Vibrations of Structures
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Lecture - 30
Vibrations of Circular Membranes – II

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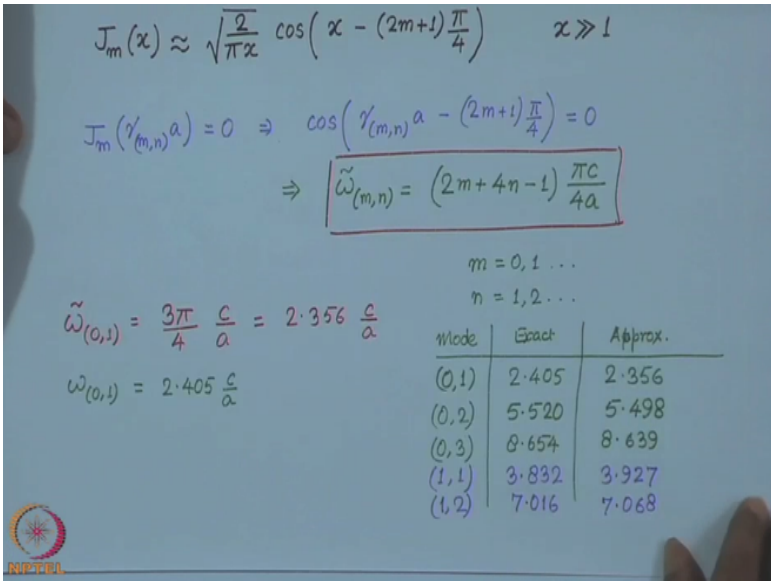

So, you can write this approximately as –when x is very large. So, for large arguments we have this approximation and let us see how good this approximation is?

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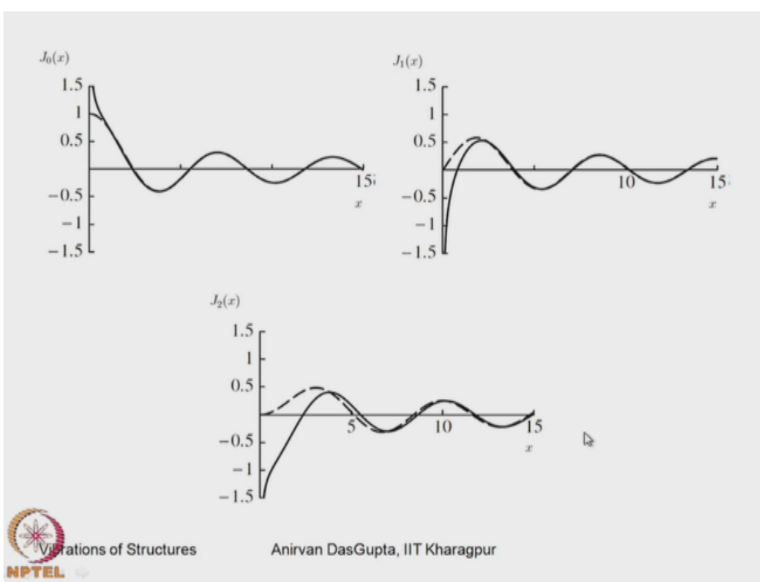

So, in this figure I have plotted out the Bessel function here of the first kind of order zero, order one and order 2. So, this dash line is actually the exact Bessel function and this solid line is approximation. So, you can see here for example the sold line and the dash are almost indistinguishable except for this. But we are interested in this approximation only to find out the roots of the Bessel function.

So, we expect for example for J0 the approximation to work very nicely even starting from the first root. Whereas for J1 there is slight error in the first two but then from the second and the third and the fourth etcetera these are very good. Now, for J2 this approximation is not so good for the first root. Well, there is some error in the second root, third root is closed and from the fourth root it is quite good.

So, you can expect that this approximation is going to give us the roots. Now, for large arguments we say so let us see what happens we want to have –so this must be zero. Now, this gives us, so if I write this as omega m, n over C and write out—so this is an approximation. Let, me put a tilde to denote that this is an approximation. So m equal to zero as I said is the axis symmetric mode.

So we can have for each of these modes we have infinitely many roots so we have circular eigen frequencies corresponding to these modes. Now, let us see how good is these approximations. So let me calculate. Let say omega 0, 1 with a tilde. So, this is m is zero. So, n is one so this gives me three –and this turns out to be –Now the exact value, exact means solution of the Bessel that obtained by solving the roots of the Bessel function.

This turns out to be 2.405 c over a. So you can see they are quite close so let me just –so 0, 1 this factor is 2.405. This factor is 2.356. If I take 0, 2 the exact solution is 5.52 and this one turns out to be 5.498. 0, 3 the exact is 8.654 and the approximation obtained from here is 8.639. So, you can see progressively you are approaching the exact solution. Similarly, if I have 1, 1 the exact solution 3.832 and approximation obtained from here is 3.9 27. 1.2 the exact is 7.016 and the approximation obtained from here is 7.068.

So like this you see that as you go to higher modes in each case you are approaching the exact solution. So that is for large value of argument. Now, if you go for m equal to let's say 2 then initial approximation will be in some error. But as you go to higher values of n therefor you will have better approximation. So that will tend to the exact one. So, this is the calculation for the eigen frequencies. Let us look at the eigen functions.

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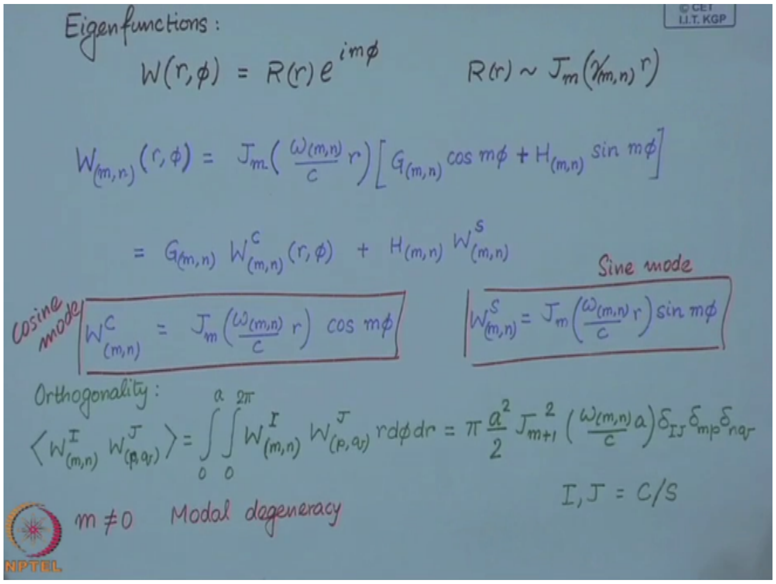

The solution that we have this R is proportional to this function therefor and this complex function is cosine m phi plus i times, Sine n phi. Now, as we have discussed before we can take the real part or we can take the imaginary part or we can take linear combination of these two parts. So, we can write the eigen functions as a linear combination of cosine and the sine. Now this I will write this as omega m, n over so c into r.

So that multiplied by –so here I can have linear combination where the indices also get indexed. So, it is a linear combination of the cosine m phi and the sine m phi and that multiplied by the Bessel function of the first kind in this form. So that is the eigen function m, n. Now, let us look at the –so here I can write this as –so I am introducing another notation so in the subscript C, over on W indicates this Bessel function multiplied by the cosine and similarly.

So, I am introducing this notion with the subscript c or s to indicate this cosine or the sine. So, these are also functions of r and phi and you know that here if you look at the orthogonality, which in this case is defined –so using the property of the Bessel functions you can write so here this capital I, capital J they can have this values C or S. So corresponding to whether it is cos or whether it is sin.

So, you can see immediately from these two even m, n, w, m, n, c and w, m, n, s they are orthogonal these are orthogonal modes and they are actually two distinct modes corresponding to a single circular eigen frequency omega m, n. So, this is called the cosine mode and this is known as the sine mode and so we have two orthogonal modes corresponding to a single eigen frequency omega m, n and this happens only when m is not equal to zero as you can very easily see.

If m is equal to zero then you have only this mode, you do not have this sine mode. So for m not equal to zero you have modal degeneracy, so which we have discussed in our previous lecture. So, in the case of circular membrane all unsymmetric modes. So this m not equal to zero I have symmetric modes. So all unsymmetric modes are modal degenerate. Now, let us see how the modes look like? So in this figure

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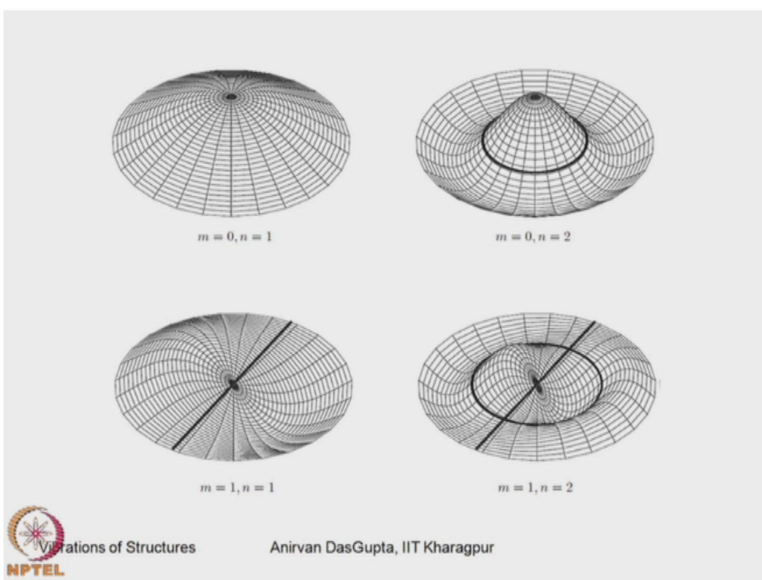

I have plotted out some of these modes. These two are the symmetric modes m equal to zero and this is the corresponding to the first root so n equal to 1 and this is n equal to 2. So, as n increases you have these nodal circles. So this is a nodal circle and these are axis symmetric modes as you can see. So, n equal to two has one nodal circle, n equal to three will two nodal circles. Now, as you increase m you generate a nodal line, a nodal diameter as you can see here.

So, n equal to one you do not have any nodal circle whereas n equal to two you have one nodal circle and for m equal to one you have this nodal diameter. Now, you see we have m not equal to zero which means say for example m equal to one so these two modes they are degenerate modes. So, corresponding to omega one, one there are two eigen functions. So how do these two eigen functions look like.

So, one is the cosine the other is the sine that is the only difference. So that sets the orientation of this nodal diameter then it is the cosine mode so when m equals one and is the cosine mode so it is zero at phi equal to pi by two then the sine mode will be at phi equal to zero. So which means the nodal diameter just gets rotated 90 degree. So that would be the sine mode. So why do we have degeneracy?

Now, it is very clear from this figure what we had discuss for the rectangular membrane, we had discussed about the isotropic in the modal space. But now we can see this isotropic also in the physical space. You see the reference from where phi is measured that is arbitatory. So you can put that reference line here then this is the cosine mode or if you take this as the reference line then this becomes a sine mode.

So, since there is arbitrariness there is isotropy in the rotational direction of the circular membrane. So, we have model degeneracy. So, here in this case we can look at this modal degeneracy or isotropy in the physical domain and this definitely is there in the model space. So, to understand this modal degeneracy we have understood this concept in our previous lecture in terms of isotropy of the modal space.

So there are two orthogonal eigen functions which are eigen functions of single eigen frequency. So for a single eigen frequency there are two eigen functions which are independent which are orthogonal. So this leads to the degeneracy so any combination of these eigen function is also an eigen function, any arbitrary combination is an eigen function. So, we call this as an isotropy in the modal space or the configuration space of the membrane.

Now, in the circular membrane this is also in the physical space. Now let us write down the general solution then.

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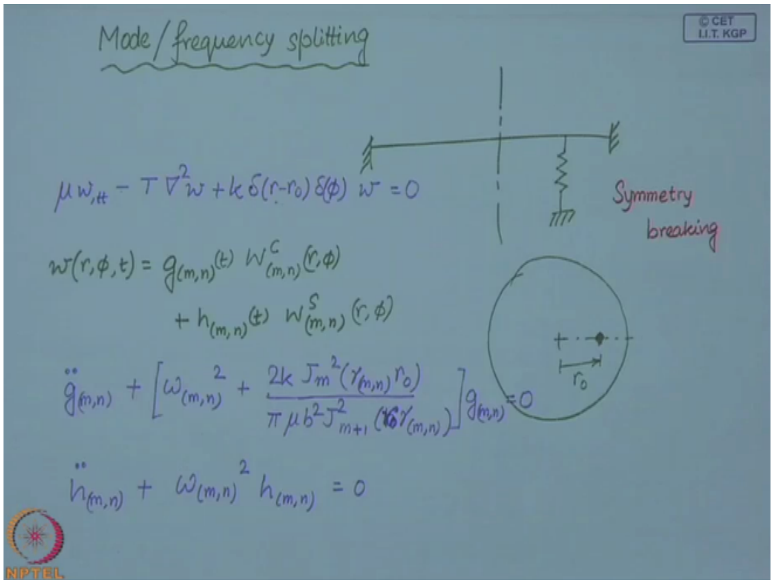

So the general solution of –for the circular membrane may be written like this. So here we note that m starts from zero which indicates the axisymmetric modes. And there are these constants. So this is the coefficient of the cosine mode plus we have also for the sine mode. So, that is a general solution here these coefficients the constants they are to be determined from the initial conditions.

So, from these initial conditions then we can determine these constants using the orthogonality property of the eigen functions. Now, let us look at an interesting property I means which comes because of this modal degeneracy which is called mode splitting or frequency splitting. So, if you consider a circular membrane and you put an external interaction in this case I have put a spring. So let us consider that we have this is a top view I have put the spring here.

So, at phi equal to zero so this is a reference line phi equal to zero and r equal to r not. So, in that case the equation of motion I can write the equation of motion. So here additionally I have this stiffness at r equal to r not and phi equal to zero. Now to understand what happens in this case let us just look at a single mode expansion. So this is the modal coordinate. This is the eigen function of the cosine one.

So, I have taken a single mode so actually corresponding to a single frequency so there are two degenerate modes. So, I have taken these two and if you substitute here and take the inner product and discretized the equation so that you get the dynamics of these modal coordinates then it looks like this. So you see that this corresponds to the sin mode and the sin mode has the node here so the sin mode remains unaffected because of the spring.

The sin mode remains unaffected whereas the cosine mode has to get effected and this is the additional term that comes with omega m, n square. So which means that the frequency of these two modes now get separated this is called the mode splitting or the frequency splitting. So this takes place because of an external interaction which now as you can understand breaks the symmetry.

So there is a symmetry breaking as you can understand and I told you that normal circular membrane has a geometric symmetry but that is now broken because of this external interaction and that immediately what it does is it splits this frequency. So there is no model degeneracy now for corresponding to omega m, n. So, in this situation this is going to split the natural frequencies of the membranes.

So to summarize what we have looked at we discussed the vibration of a circular membrane we looked at model degeneracy and we have looked at this interesting consequence that because there was degeneracy and when you break the isotropic or the symmetry of the system then you have mode splitting or the frequency splitting. So with that I conclude this lecture.