Vibrations of Structures
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Lecture - 31
Dynamics of Plates

We are going to discuss the dynamics of plates. So we will be in the next few lectures we are going to actually discuss the vibrations of plates. So today we are going to initiate some discussion on the modelling of plates. Now what is a plate? So we have seen that a membrane is a two dimensional continuum which does not transmit any bending moment. Now when you think of a plate it does transmit bending moment or it does resist bending.

So which means a plate is a two dimensional elastic continuum which resists or transmits bending moment. So first where do we find plates. So plates are found in various machines in civil structures etcetera. So we are interested in first the dynamic model. We are interested in setting up the equations of motion or modelling the dynamics of plates. So as it happens with any dynamic modelling we make some simplifying assumptions.

So that we can have two dimensional theories for plates so for continuum two dimensions. So what assumptions do we make to simplify our models.

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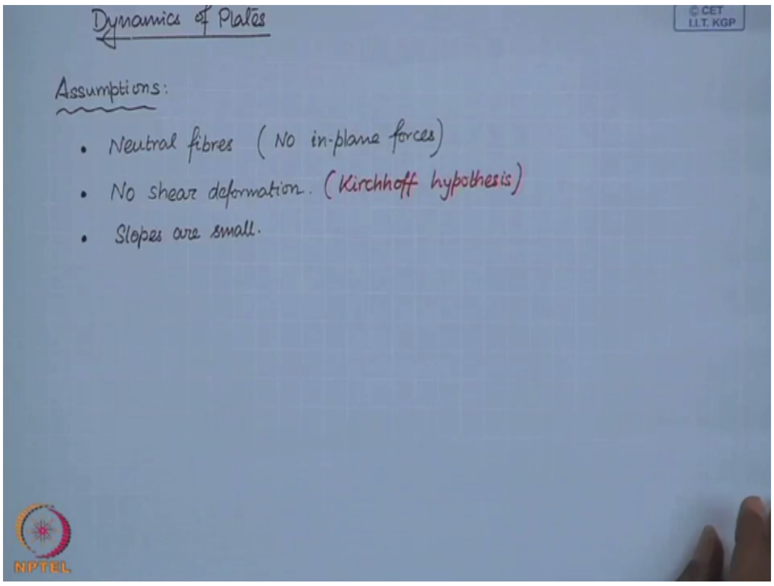

So the first thing that we assume that there is a plane or there are fibers which are unstressed these are called neutral fibers. So we assume the presence of neutral fibers which occurs if you do not have in-plane forces in the plate. So the plate is not subjected to any in-plane forces then you have when the plate undergoes transverse small transverse vibrations then there are fibers which remain unstrained.

The second assumption we make is that there is no shear deformation. So this is known as Kirchoff hypothesis. So this corresponds to the Euler-Bernoulli hypothesis. The third assumption we make to keep our model linear is that the slopes are small. So when the plate deflects the slope because of this transverse deflection they are small. So we are going to model under these assumptions.

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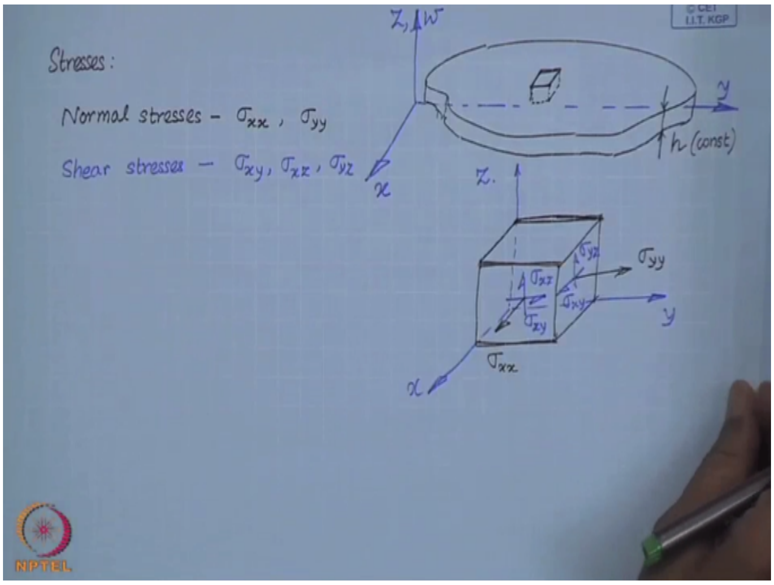

So let us first look at so let us consider a plate and we look at a little element in this plate. So what are the stresses on this element? We will make further assumption that the plate thickness is constant. So this plate is line in the XY plane and its deflection is transverse to this plane. Now we consider that the stresses that are acting are the normal stresses so we have stresses sigma XX sigma YY this is a normal stresses and we have shear stresses.

So sigma XY sigma XZ and sigma YZ. So I am showing the stresses only on these two surfaces. Now there can be a distributed force on the plate, but at present we are going to drop that. So essentially we have only these stresses which are non 0. Now when we want to construct a theory in two dimensions then we integrate over the thickness of the plate. We integrate over the thickness of the plate.

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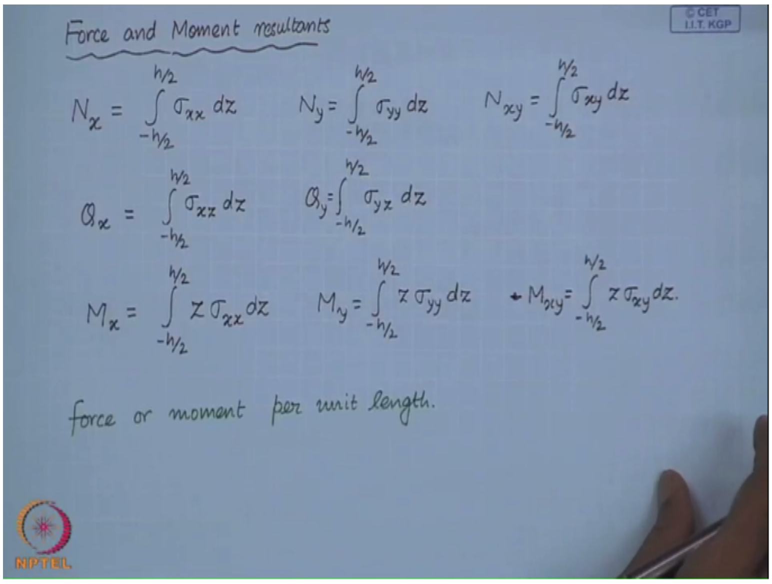

And we define what are known as the stress resultant the force and moment resultants. They are also known as the stress resultant. So the stress resultant due to the normal stresses. So these are because of normal stresses and this is because of the shear stress so this is in-plane and then we have out of plane shear stresses which we denote by QX and QY and we have the moment resultants because of the moments due to the normal and the shear stresses so these are the in-plane.

So we have defined here as MX as Z time sigma XX, but actually this moment is along the Y axis, but still we call it MX because it is the moment with sigma XX. Now these resultants they have the units of force per unit length or moments per unit length.

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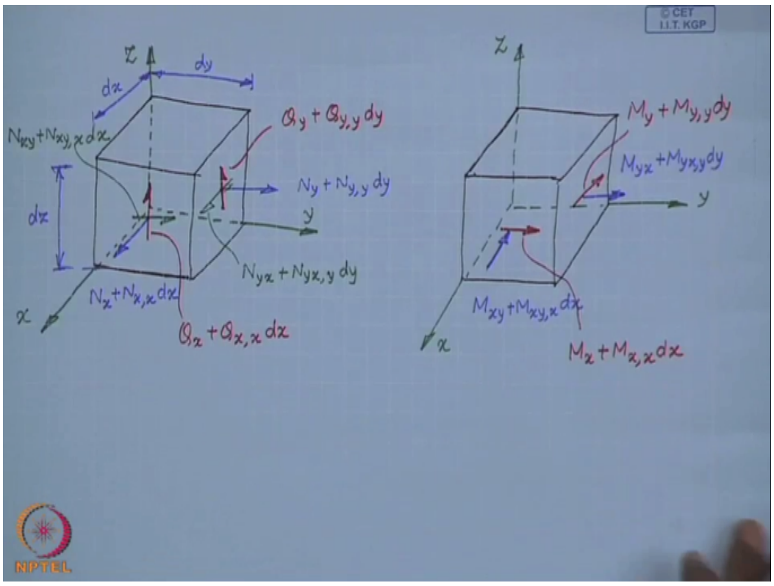

Now let us look at these force and moment resultants on the infinitesimal element. So this is an infinitesimal element of the plate so I will mark out the resultants. So the first is NX so I will show the so this phase at X = 0 there you have NX in along the negative X direction and on this phase you have NX plus NX derivative with respect to X DX so this length is DX. Similarly, this is DY. So this is the normal stress resultant on this phase.

Similarly, you have on this phase. Now the resultant because of the shear stress on this phase is up on the other opposite phase it will be down that will be QX and this is QX plus Del QX Del X DX. Similarly, here up this is QY plus and the in-plane shear stress resultant which is NXY. So this of course is similarly this will be now on the other two phases you can imagine that they will be without this additional part and they will be opposite in direction.

So these are force resultants now let us look at the moment resultants now. So here we have now this moment in this direction on the normal to this phase. So this is the moment because of the normal stress sigma XX so that is going to follow this right hand rule so this is the moment on this phase and the phase X = 0 you have just MX in the opposite direction and on this phase it is because of sigma YY so you have these as the moment resultants.

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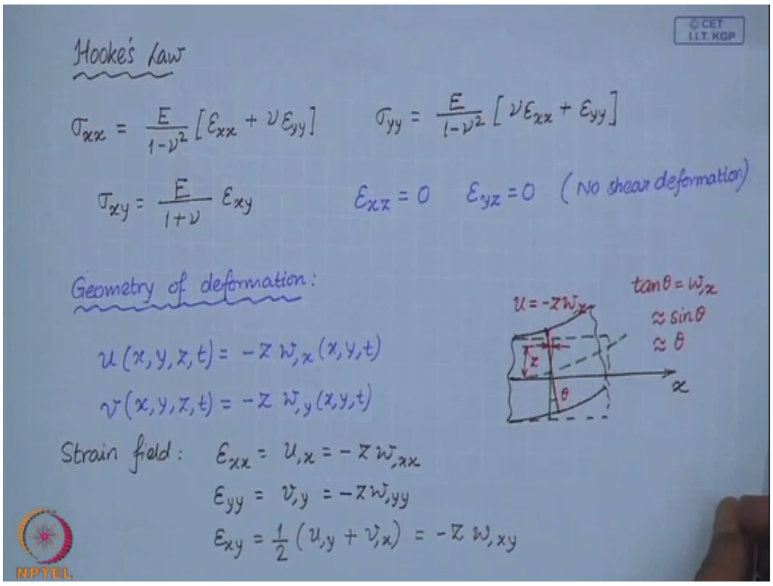

Now we will come back to this figure again now let us write down the constitutive relation for the material so these stresses are related to the strains. Now since we have considered that the plate is infinitely stiff in shear there is no shear deformation of the element along the Z. So therefore the corresponding stresses actually cannot be calculated from any material constitutive relations we have to determine them from the equations of motion.

Now let us look at the geometry of deformation so that we can calculate the strain in terms of deflection of the plates. So let us consider the plate initially undeformed represented by this dashed curve that along the X axis and this deflects in this manner. So if this is a line which is initially along the Z so it is vertical then this line deflects to this configuration. So the displacement in the direction of X can be written approximately if you call that displacement as U is minus Z times Del W Del X.

As you know Del W, Del X is stand of this angle so tan and for small theta this is also equal to and that is equal to theta. So if Z is the location of this point from the neutral line, neutral surface then the deflection in the direction of axis given by minus Z Del W, Del X. Similarly, for the Y in the Y direction we can write now using this we can calculate the strain field so that is a strain field. Now if you have this strains so this are linear in Z.

Now this W is independent of Z. So strain is linear in Z so epsilon XX and epsilon YY are linear in Z so therefore sigma XX is also linear in Z as you can see from these expressions so that will be linear in Z. So if you back to this calculation of the resultants if this is linear in Z then Z integrated from minus H/22 plus H/2 is actually 0 so which means that these terms are going to vanish so these are NX and Y and NXY they are of 0 so the non 0 resultants are Q and M.

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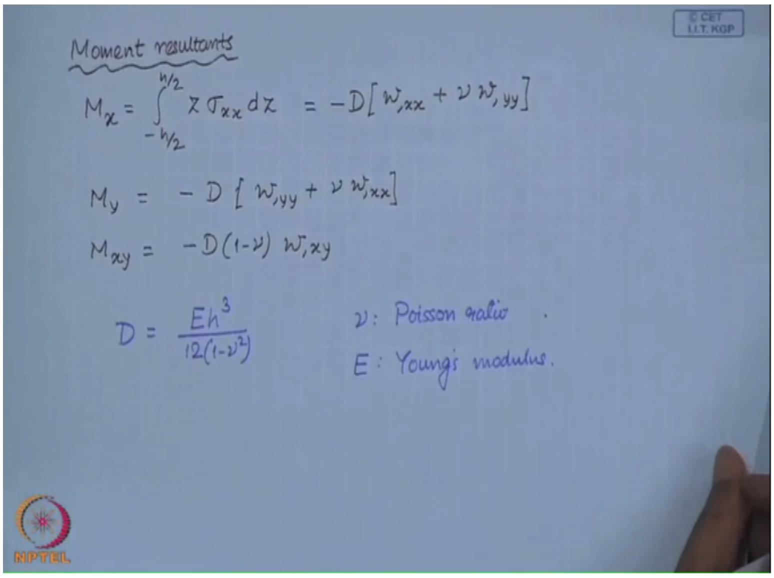

So let us calculate then M because Q will ultimately come from the equations of motion so the moment resultant theory called this is the expression. So if you substitute so use this expression of the stress where these strains are written from here and if you substitute in this expression and simplify then you can see that this leads to similarly so here D and of course Nu is the Poisson's ratio and E is the Young's modulus.

So these are the moment resultant now we can write down the equations of motion.

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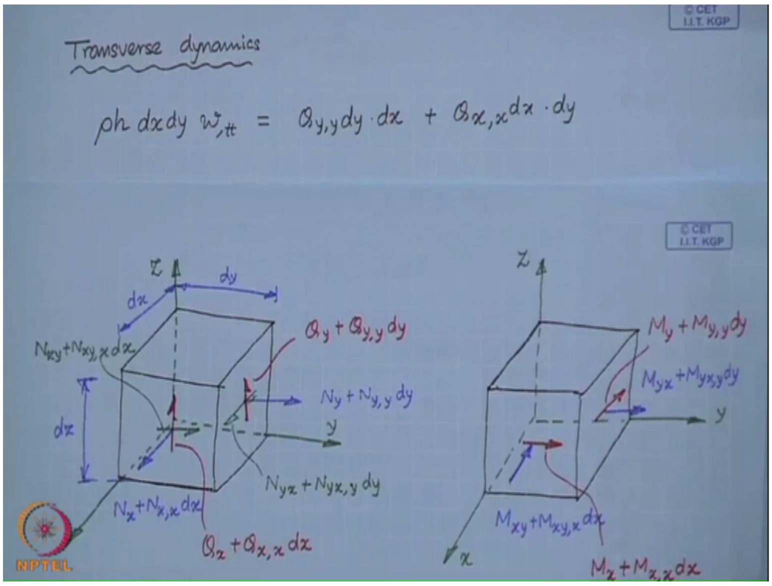

So we write down the transverse equation of motion in the transverse direction, So if rho is the density of the material and H is the thickness so this is mas per unit area times the area of the little element times the accelerations. So if you now look at this figure so we have in the transverse direction these forces Q. So here it is upward so here there will be a downward which will be minus of QY in the equation of motion and since remember these are all forces per unit length.

So this and from this the contribution is so these are the forces in the transverse direction.

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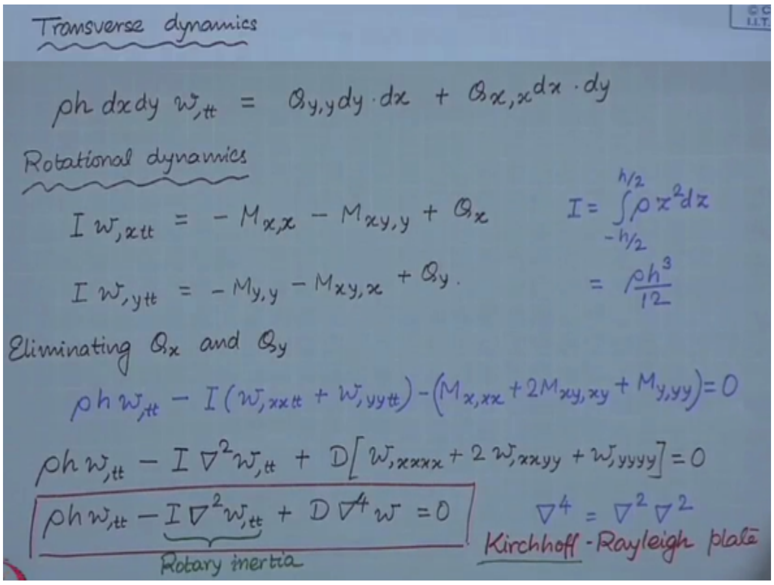

Now let us look at the rotational dynamics so for that we will refer to this figure. So I will directly write so Del W, Del X is the small angle so this is the rotation about the Y axis so this double derivative with respect to time will give us theta Y double dot. So rotation about the Y so angular acceleration about the Y axis. So this equals so we have these moments so it is a rotation about the Y axis we have for this moment then we have the moment this one because of YX.

So let me write this then and because of this for QX you have another moment. So this is the rotation about the Y similarly you can write down the rotational dynamics about the X axis and this is the I is the moment of inertia per unit area. So for this element we have I equal to row HQ over 12. So these two equations correspond to the rotational dynamics whereas this equation of course I will divide this whole thing by DX, DY so this corresponds to the traverse dynamics.

Now I am going to eliminate Q, QX and QY in the transverse dynamics using these expressions. So if I do that so this is what I obtain now here I will replace these moments using the expressions that you have derived the moment resultants in terms of the field variable. So if you do that then you can simplify the equation this can be written as so the Laplacian of Del square W Del T square plus D now this turns out to be so this is written as so we can write this in a compact form.

So this is the equation of motion of the plates. So here Nabla 4 is the square of Laplacian. Now we need to talk about the boundary conditions. So this model the way this is known as the Kirchoff-Rayleigh Plate Model if you this is the Rotary Inertia term. This is because of the bending and this is the inertia term. So if you drop rotary inertia term assuming that the moment of inertia is very small in that case if you drop this term then you have Kirchoff Plate Model.

So if you drop the rotary inertia term then you have Kirchoff Plate Model with the rotary inertia term is called the Kirchoff-Rayleigh Plate Model.

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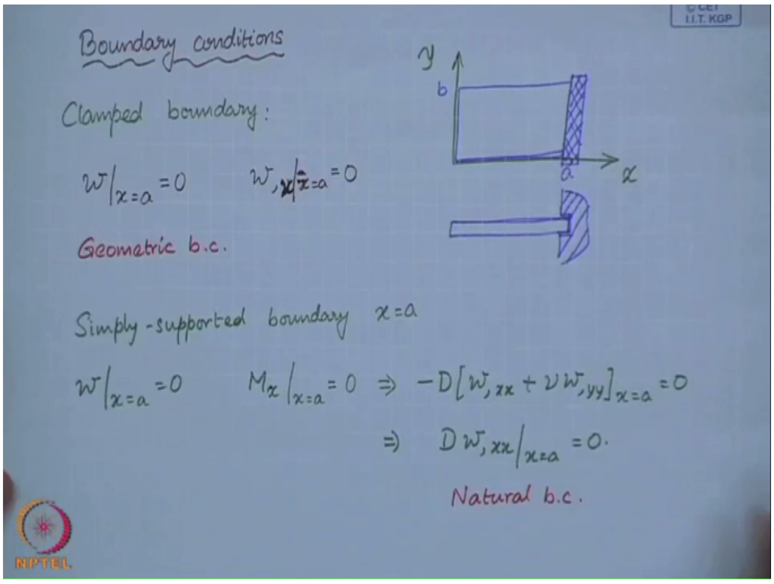

So now, we discuss about the boundary conditions so we can have various kinds of boundaries. So suppose let us consider a clamped so suppose you have a plate and you have a boundary here which is clamped boundary. So if the boundary is clamped so it is actually quite simple then the displacement must be 0 and of course the slope must be 0. So these are geometric boundary conditions then if you have simply supported edge at X equal A so we have the displacement 0 and the moment equal to 0.

So this implies if you use the expression of the moment, but if the edge is simply supported and is straight so that W is at X equal to A for all Y is 0 then there is no curvature so this term is also 0 so this will imply now this is a natural boundary condition. Now we come to this interesting case of a free boundary so we have discussed clamp boundary and simply supported boundary.

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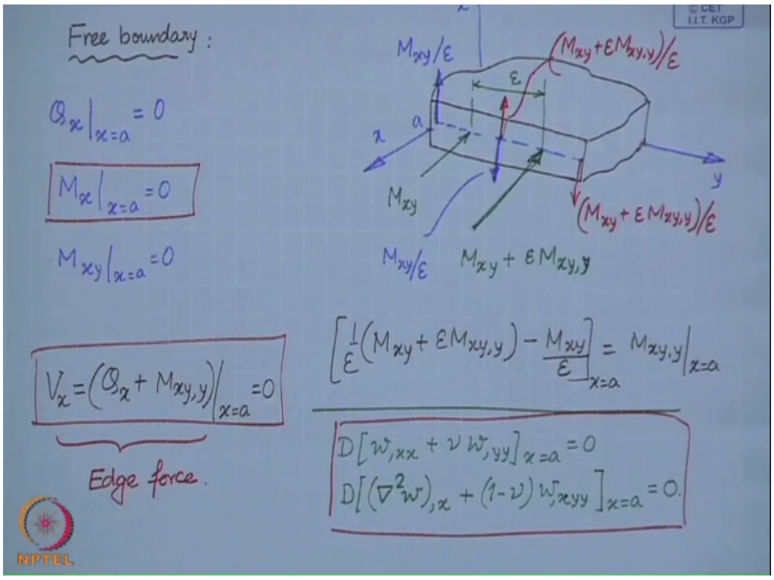

Let us discuss the free boundary. So if a plate has a free boundary then intuitively we may guess that so suppose this is X equal to A. So this boundary is at X equal to A. So intuitively we might guess that so this shear stress resultant so this QX is going to be 0 so QX is along z on this space so that should be 0. The moment = 0 and in addition we have another moment because of the shear stresses in-plane stresses that also has to be put to 0.

But you see this differential equation of the plate is 4th order. So it cannot support 3 boundaries conditions at an edge like this and a boundary cannot satisfy three boundary conditions. So there must be something wrong about this these boundary conditions at least some of them so there must be some combinations so now we are going to discuss how they are actually combined. So let us see this MXY.

So let me refer to this figure once again so this was MXY on this phase let say at X equal to A so this MXY. Now this is a moment because of the in-plane shear stress so this moment can change as you move in the Y direction. So at another location this can be M. So I am considering two locations separated by a small distance epsilon. So we are moving in the Y direction so this is the moment at the location a distance epsilon along Y.

Now this can be equivalently represented as a couple so this can be replaced by a couple and similarly this can be replaced by another couple. So therefore at this point you can imagine that the force resultant here which is now a transverse force is given by so all these things are calculated at X equal to A. So this is an additional edge resultant force which is in the transverse direction which is same as QX. So we can combine now this must be 0.

This is defined as the edge force this is known as the edge force. So this edge force must vanish. So it is a combination of the force because of the out of plane shear stress and the moment because of in-plane shear stress. So these two they combine to give us what is known as the edge force. So we have the boundary conditions as this and this if you write this down in terms of the field variable.

So and if you calculate this edge force so these are the boundary conditions for the free edge at X equal to A. So let us summarize we have looked at the equation of motion of small amplitude, small slope vibrations of flat plates. So plates are two dimensional elastic continuum which can transmit or resist bending moment. So we have looked at the equation of motion and the boundary conditions.

And we have for the standard boundary conditions which are the clamped and the simply supported the boundary conditions are quite simple whereas for the free boundary we have discuss about the edge force that must vanish. So the boundary conditions for the free boundary have to be carefully determined. So with that I conclude this lecture.