognize now.

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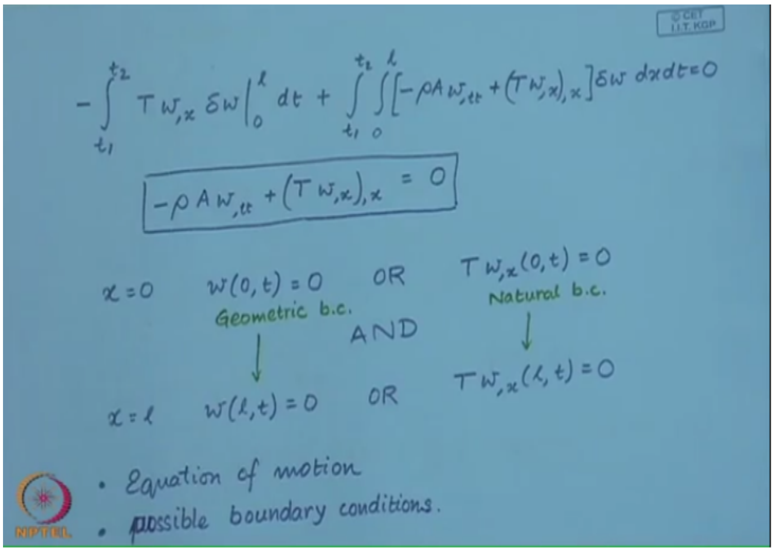

And this must vanish at the two boundaries. Now what this says is that I mean this can vanish in two ways say F equal to 0 let us look at X equal 0. This can vanish if delta W vanishes which means that W is fixed and for all times it maybe 0 or this may vanish at X equal 0 if system is equal to 0. Similarly, that X equal to L this can vanish if W at L is fixed or this term is O at X equal to L.

Now since this has being evaluated at X equal to 0 and X equal to L and both terms now must vanish. So both these things so either this equal to 0 and or this equal 0 or this equal to 0 and this equal to O. So various combinations are possible at the two boundaries and you can now easily recognize that this condition on 0 displacement which is like a fixed end is a geometric boundary condition whereas this is a forced condition.

And same thing here. So what are the various possibilities? Possibilities are both ends of the string are fixed or one end is fixed or one end is sliding or both ends may be sliding. So what we find here is that not only we get the equation of motion, but we also obtain all the possible boundary conditions. So when we use the variational formulation of the string or any continuos system not only we get the equation of motion, but we also get possible boundary conditions.

Now this is an extremely powerful method for formulating the equation of motion of very complicated systems. Now this is not this variational formulation is not just an approach for finding out equations and boundary conditions, it also as we will see later a method which will help us or lead to numerical methods for computational purposes and very powerful methods have been based on this variational principles.

So to summarize what we have discussed in today's lecture the variational formulation of dynamics of the continues systems and we have looked at the Hamilton's principles which forms the basis of this formulation and before that we have looked at what a configuration space of the continues system based on this Hamilton's principles works then we are finally taken an example of a string and derived its equation of motion and also obtained the possible boundary condition for the string. So we end the lecture here and we will continue further.