Vibrations of Structures
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Lecture - 29
Vibrations of Circular Membranes – I

So, we have been discussing about vibration of membranes in the present lecture. So, today we are going to look at the vibration of a circular membrane. So, we have in the previous lecture looked at the rectangular membrane. So, today we are going to look at a different geometry which is the circular geometry.

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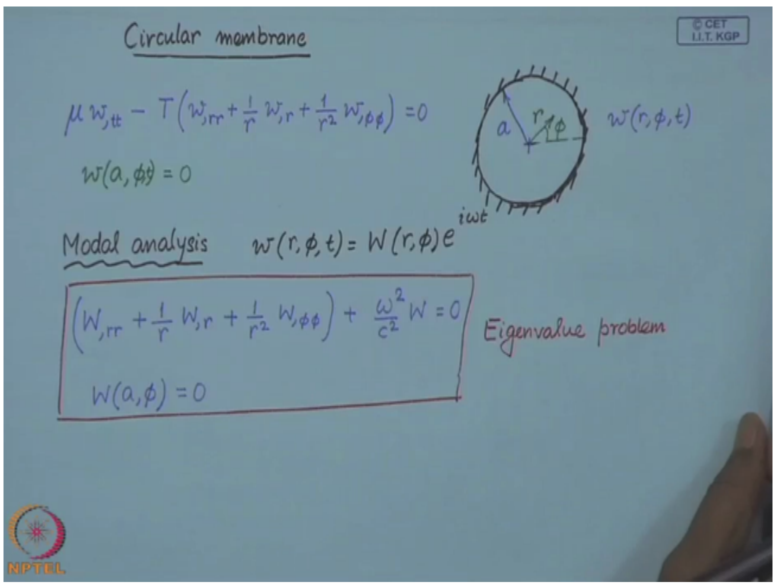

So, consider this as a circular membrane. We are going to consider membrane with fixed edge. So, the radius of the membranes let it be small a, any point on this membrane is denoted using the coordinates r and the angular coordinate pi. So, using this and our field variable which is the displacement of the membrane from its equilibrium position measured at a coordinate location r, phi, at time t. So, this is a transversed displacement of a point at r, phi at time t.

So, the equation of motion so if mu is the areal density and T is the force per unit length. So, in the fuller coordinates the Laplacian operator along with the boundary condition. So the displacement on the boundary is zero. So, we will be looking at the model solutions of the circular membrane. So we are interested in solutions of the form of this structure. So, if I substitute this solution in the equation of motion and in the boundary condition and I do a rearrangement by removing this exponential i omega t.

So, this along with the boundary condition so then this defines our eigen value problem. So, this is our eigen value problem which we must solve in order to determine the circular natural frequencies and the modes of vibration of a circular membrane. So, we search for separable solutions once again as we did for the rectangular membrane we look solutions

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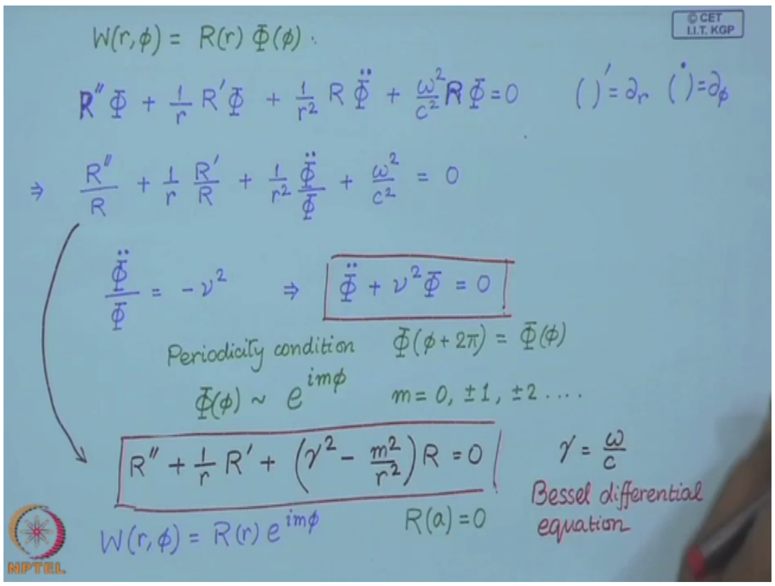

Which are separable in r and phi and this is motivated by the fact that these are independent coordinates. So, they can be separated out. So, we are searching for solutions with the structure. So, let us see what happens when we substitute this in here. So, let me indicate delta delta R with the prime and derivative with respect to phi with dots. So, here I have used delta delta R by prime and dot indicates delta delta phi.

Now, I do some rearrangements I divide this through by this product R into phi. So, then now you see that this capital R is only a function of the coordinate R while phi is only a function of the coordinate phi. So, if this which is a function of R and this which is purely a function of phi. So, and that must add up with a constant which is independent of R and phi and that has to be equal to zero for any R and any phi then it is natural that these are all constant.

So, which means I will write it as minus of nu square for reason that will be clear later since I want you see this is going to give me this equal to zero. So, this is purely a function of the angular coordinate phi. And we must have periodicity in this function otherwise at phi equal to zero and phi equal to two pi this function will not match. So, we must have now that can be ensured when phi is proportional to i m phi where m can be zero, plus or minus one, plus and minus two etcetera.

So, we must have essentially what this means is we must have solutions in terms of cos m phi and sin m phi. So, that can be written in this complex form. Now, this solution is possible only if I choose this constant to be minus of nu square and nu cannot be arbitrary we know that nu has to be an integer so that this precocity condition is satisfied. Now if you have this as minus m square so I can write.

So this equation then becomes can be written as let me define this as gamma square and this is minus of m square. So, that must be zero where gamma is omega over c. So, this is the equation governing our radial function. So, this equation was for the angular function pi so this is for the radial function r. So, for the angular function we must have solution like this so our solution till now what we have is like this.

So, this function. Now, we have to solve for this radial function. This has in addition the boundary condition so if you use the boundary conditions then you have R at the periphery of the membrane at R equal to a must be equal to zero. So, this equation has to be solved with this as the boundary condition. Now, this is a second order ordinary differential equation but we have only one boundary condition so let us see what happens.

Because this is going to lead us to something interesting. Now, this differential equation is known as the Bessel differential equation and we have come across this differential equation when we discussed the hanging string if you recall there also we had the Bessel differential equation except that this term was zero. So, let us see then we can draw an analogy from the hanging string and you can understand why only this one boundary condition will suffice.

The other condition comes from the finiteness of the solution. So let us see what is the solution of the Bessel differential equation? So, let us see what is the solution of the Bessel differential equation?

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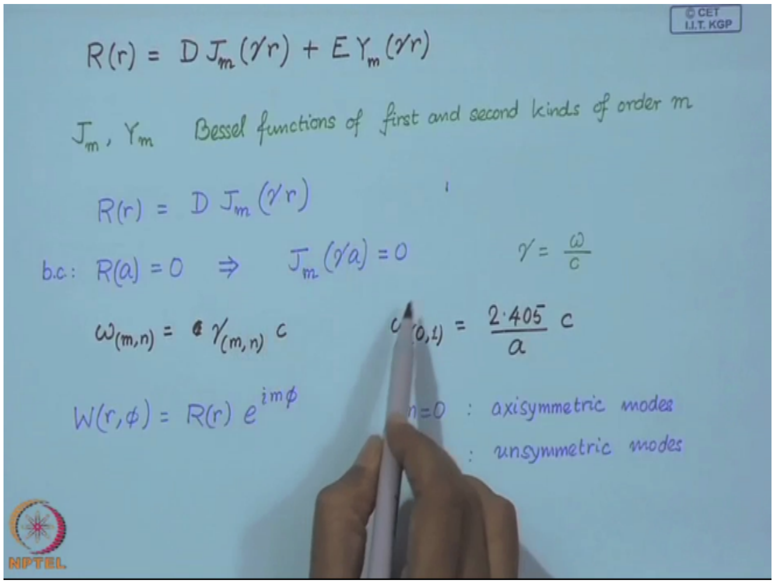

So, the solution of the Bessel differential equation so this is a very standard differential equation in mathematics and the solution can be written as some constant in this form. So this is called the Bessel functions of first and –so these are of order m Bessel functions. This is the first kind and this is the second kind and they are of order m. Now, since we have the solution as a liner combination of these two functions let us once have a look

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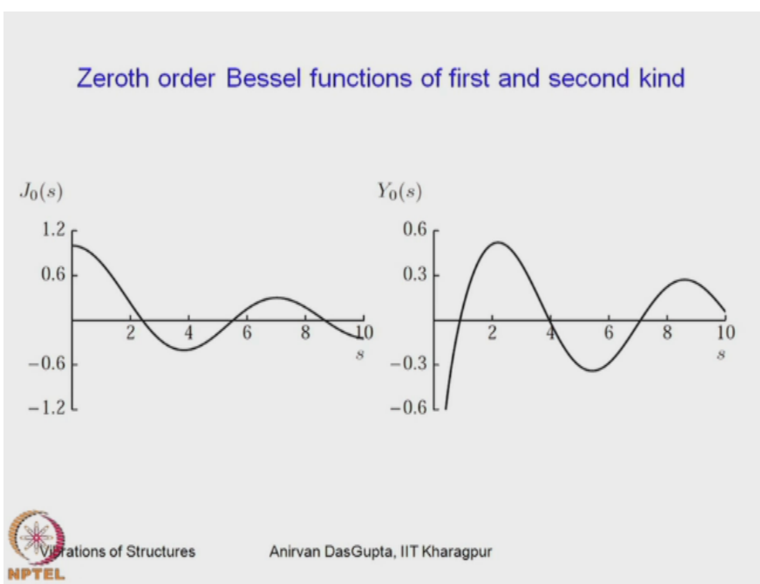

To just have an idea of these functions so this figure shows the Bessel function of first kind here I have plotted only the zeroth order Bessel functions Jo and Yo as a function this argument S. So, here actually for integral values of the order this Bessel function of the second kind has a logarithmic singularity at argument equal to zero. So, this is known from theory of Bessel functions.

So, what this means for our membranes is that at R equal to zero this function is going to introduce logarithmic singularity that means this is going to go to minus infinity which we do not want. I mean physically we do not have this kind of a behavior. So we must drop this function from our solution and Bessel functions of all order of second kind they have this kind of a singularity.

So we must write our solution or express our solution only in terms of the Bessel function of the first kind. Now if you do that now we have just one boundary condition. So R a is equal to zero and that implies –so a is the radius of the membrane. So, we must choose gamma. Now remember that gamma is omega over c. So, we must choose this gamma so that this condition is satisfied. Now, let us look once again at the Bessel function of the first kind.

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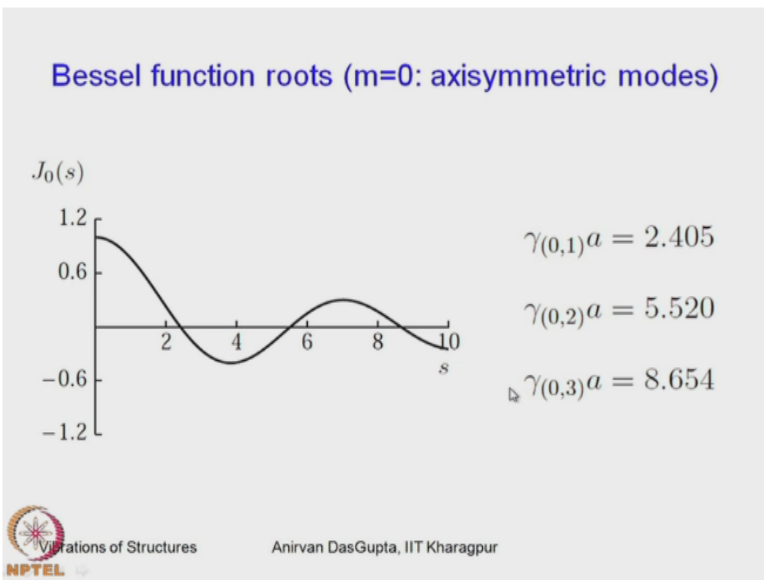

Let us say with m equal to zero. So, zeroth order, this function goes to zero at these points. So, in these figure there are only three visible but then this function has infinitely many roots so this goes on and on. So, the first three values of gamma times a which are now indexed with m and the root numbers. So, this takes the number one, this is the second root, this is the third root. So, one is 2.405. So, this approximately that 2.405.

This is 5.52 and the third root is 8.654. So, the first three solutions of this condition therefore obtained here. So therefore our frequency is also now get indexed. Let me first write this in this form. So, this is gamma m and times c. Now gamma m, n let's say omega 0, 1 as we have seen is 2. 405 over a, times c. So, that is the circular eigen frequency corresponding to the mode 0. 1 and in this way you can determine the higher modes. So, here I have listed out.

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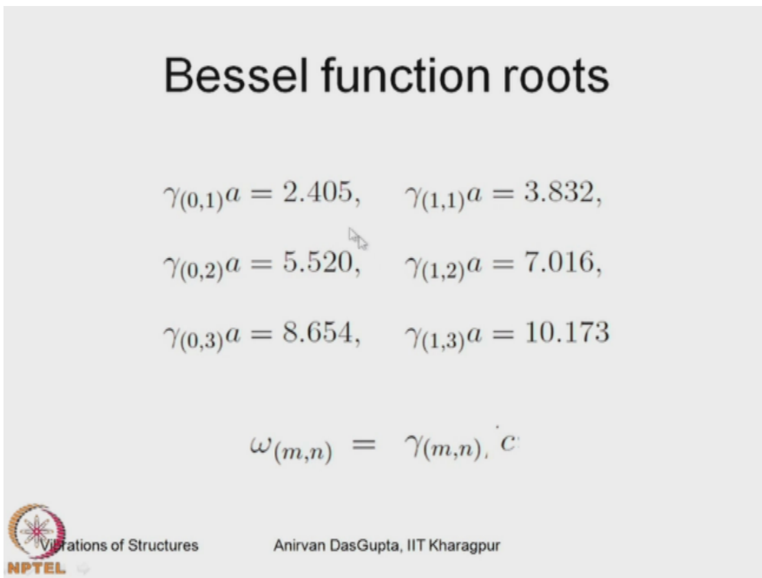

The some of the higher modes as well so these are with m equal to zero these three what we have seen. And these are with m equal to one. So, one the first root with one, the second root etcetera and therefore the circular eigen frequency or national frequency of the circular membrane is given in this form. So let us once again have a look therefor so our solution was in this form. So when m equal to zero, we have what are known as the axisymmetric modes.

And for n not equal to zero we have the unsymmetric modes. So this is essentially finding roots of the Bessel functions of the first kind. Now, finding these roots is actually a little cumbersome though in various software or numerical analysis program. These are coded.