Vibrations of Structures
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Lecture - 32
Vibrations of Rectangular Plates

Today, we are going to discuss the vibrations of rectangular plates. So in the last lecture, we had seen the mathematical modelling of plate vibrations under small transfers displacements. So today we are going to look at the vibrations of rectangular plates.

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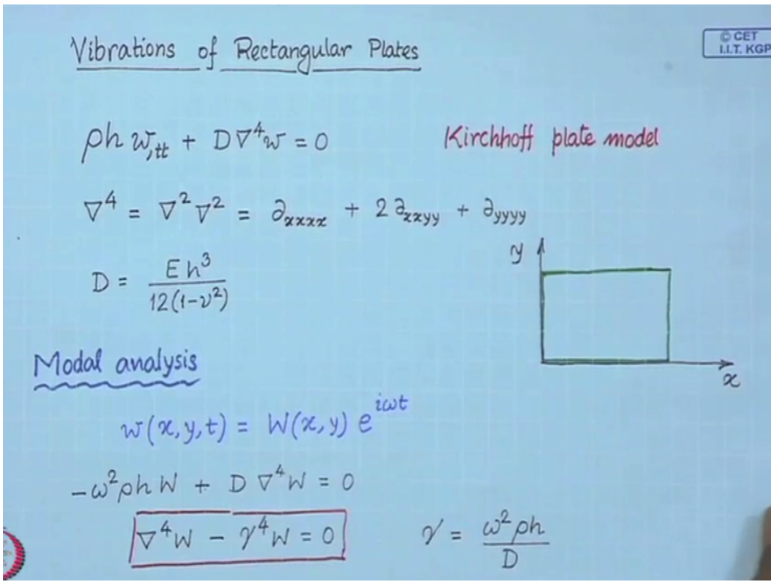

So let us recall that the equation of motion for a Kirchhoff plate. So this is the Kirchhoff plate model where the nabla 4 is square of the Laplacian that turn out to be this form and D is in terms of the Young's modulus E, thickness cube, thickness of the plate h cube divided by, the nu is the personal ratio. So here of course rho is the density of the material of the plate. So this is a Kirchhoff plate model for plate with constant thickness.

Now along with this we will have of course the boundary conditions, which we will look at as we proceed. So we are interested in the modal analysis, so we will be looking for solutions. So we are looking for solutions for rectangular plates, so this is in the Cartesian coordinate. So we look for separable solutions, space time separable in this form. So suppose we have a plate lying in the xy plane and the displacement.

The field variable w is measured in the transfers to this plane, which means perpendicular to the plane of the paper. So if you substitute this solution form in the equation of motion, so we can write this as, and we will make redefinition of, so we will rewrite this as, where we have defined this gamma as omega square rho times the thickness divided by this constant D. So this is our differential equation of the Eigenvalue problem.

So the complete Eigenvalue problem description will also have the boundary conditions along with this differential equation. So let us look at this differential equation first.

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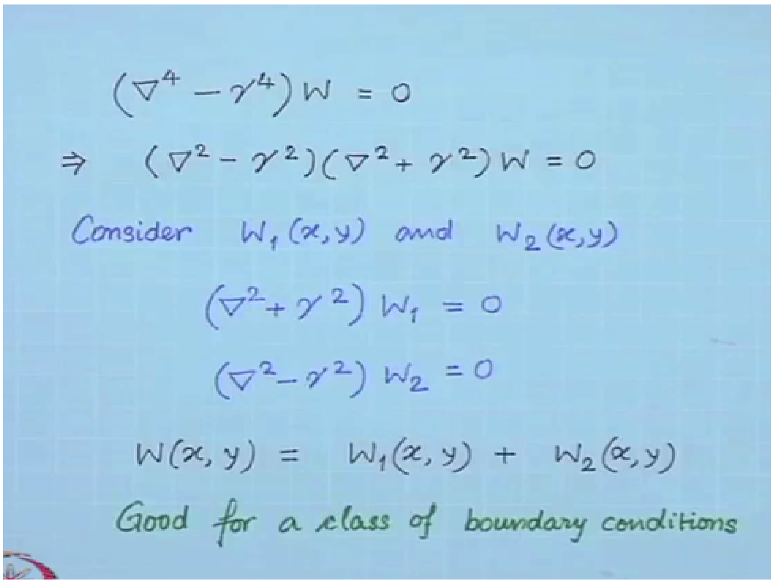

So we have, I can write this as, this operator in the equation of motion. So I can write this as nabla power four minus gamma power four operating on W and that is zero. And I can factorize this, in this form. Now these two operators they commute, so this can operate first or this can operate first, that does not matter in which order. So then, if I consider two functions such that this operator operating on W1 is zero, and W2 is such that.

So these two functions satisfy these differential equations. Then I can say that the solution W can be written as a combination of W1 and W2. So this can be very easily checked that you can, if you construct a solution like this, then this is going to satisfy our original differential equation of the Eigenvalue problem. Now but this, it so happens this is as we will discover very soon that this is not the most general solution structure that is possible.

So this solution is valid or good for a class of problems. So class of problems, by class of problems I would, I mean the class of boundary conditions. So this structure can be used to satisfy a class of boundary conditions. Now, then let us look at, then let us search for solutions of this class, then we have to look one by one at these two differential equations. So let us first take this differential equation which.

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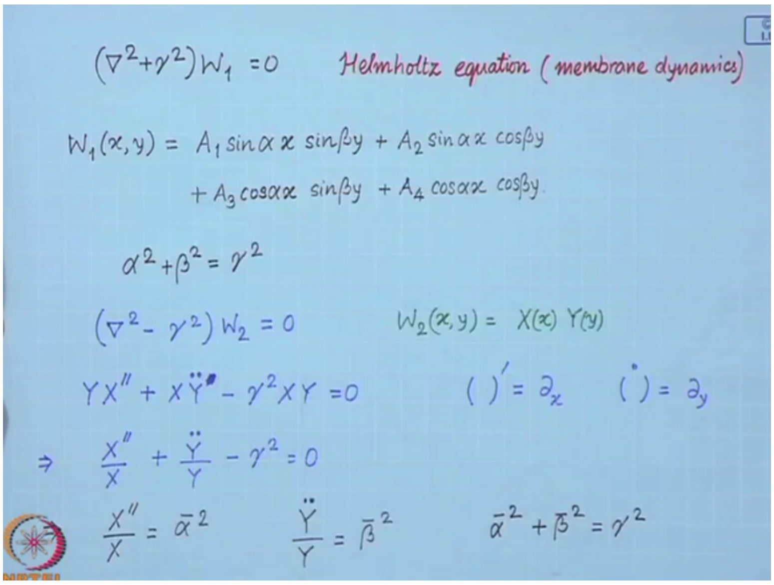

So this differential equation, now this differential equation is also known as the Helmholtz equation, which we have come across when we discuss dynamics of membranes. So when we studied the Eigenvalue problem for membranes, then we have encountered this differential equation. And the solution, you can recall is in this form. So the general solution of this differential equation is, so that is the general solution of the Helmholtz equation.

Now where this alpha and beta, they satisfy the condition that alpha square plus beta square must be equal to this gamma square. So this we have already discussed, now let us look at the other differential equation. So this nabla square, the Laplacian minus gamma square, so operating on W2 is zero. So then let us look for solutions with the structure, which are separable in x and y.

So if you substitute this solution form here, then we can write, so let be indicate this by dot. So what I have used is, this is a del x, ddx and dot indicates ddy. So if I divide this equation throughout by xy, then, so I have this structure of the equation. Now you see that this term is solely a function of X, this term is only a function of Y, and this is a constant. So orbitary xy if this equation has to be satisfied, then each of them must be constants.

So that would imply, let me indicate this constant by alpha bar square. So and Y, this constant by beta bar square. So in that case what I have is alpha bar square plus beta bar square must be gamma square. So now let us look at these differential equations for capital X and capital Y.

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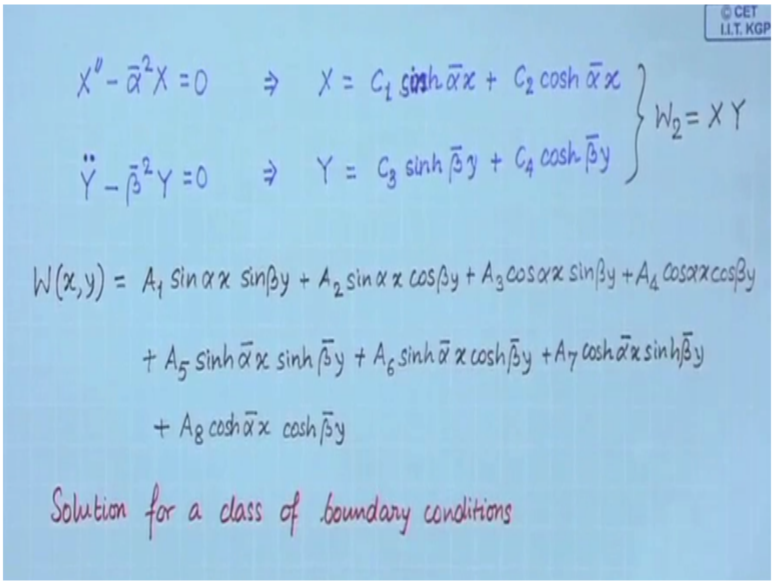

So these differential equations they read double derivative of capital X minus alpha bar square X must be zero and we know that the solution of this differential equation may be written as, so let me write this as the first term, as the sin hyperbolic. Similarly, from the second equation, so the solution of this can be written as C3 sin hyperbolic beta bar y plus C4 cos hyperbolic beta bar y.

And of course this with the condition that alpha bar square plus beta bar square is equal to gamma bar square, gamma square. Now then, let me write down the solution so far, so our W is W1 plus W2 therefore we have all these terms. Now you see the actual solution of W2, so W2 is X multiplied by Y. So now this product I can write therefore product f say C1, C3, if I write as A5, then this is A5 sin hyperbolic alpha bar into sin hyperbolic beta bar y.

Now you can see here that we have product of the trigonometric functions in xy direction and product of hyperbolic functions separately again in xy directions. In this class of solutions, there is no product of trigonometric and the hyperbolic functions. So this we can, I mean we can intrudively have an idea that this is not general enough. So, but this solution this is nevertheless a solution, which can solve class of problems.

So this, you can use this solution for a class of boundary conditions.

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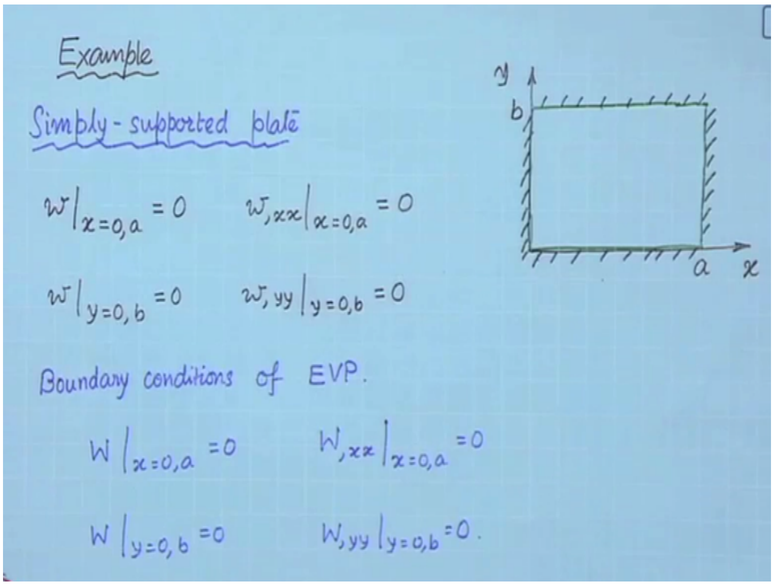

So let us look at certain example, so the first example that we are going to take is that of a simply supported. So let us consider a plate, which is simply supported on all four boundaries. So let us consider this as a and this as b, so the boundary conditions for this plate we have seen, discussed in the last lecture. So the boundary conditions, so at x equal to zero which means this edge and x equal to a means this edge.

We have the displacement zero and since they are simply supported the movements are also zero and the movements in this case happen to be the double derivative of w with respect to x. Similarly, at these two edges the displacement at y equal to zero and y is equal to b, so these two edges, the displacement again equal to zero and the bending movements are also zero. So these are the boundary conditions for the simply supported plate.

Now the corresponding boundary conditions of the Eigenvalue problem, so they can be easily determined from here, so W, so these are the corresponding boundary conditions for the Eigenvalue problem. Now let us look at the solution and these boundary conditions. So this was our general solution and these are our boundary conditions. So suppose when x equals zero so these terms will vanish, so we are left with these four terms.

Similarly, when you take double derivative with respect to x and look at x equal to zero, so again you will find that those terms will vanish. Similarly, when you consider y equal to zero, the displacement and the curvature so double derivate with respective to y, so if you look at all these conditions that you obtain, then finally using these conditions you will come to the conclusion that this the solution will boil down to.

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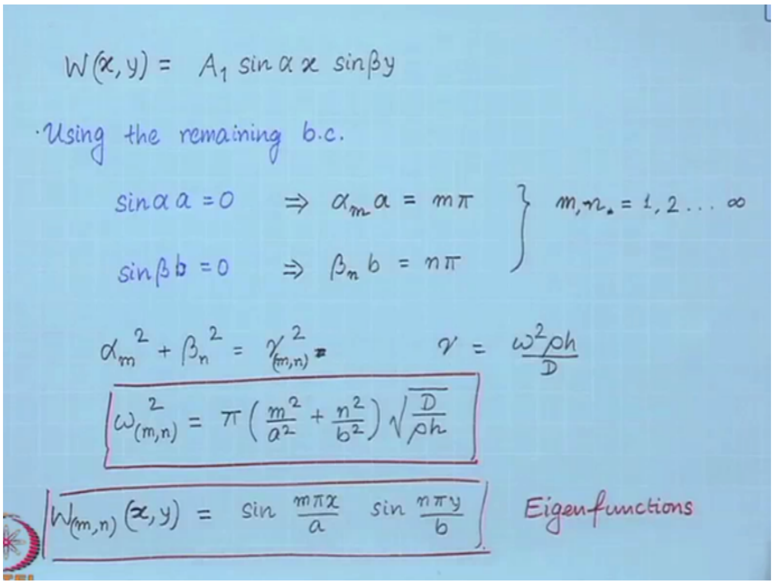

Now here we have used the conditions for x equal to zero, the displacement and the double derivative of the displacement and again at y equal to zero the displacement and the double derivative of the displacement. So we have some further boundary conditions, which we must satisfy with this equation. So then if you use those condition, then the remaining boundary conditions.

So if you use the remaining boundary conditions, then you will obtain these conditions. So for example when you use W at x equal to a, equals zero, so sin of alpha a for all y must be zero and when you take the double derivative with respect to x again you will get sin of alpha a equal to zero. So then you have this condition, and you have another condition similarly sin beta b must be equal to zero.

So this is for W at y equal to b and W dot at y equal to b vanishing. So that will give us this condition, so the first condition implies that alpha, now this gets indexed, because there are countably infinitely many solutions of this equations. So alpha m times a equals to m Pi and from here, another index, so m and n can have values from one to infinity.

So if you recall the definition of the condition or the constraint on this alpha and beta, that must be equals to gamma square and, if you look back, then this gamma square. So you have this gamma was defined as, so this gamma had this omega square. Now this gamma also gets indexed because of this m and n. So then omega also indexed as m and n. So omega mn, if I now use these expressions of alpha m and beta n, so this is m Pi over a and, so that is what we are going to obtain as, so this is omega mn square is given by this.

Now so we obtain the circular natural frequencies of the plate and using and the Eigen functions are obtained from here, they also get indexed, so these are the Eigen functions.

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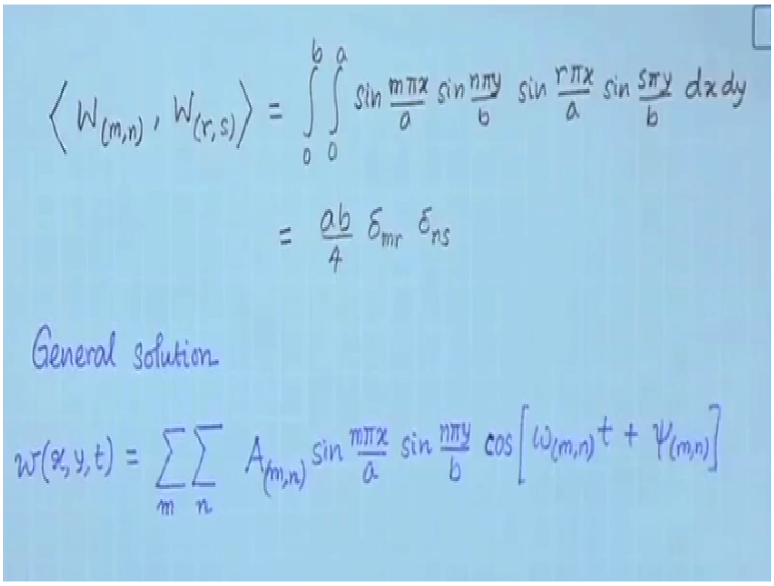

Now you can quickly see that these Eigen functions satisfy the orthogonality conditions. So that is ab over four, these are the chronical delta functions. So they take the value when the two indices are equal. So these Eigen functions they are orthogonal and the general solution, we can write down the general solution of the plate using these Eigen functions, we have converted this to an amplitude and phase form the temporal function.

So we can have this general function, now let us look once again at these Eigen functions. So these are, we have obtained these Eigen functions even for membranes and the modes look very, the modes of vibration are the same. So for the simply supported plate and that of the membrane. Now let us look at another example of the plate, with mixed kind of support, sp on two edges we consider simply supported edges.

And the other two opposite edges we will consider as clamped. So let us consider this membrane, this plate.

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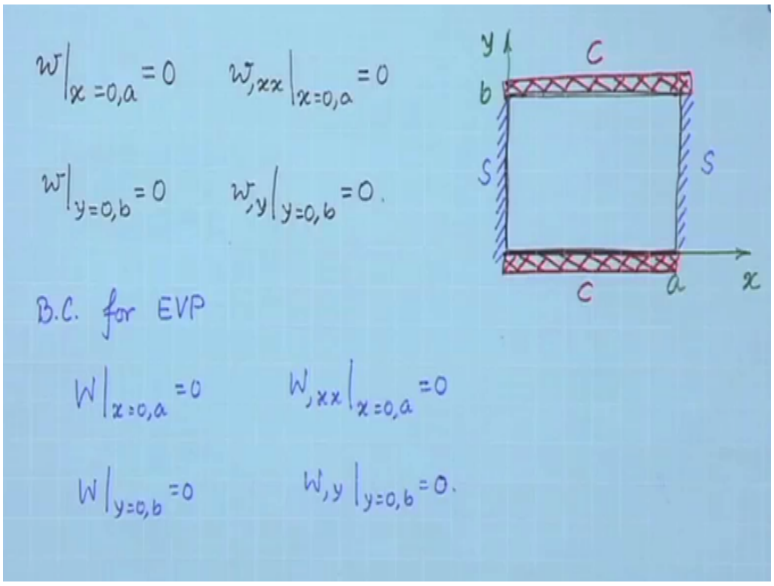

So in the x direction the length is a and here it is b again. Now we will assume that this side simply supported and these two sides are clamped, so these two are clamped, and these two edges are simply supported. So the boundary conditions for such a plate with such a boundary condition. So the mathematical representations are the displacement at x equal to zero and x equal to a they must be zero.

The moment, these also must be zero at the simply supported edges, for the clamped edges we have the displacement at y equal to zero and b as zero and we have the slopes at these two edges as zero. So corresponding to these boundary conditions, the boundary condition for the Eigenvalue problem. Now if you look at these boundary conditions, and also look at, so these are our boundary conditions now.

And the kind of solution that we had here so, if you now use these boundary conditions for this solution, then you will find that this solution can not satisfy this set of boundary conditions, that can be checked. You have here at y equal to zero and once with single derivative of y. So because of this structure being a little special, these boundary conditions cannot be satisfied by this class of solutions now.

So we have to start a fresh for this set of boundary conditions, we have to look at the Eigenvalue problem a fresh.

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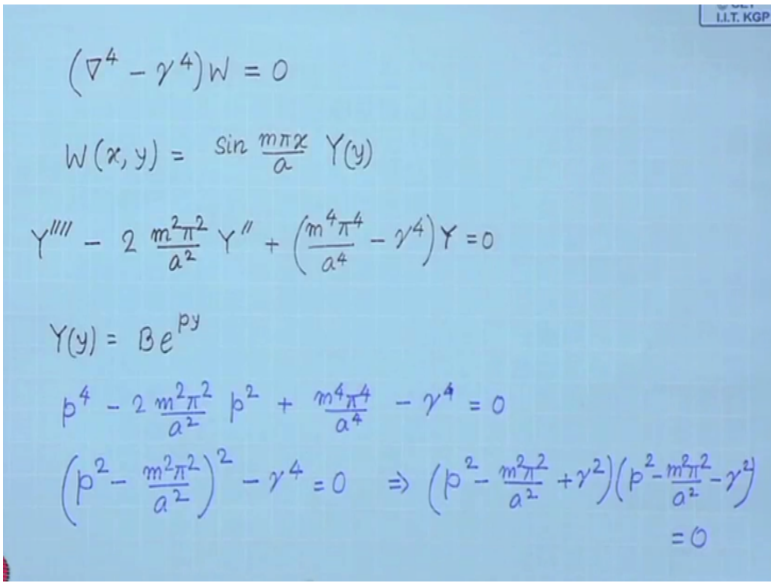

So let us look at, so this was the differential equation of the Eigenvalue problem. Now here we have simply supported edges at x equal to zero and x equal to a. So let us try a solution which already satisfies these two boundary conditions at x equal to zero and a, the displacement and the movements being zero. And we know this that sin m Pi x over a is a functions which satisfies these four boundary conditions, two on each edge.

For the y coordinate, let us have this function unknown function as yet unknown, capital Y, which is the function of the y coordinate. Let us try the solution in the differential equation of the Eigenvalue problem. So if you substitute this in here and make some simplifications, then so this is what you are going to get. Now we can try a solution for this, let us say so if you try a solution like this, then p will have as you can see that this differential equation.

So this will reduce to. This can be decomposed as, that implies that must be zero. So we have two solutions of p.

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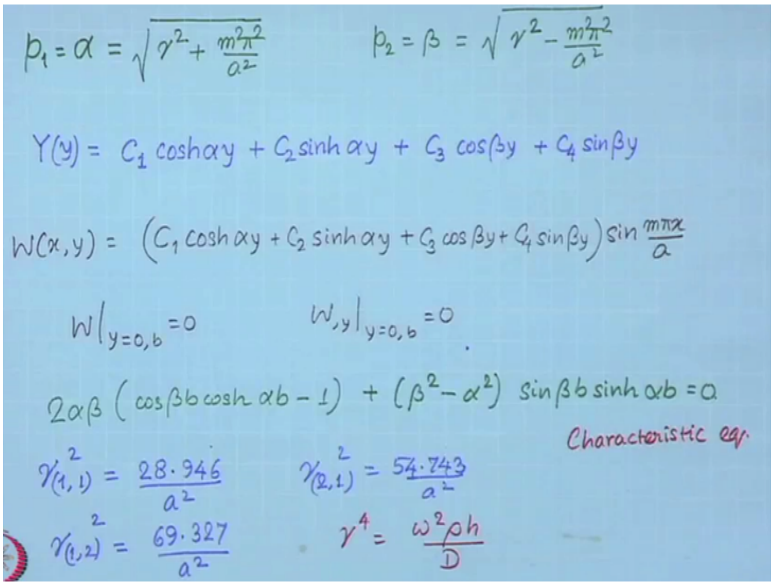

Let me name this solutions of p as let say alpha, so p1 is alpha equals, so let me consider this gamma square plus and p2 I call that beta. So we have these two solutions, so correspondingly we can write y as, so if you define this beta in this form and alpha in this form, then I can write the solution, so because of this definition of beta which is, you have both real and imaginary solutions for p.

And by defining p2 in this form I can write this in terms of trigonometric functions. So then my solution stands as, now we have satisfied four boundary conditions by choosing the function in x, we are left with these four boundary conditions as yet. Now if you substitute in this solution from in these boundary conditions, finally you will get the characteristic equation, so for non-trivial solutions of C1, C2, C3, C4, so this is our characteristic equation.

So if you solve this equation numerically, and you already have the relation between gamma and the frequencies so you can find out the natural frequency. Now this, I have written out the first three modes and if you look at this definition, so this is also square. So gamma power four is, so from here you can determine the natural frequencies.

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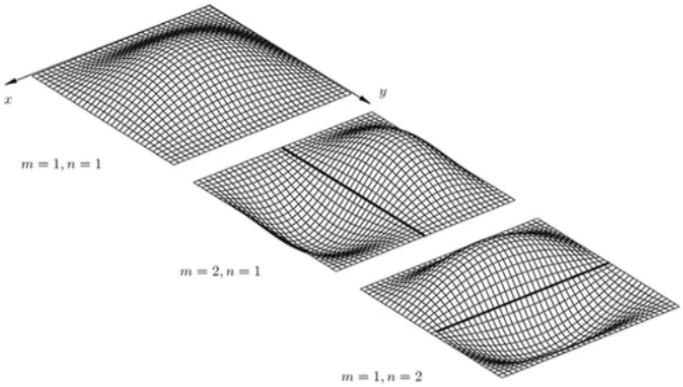
Modes of vibrations



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Now this figure shows the first three modes of vibration of this plate. So in the first mode there are no nodal lines where as the second and third they have these nodal lines. Now here you can see these two are the clamped edges, so the slopes are zero where as these are simply supported edges.

So to recapitulate, we have today discussed the vibrations of rectangular plates and we have seen that the solution is, I mean determining the solution is little complex, we have looked at two kinds of boundary conditions, or two classes of boundary conditions and we have solved these problems. And determined the Eigen frequencies and the modes of vibration. So with that I conclude this lecture.