Vibrations of Structures
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Lecture - 33
Vibrations of Circular Plates

Today, we are going to discuss the vibrations of circular plates. So in the last lecture, we have seen the vibrations of rectangular plates, so today we are going to discuss the circular, the polar geometry.

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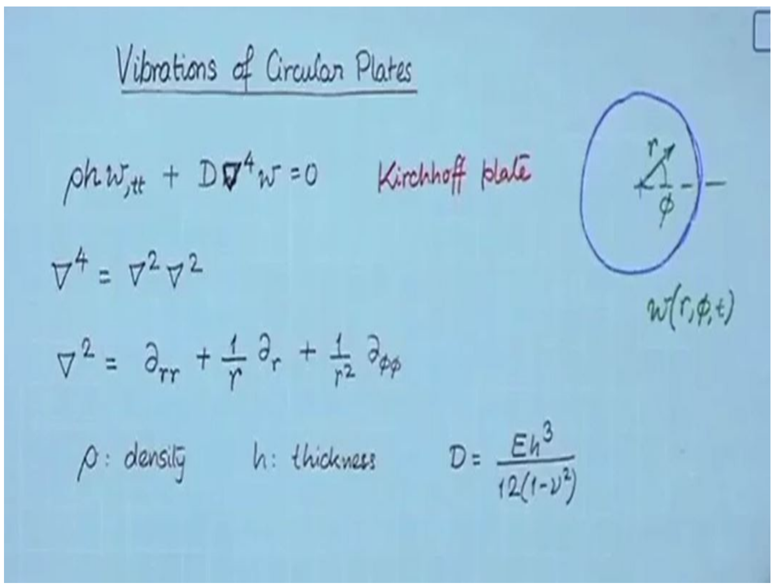

So let us consider this plate, so at any, so you are using the polar coordinates r Phi. So at any point r Phi at any time t, the field variable is represented by the W (r, Phi, t). Now the equation of motion reads, so this is the equation of motion for the plate where now for the polar, where this Nabla power 4 is of course square of the laplacian and the laplacian in the polar co-ordinates, plane polar coordinates is given by this operator.

So the square of this operator is Nabla power 4, which appears in this and here Rho is the density, h is the thickness, thickness considered constant in this equation and D. So, E is the Young's modulus and Nu is the Poisson’s ratio. Now, this is the Kirchhoff plate model, so we are going to look at the vibrations of circular plates, so essentially we are going to solve the Eigen value problem for various kinds of boundary conditions.

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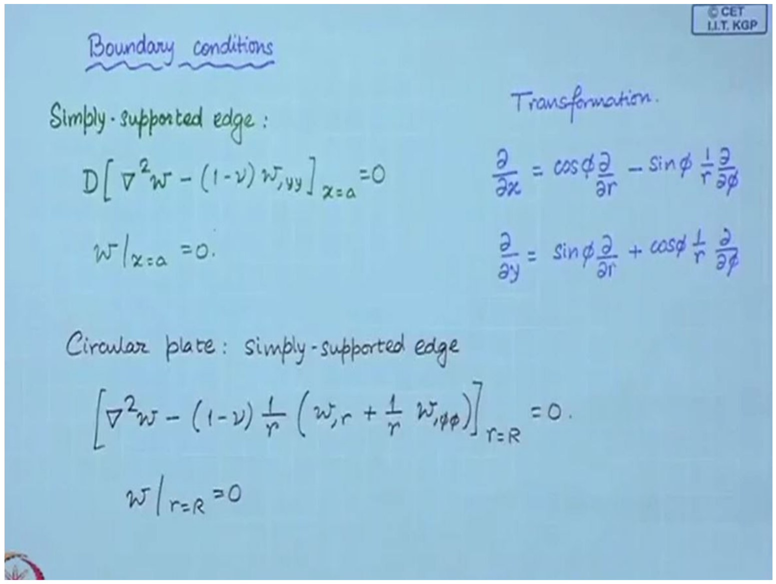

So let us have a look at let us say the boundary conditions of, so in the case of rectangular plates we have seen, for example for a simply supported edge we have, so for this is for the, so for the rectangular case, this actually simplifies, so this is the natural boundary condition, so this along with, so in the case of a rectangular plate this was a boundary condition. Now this of course because, there is no variation with y at x equal to a.

So you actually get the second derivative with respect to x at x equal to a is 0. But when we come to the simply supported condition for the circular plate what we can do is, we can use the transformation from the rectangular to polar coordinates, so we replace all these derivatives in terms of, the derivatives with respect to r and Phi. So this is the transformation, now when you use this transformation on these boundary conditions, then for the circular plate these.

So if the periphery is simply supported at r equal to capital R then this term will vanish of course, but still then this is the contribution from this term. So for the circular plate the natural boundary condition is given here and the other boundary condition, of course is a zero displacement condition.

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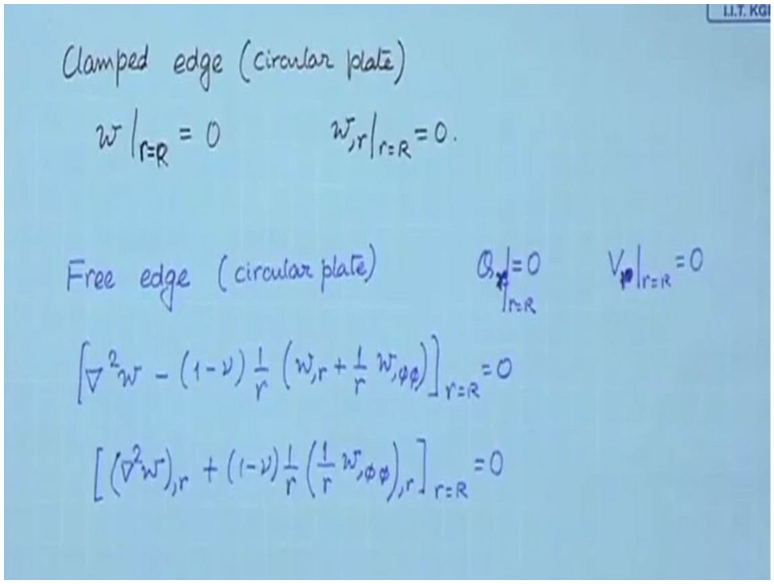

Now if you at the Clamped case, for a circular plate, then these boundary conditions are simple, so the displacement at r equal to capital R is 0 and the slope is also 0. Now for the free edge again for the circular plate, you can derive this once again from the boundary conditions for the rectangular plate using the transformation.

So for the free edge, we had the stress resultant because out of plane sheer equal to 0 and the edge Force, so, that must also vanish, so corresponding to these boundary conditions for the circular plate, they are, so this comes from the out of plane, so this should actually be r. Now here so this comes from the first condition and from the second condition, so this is the boundary condition for the zero edge force at r equal to capital R.

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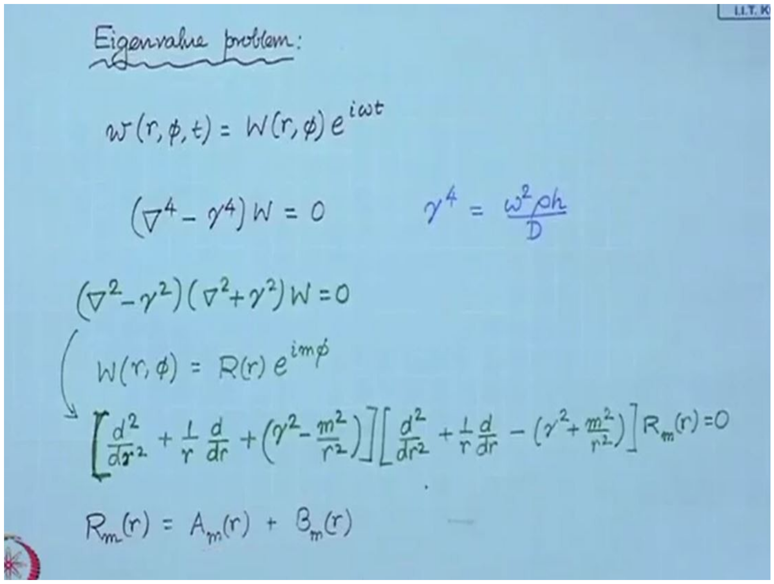

Now let us then look at the Eigen value problem that we obtain, when we do the modal analysis. So we would be looking at solutions of the form, so the separable in space and time. Now, when you substitute this solution in the equation of motion, so this is the differential equation of the Eigen value problem, so along with this, we will have the boundary conditions, where of course this Gamma power 4 is, now let us look, so this can again be decomposed as we had done for the rectangular plate.

We try a solution again separable in r and Phi because for a complete circular plate this function must be periodic in Phi, so we already know that this then, must have a structure like this. So if you substitute this solution form in here in this equation, then we can write, now because of this m, I can put an index m. So this is the differential equation that we have.

Now this function R, can then, we can consider as we have done for the case of the rectangular plate, so if you consider this function R to be constructed out of two functions such that A is a solution for this operator and B is a solution for this operator, then I can construct the solution of R by combining these two functions.

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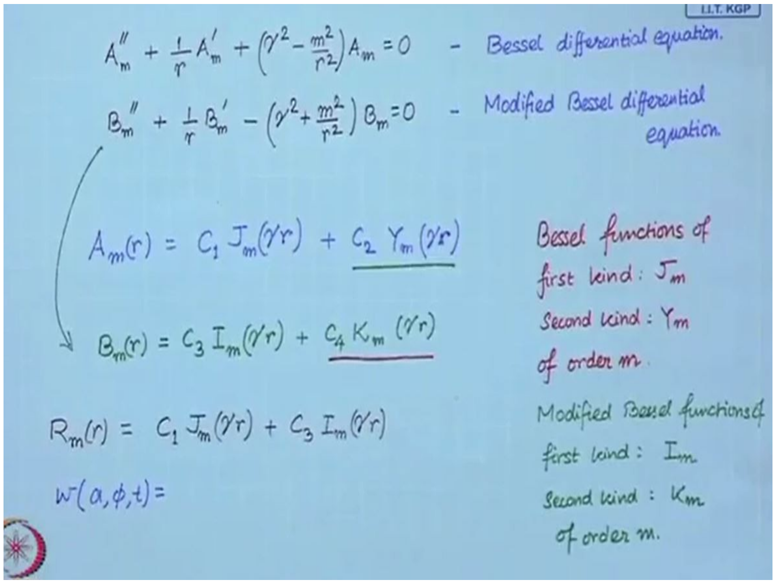

So what we have, so A satisfies this differential equation and B satisfies this differential equation. So now we are going to look at the solutions of these differential equations. So let us start with the first one. This we immediately recognize this is the Bessel differential equation and this actually is the Modified Bessel differential equation. So the solution of the Bessel differential equation.

We can straight away write, where these C1, C2 are constants and these are the Bessel function of first and second kind. So these are of order m, now the solution of the Modified Bessel differential equation this is written in terms of the Modified Bessel functions. So these are the modified Bessel functions. So, we have come across the Bessel functions J and Y and we already know that Y has a logarithmic singularity at 0 argument.

So at r equal to 0, this has a logarithmic singularity. So this function cannot appear in the solution of a complete plate, complete circular plate. If it is an annular plate, then this function can be present but for a complete circular plate this function this cannot appear in the solution for A m. Now let us look at these two functions the Modified Bessel.

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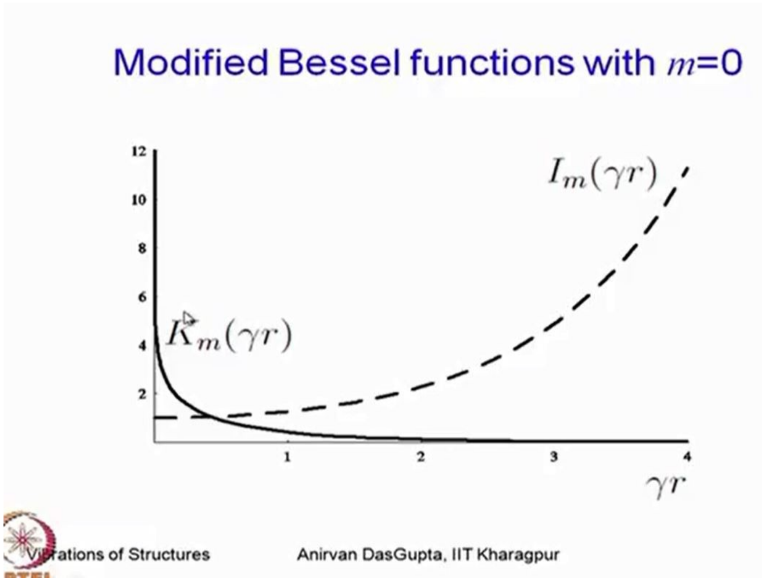

So this figure shows the Modified Bessel functions, for ‘m’ equal to 0, so this is K 0, this is I 0, so you can see that this K 0 and all Modified Bessel functions of the second kind, they have a singularity again at this argument 0 and I is this dashed curve. So, in a complete circular plate again, this function cannot appear in the case of a complete circular plate. For annular plate once again this can appear.

So since we are considering a complete circular plate so our solution for R therefore is, so this is the solution for R. Now we have to satisfy the boundary conditions in the case of ‘a’, let us say.

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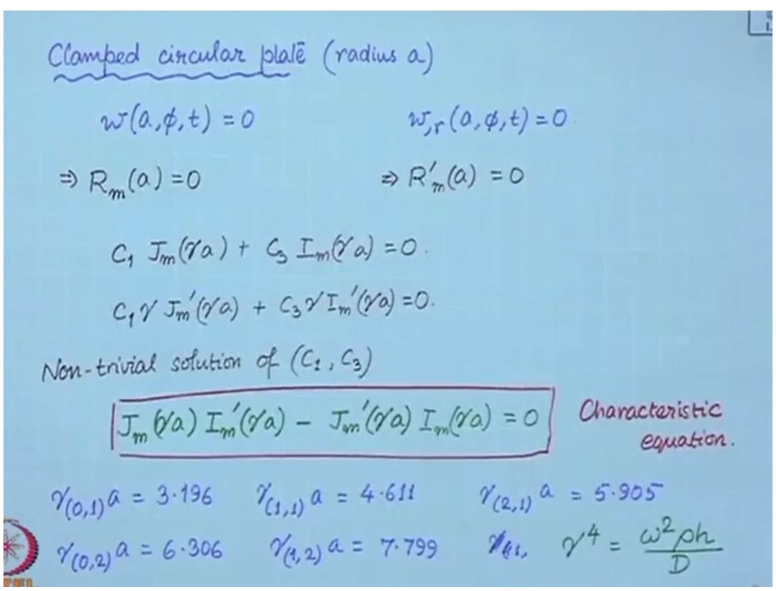

So let us consider a Clamped circular plate, so for the Clamped circular plate, the displacement at R equal to ‘a’ is 0 and this is also, the slope is also 0. So if you use these conditions with so R m at (a) must be 0 and the slope of so derivative with respect to R, at R equal to (a) must also vanish. So if you use these conditions you have and these two equations. Now, for non - trivial solutions of C1, C3 we must have the determinant of this matrix, so that turns out to be.

So this is our characteristic equation from where we are going to solve for this Gamma, so Gamma is in terms of the frequencies, so if you can solve Gamma then we know the frequency, so this equation has to be solved numerically and if you do that then the first few values of Gamma ‘a’, so these are some of the initial values of Gamma and we already know that this Gamma power 4 is, so if you know Gamma, then you can determine Omega from here.

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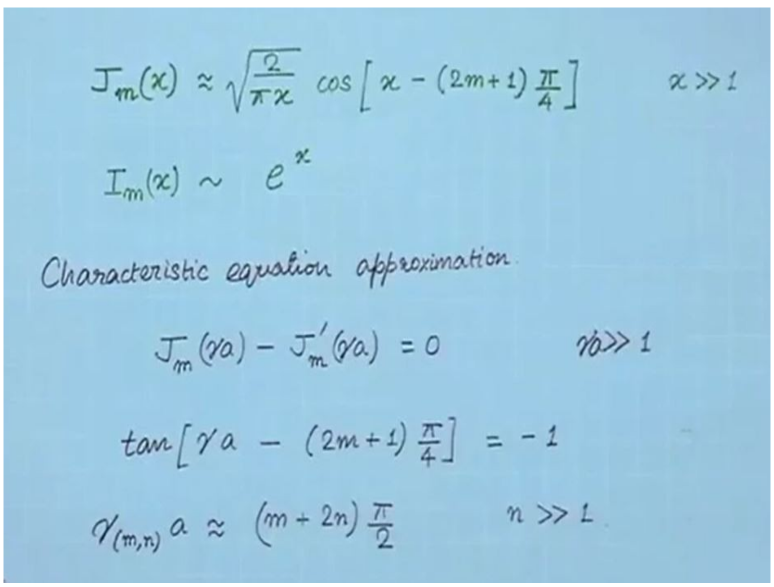

Now this characteristic equation efforts and approximation so as we have seen before that this Bessel function of the first kind, this has an approximation for large arguments, so for large value of argument of the Bessel function of first kind, I can write for x large. So we have this approximation. Now if you look at the function ‘I’, which is the Modified Bessel function of the first kind, then this is an increasing function.

This is proportional to exponential x so this function is dominated by this increasing function so the slope of this therefore is also some function of x times this exponential function. Now so far large arguments of this Modified Bessel function 'I', these derivative as well as the function itself they are like exponential so they can be dropped. So that we can simplify the characteristic equation as, so when Gamma is very large so let me put it as Gamma 'a' is very large.

Now if you use this approximation then this characteristic equation can be approximated as so again for large arguments and the solution, so the solutions of this gamma a times a, so for n much much larger than 1. So this can be used to approximate the circular Eigen frequencies of circular plate.

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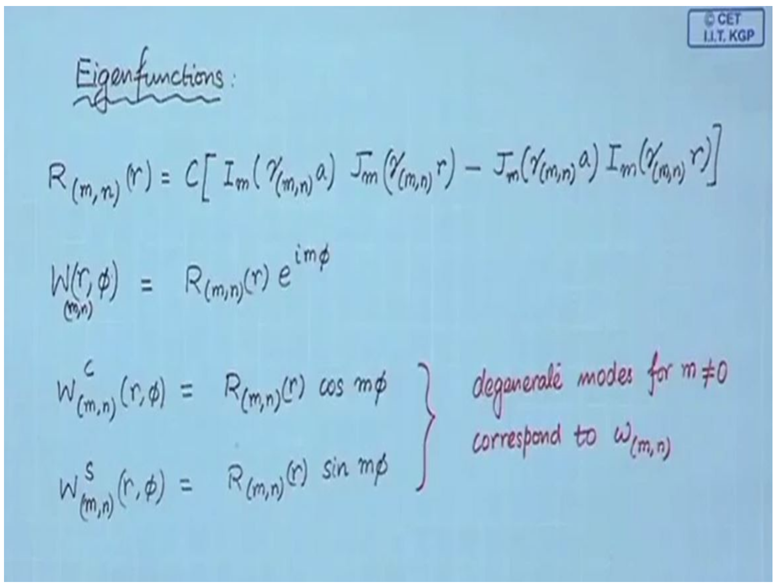

Now let us look at the Eigen functions so the radial, first the radial Eigen function this was initially we had an index 'm' now we have two indices m and n, we can solve so essentially what we have to do is we have to solve for C1, C3 and from these two equations and then we can write, so we can put these coefficients as some constant times this times J m and this coefficient as minus of some constant times J m Gamma m, n in to a and we have this as I m.

So this is the radial Eigen function. Now our original Eigen function so which also actually now should be indexed was written like this. Now here we have the Cosine and the Sin now both of them, so we can take either the Cosine part or the Sin part or any linear combination of these two. So we can say that we can have just like for the membrane, circular membrane we have the Cosine mode and we have the Sin mode.

Now these two modes are of course orthogonal if you see, if you integrate the product of these two then that you can very easily see, so these are orthogonal and these are Eigen functions, these are orthogonal and they correspond to the same Eigen frequency Omega m, n. So these are degenerate modes for m not equal to 0. So both these modes they correspond to the circular natural frequency Omega m, n.

So we have, so m equal to 0 is the axisymmetric mode, so if m is equal to 0, then the Eigen function becomes only a function of R, there is no dependence on Phi, so there is a axisymmetric mode, when m is not equal to 0, then we have the unsymmetric modes.

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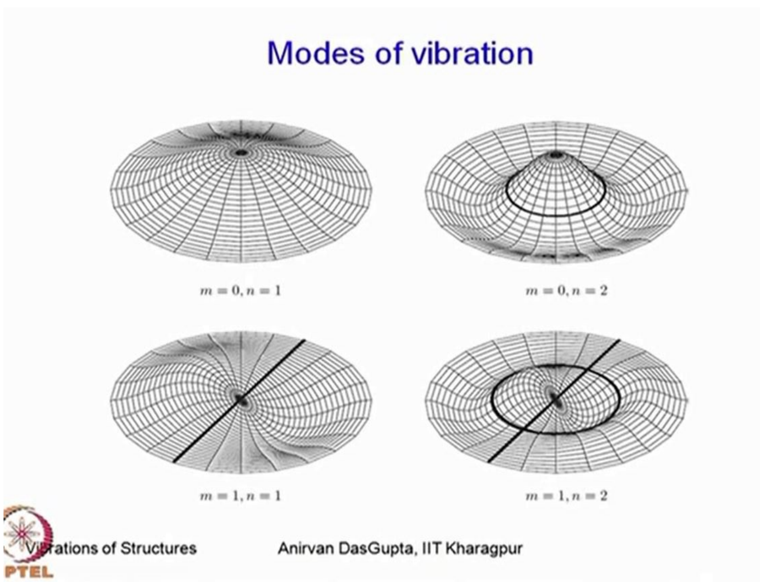

So let us now look at these modes of vibration of a Clamped circular plate, so this is the first mode, so m equals 0 in this case, so this is axisymmetric and similarly this is axisymmetric, this is the fundamental the first mode, so this has no nodal curves, nodal diameter or nodal circle whereas this is the mode with 1 nodal circle, now this is 0, 2 but this is not the second mode as such the second mode is so you can see here that 0, 2 so this is 0, 1, this is 0, 2 this is the 4th mode.

Now we also have 1, 1 and 1, 2, so these are the second and the 5th mode. So in this figure once again you can see that 1, 1 so this is a mode with one nodal diameter and 1, 2 has one nodal diameter and one nodal circle, so these two are unsymmetric modes, so they have modal degeneracy. This is because you can have one Cosine mode and Sin mode, which will actually rotate this nodal diameter by Phi / 2.

So if this is the Cosine then the Sin 1 is with the nodal diameter orthogonal to this, so once again we can understand this from the geometric symmetry or isotropy of this circular plate since there is no particular choice of any reference line from where Phi can be measured, so this reference line is actually arbitrary for a perfectly circular plate. So you have this degeneracy. So you have one Cosine mode and the other Sin mode.

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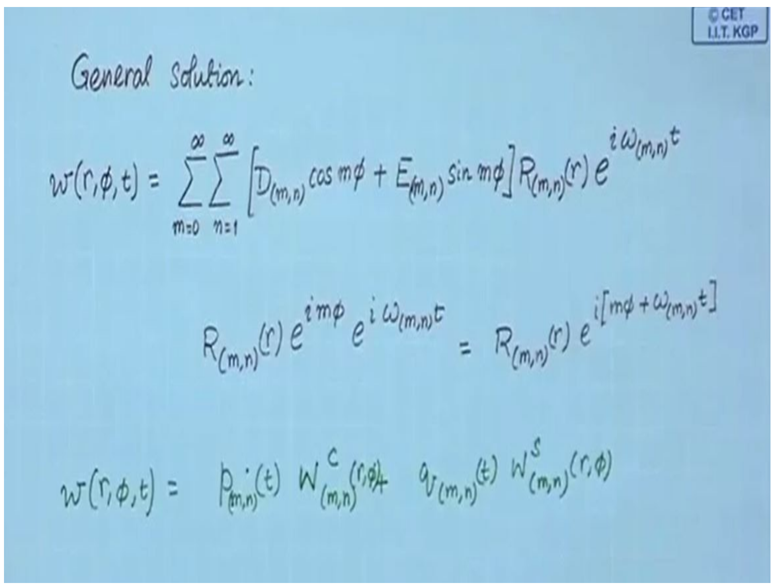

Now you can write down, using this Eigen functions you can write down the general solution, so here we can have combinations of again Cosine Omega m, n, t or Sin Omega m, n, t. So in general, we can have linear combinations of the Cosine and the Sin Omega n, t terms. Now so you see we had this solution.

We have this complex form a solution so this can be written as - so because of this modal degeneracy, you can have travelling waves in the circumferential direction as this solution tells us, also you can have depending that depends on the initial conditions you can have completely separable solutions like this. Now this occurs because of this again modal degeneracy so if you look at the solution the Cosine solution and the Sin solution they can come in arbitrary combinations.

So, for example I can write, so you can write the solution like this, so this is of course, so this is the Cosine mode and this is the Sin mode. Now just as in the case of the circular membrane or membranes with modal degeneracy we have discussed that the degenerate mode since they had the same frequency the nodal structure, the nodal curves in this case, the nodal lines or the nodal so nodal diameters and the nodal circles they need not remains steady.

If for example, for the mode with one nodal diameter, so if you have the function p as I wrote here suppose Cos and the function q as Sin, then you can see that this nodal diameter will actually be rotating. So there is no steady kind of solution will be observed as we expect for in the case of a modal solution so it might look unsteady. This is because of this modal degeneracy that is present in the case of the circular plate.

So finally to summarize we have discussed today the vibrations of circular plates we have looked at the some of the boundary conditions, now for the circular plate they happened to be more complicated than the case of rectangular plate. So we have looked at some of these boundary conditions and we have looked at the modal analysis of Clamped circular plate, so with that I conclude this lecture.