Vibrations of Structures
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Lecture - 28
Vibrations of Rectangular Membranes

In the last lecture, we had discussed the dynamics of membrane, a flat membrane and today, we are going to carry forward our discussion on this and look at the Vibrations of rectangular membrane, so just to recapitulate what we discussed in the last lecture let me write down the general mathematical formulation of a flat membrane.

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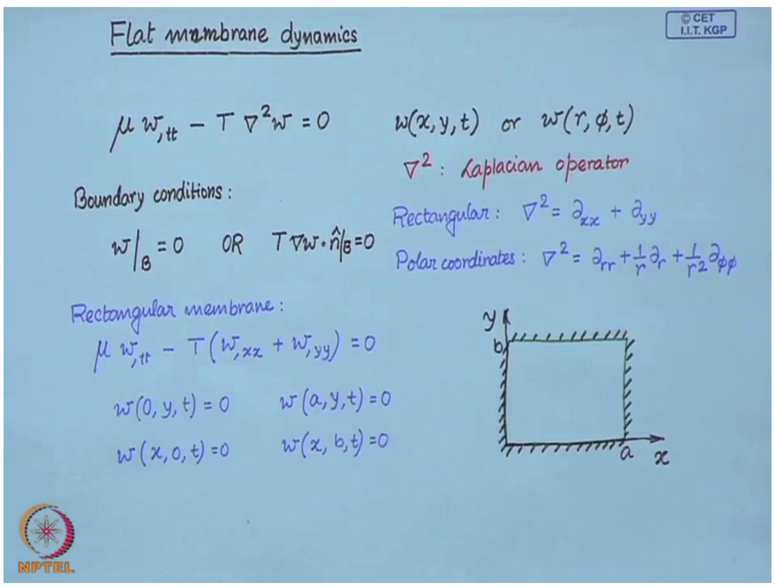

So, we had derived the equation of motion of a membrane with density per unit area or the areal density mu undergoing small transverse vibrations where this w is the field variable, which is the function of x y t r w could be in the polar coordinates, for example could be function of r pi t and Nabla square is the Laplacian.

So in the case of rectangular membrane so in the Cartesian co-ordinate system, Nabla's square is double derivative with respect to x + double derivative with respect to y, in the polar coordinates Nabla square is derivative with respect to r 1 over r single derivative with respect to r + 1 over r square double derivative with respect to pi, so we have this Laplacian operator in two co-ordinate systems.

And along with this we have boundary conditions, so we can have a fixed boundary or a sliding boundary, so this is the force per unit length T times the gradient of the field variable dot the unit normal at the boundary, so for example for the rectangular membrane which we are going to discuss today, and we will assume that the boundaries are fixed.

So let me first so we have w at x equal to 0 which is the y axis and for all y and all time at x equal to a for all x at 0 which is the but y equal to 0 which means the x axis so this is our mathematical formulation for this rectangular membrane. So we are going to look at the modal vibrations of the rectangular membrane.

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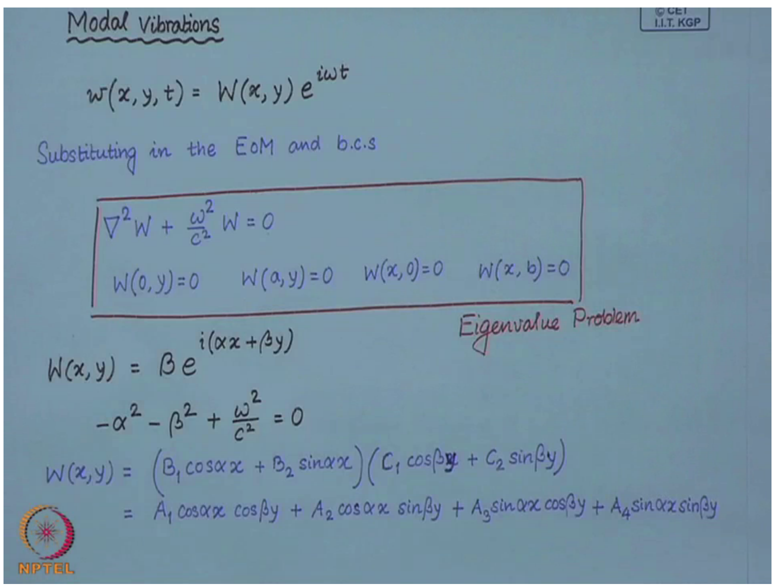

So let me write down, so we are interested in a solution, as we have discussed before so this is separated in space and time so we have this spatial function w which is the function of x and y and that complex time function, now we substitute this solution form in the equation of motion and also the boundary conditions we obtain so upon simplification so this is our Eigenvalue problem.

Where of course this Laplacian is once again the double derivative with respect to x + the double derivative with respect to y, so in order to solve this, let us consider solution of this differential equation of the Eigenvalue problem has some constant exponential i times alpha x + beta y where alpha and beta are as yet unknown constants.

So if you substitute this solution form in here since we have double derivative with respect to x, so that gives as alpha square, minus of alpha square, minus of beta square from the double derivative with respect to y +, so this is what we obtained so alpha and beta have to be chosen such that this equation is satisfied, but that is not all as you know that we have to also satisfy these boundary conditions.

So we expect the solution in the form let's say since we have considered separable solution even in x and y, so I can write, this is y, so this may be simplified and written out like, so I multiplying out and redefining these products as A1 A2 etc. so we have this as the general solution of the spatial function w x, y. Now this general solution must also satisfy the boundary conditions which are going to give us further conditions so let me substitute this solution in the boundary conditions.

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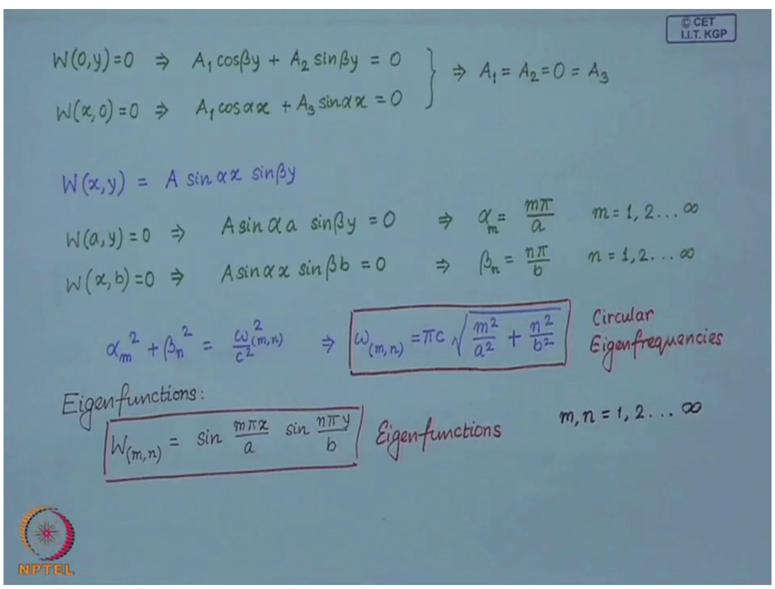

So you can see that, so this boundary condition would imply, so x is put a 0, so this follows from this term and here its sin alpha x, sin alpha again sin alpha x here so at x equal to 0 these two terms vanish, so we are left with this, then let me take this boundary condition so at y equal to 0, so this term will vanish and this term will vanish, so this - so these are the two conditions I obtained from these two boundary conditions.

Now it is very easy to see that these two boundary conditions imply, so for all y and all x, so for all y this condition should be true, which immediately tells us that A1 and A2 must be 0 and this also tells us that A1 and A3 must be 0, so A1 A2 A3 or all of them are 0, so we are left with only A4 so which means in order to satisfy the boundary conditions, let me call this only A, since there is only one term in this function and we are left with two more boundary conditions.

So these are, so when I substitute replace x by A and the fourth one, now again this condition should be valid for all y and this condition should be valid for all x, so this should imply that sin of alpha x should be 0, since A cannot be 0, so therefore alpha must be m times pi over a so this gets indexed and there are infinitely many values of alpha m and similarly, from here this gets indexed by n and there are infinitely many values of beta n.

So then finally, we recollect that alpha m square + beta n square must be equal to omega square over c square, now this should also therefore get indexed I will write as m, n within brackets, so I can take this pi as well outside, so its square root of m square over a square + n square over b square so that is, these are the Eigen frequency of the rectangular membrane okay so, and then we obtain from this function the Eigenfunctions which should also now get indexed.

So these are the Eigenfunctions of course here we have these infinitely many Eigen functions, now here you can notice that the higher so the fundamental Eigen - Circular Eigen frequency is obtained when m and n are 1 and you can see that higher circular Eigen frequencies are no longer just integral multiples of the fundamental what we have seen in strings, so in the case of strings suppose I simply block this, then you obtain actually the Eigen frequencies of the string.

So there you will have seen that the higher Eigen frequencies are just integral multiples of the fundamental frequency, now in the case of membrane this is no longer true, there is another possibility that we are going to discuss shortly and that is when m square over a square + n square b square is some r square over a square + a square b square, so that a possibility that means there are two different modes which have the same circular Eigen frequency.

So that we are going to discuss shortly so before we discuss, so here let me just tell you about the orthogonality of these Eigenfunctions so which is very clear from these expressions.

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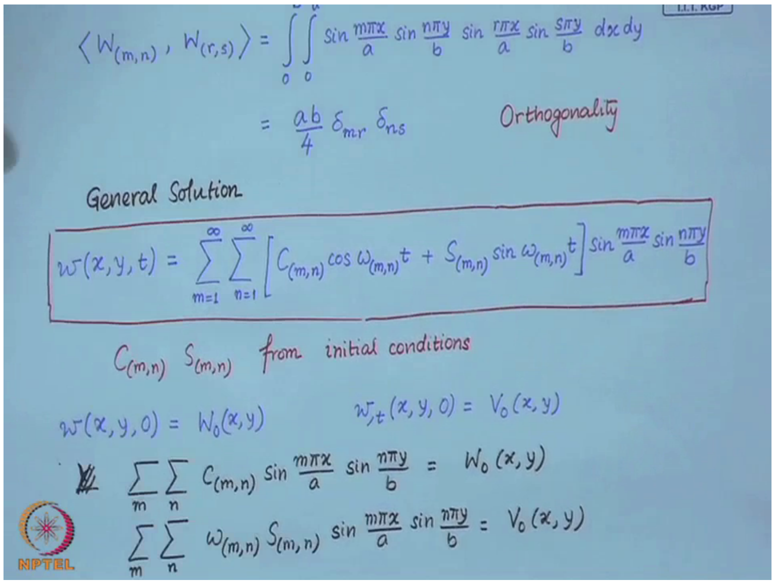

So this inner product, which is defined as the integral over the domain of the membrane and we know from this trigonometric functions, properties of this trigonometric functions that this is nothing but, so for example the integral of this over x from 0 to a is a by 2.

Similarly, this is b by 2, so this is a b by 4, but then that happens when m is equal to r and n is equal to s, so these are Kronecker delta functions, so this takes a value 1 only when m equals r and similarly, this takes a value 1 when n equals s otherwise they are 0, so this is the Orthogonality property of the Eigenfunctions, now before we move on.

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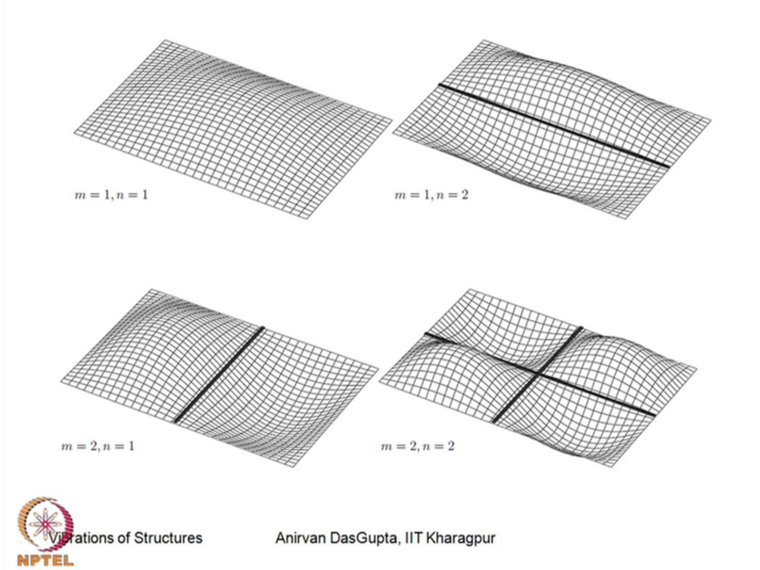

So let us have a look at some of these Eigenfunctions so here I have plotted four, so this is the fundamental which is with m equal to 1 and n equal to 1, so there are no nodal curves in this mode, here this is m equals 1 and n equals 2, so there are two lobes here so m is for this direction, n is for this direction, so we have one nodal line along, so this is parallel to the x axis this is for 2 1, so one nodal line parallel to the y axis.

And this is m equals 2 and n equal to 2, so this is the w 2, 2 mode, so you have two nodal lines like this, now the general solution, so this Eigenfunctions can now be use for constructing general solution of for any problem related to the rectangular membrane, so let me write out the general solution, so here so motion of the membrane the general motion of the membrane may be written as the summation over.

Now all these constants that I introduced we have this indices m n, so this is the general solution which can be used for solving the initial value problem. And here these constants, they are determined from the initial conditions, now in order to do that we have to use the orthogonality property so the initial conditions will be in the form, so the initial configuration of the displacement of the membrane and the initial velocity distribution.

So given these two functions W not and V not, then we can write this. And similarly for the velocity condition we have this so we can solve now for this infinitely many coefficients C and S using the orthogonality property that we have just now discussed and we have discussed this procedure, in a previous lecture as well, so we can determine these constants and finally, construct the solution using the submission.

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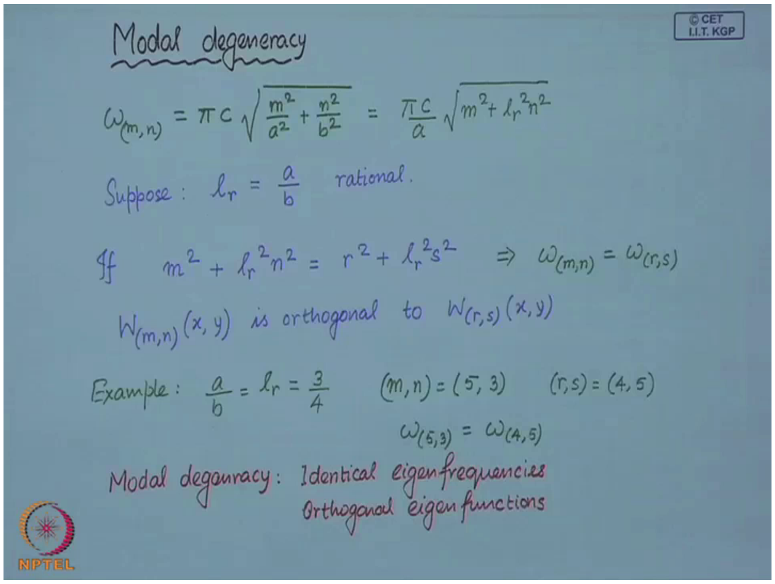

Now let us consider this situation we have an interesting property called, so here what we have let us look at the expression of the Circular Eigen frequency which reads like this, now suppose you have this ratio a over b let me call it the length ratio, suppose this is a rational number so if this is a rational number, in that case we have, we can have two integers a b such that - that equals the length ratio.

Now when this happens then we can have an interesting possibility that so we have this possibility, m square + l r square this ratio square, for a different number r and s different integers, if this equals this, so - so once we have this we have defined this ratio, so you can write this as, so I can express this in this form in terms of l r, so now if for 2, so for a given m n if there exists another mode with r and s such that these are equal.

In that case we have two identical Eigenfrequencies Circular Eigenfrequencies, so there are two modes with identical Circular Eigenfrequencies, but then remember that the Eigenfunctions they will be orthogonal they will be still orthogonal, so we have two modes with identical Circular

Eigenfrequencies but orthogonal Eigenfunctions now this is an interesting occurrence now let us see an example.

So for example if a over b which is l r is 3 over 4, in that case if you take m n as 5 3 and r s as 4 5, then you can easily check that for l r 3 over 4, these two give identical circular natural frequencies, but their Eigenfunctions are orthogonal now this is an interesting situation, so let us understand, so this is known as Modal degeneracy, so you have identical Eigenfrequencies but distinct Eigenfunctions or orthogonal Eigenfunctions.

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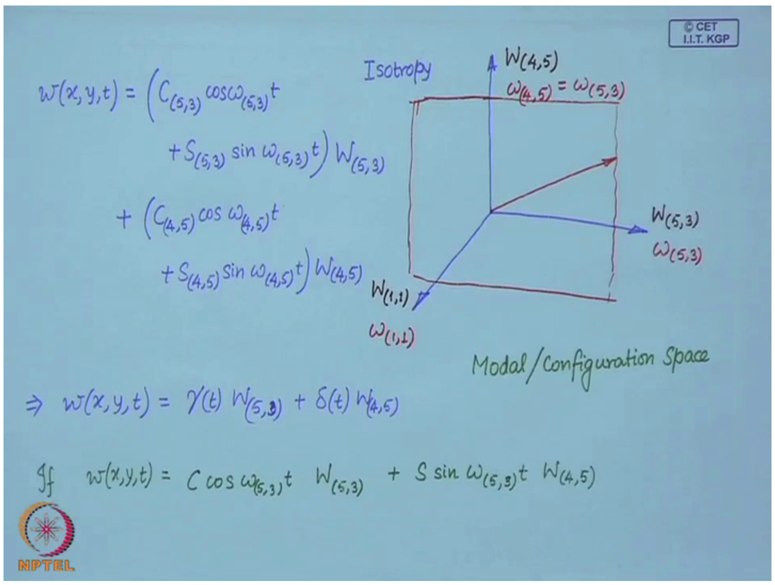

So let me graphically try to depict this situation, now we have been discussing this visualization in various contexts, so let us consider the modal space of the membrane, so this is the modal space or configuration space of the membrane, so the access so which I have drawn perpendicular to depict orthogonality, so for example this is let's say w 11 and then this is let us take the example that we have just now considered this is w 5 3.

And again there are many many such access which are - which are orthogonal to one another, I am only drawing 3 and you have to stretch your imagination a little bit, so this mode in this modal space in which I have drawn only three of this orthogonal Eigenfunctions, you see this is an Eigenfunction corresponding to Omega 5 3 similarly, this is 1 1 and this is, but this is same as Omega 5 3.

Now if this is an Eigenfunction corresponding to omega 5 3, then this is also there Eigenfunctions corresponding to Omega 5 3 so therefore any linear combination of these two Eigenfunctions is also an Eigenfunctions of Omega 5 3, so anything that lies in this plane, so this is the w 4 5, w 5 3 plane, so anything that lies in this plane any vector any function that lies in this plane is also and Eigenfunction of Omega 5 3.

So we have some kind of an isotropy in this modal space so this - this plane is isotropic any function on this is an Eigenfunction, so this property is completely missing in the case of string, so in strings for example we do not have modal degeneracy we do not have this property at all, so this is one distinction of a membrane from a string.

So once again this is the space which is spanned this function space which is spanned by these two Eigenfunctions the membrane dynamics is isotropic in this, so imagine that you are exciting this membrane close to omega 5 3, so you can expect the solution so the solution would be something like this, so this can be written as, now you can imagine that these two - so these two frequencies are actually one and the same.

Now it depends on how the - the two modes have been excited, so what will happen is since the combination of so now the actual solution is the combination of these two Eigenfunctions, but then this is the mode corresponding to a single frequency Omega 5 3, but these two modes are distinct.

So what is going to happen is you are going to see a combination of these two modes appearing in the - in the vibrations of membrane and it may so appear that the membrane is vibrating in an unsteady manner because it might be switching from possibly this mode to this mode or a combi - through a combination of these two modes, so that you can understand from this solution form.

So because of this isotropy of the modal space we have this kind of a phenomenon, so for example - for example if you excite this in a certain manner such that since 4 5 is same as 5 3, supposed the excitation initial conditions have been set such that this is the solution let's say, now in that case since one is cos and other is sin, so at time t equal to 0, have this mode you can see the nodal lines corresponding to this mode and at time 2 pi by omega you have this as 0.

And this is active so you will see the number of nodal lines corresponding to w 4 5, so it is going to continuously make transitions from w 5 3 to 4 5 and back, so but this is the modal solution so you will actually see changing profile of the membrane, there is no fixed nodal lines so they keep varying according to this, but suppose that excitation is cos cos, in that case you there will be a fixed combination of w 5 3 and 4 5 which is will be C times w 5 3 + S times w 4 5.

Because this will be cosine this is also cosine in that case that will come out and therefore you have fixed kind of a shape of the membrane vibrations, so this is an interesting feature of modal degeneracy which is present in membranes and not in strings, so in this picture.

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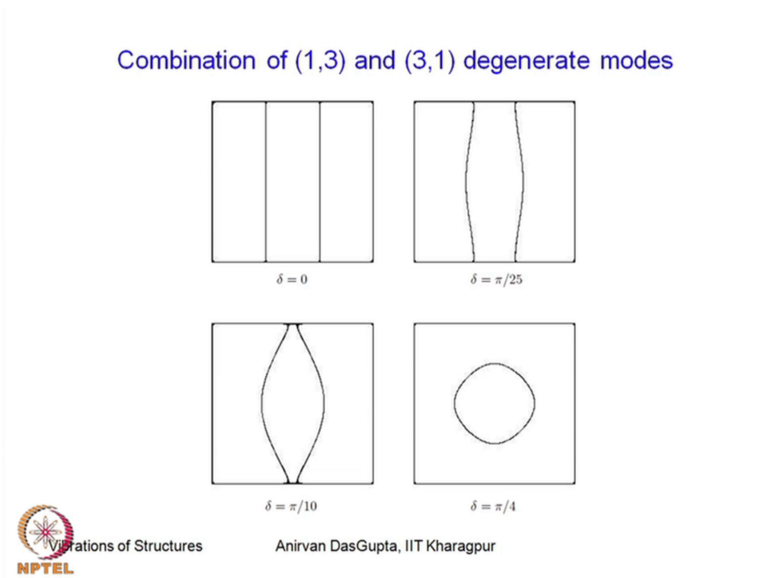

I have shown this for a - for a square membrane, interestingly for the square membrane.

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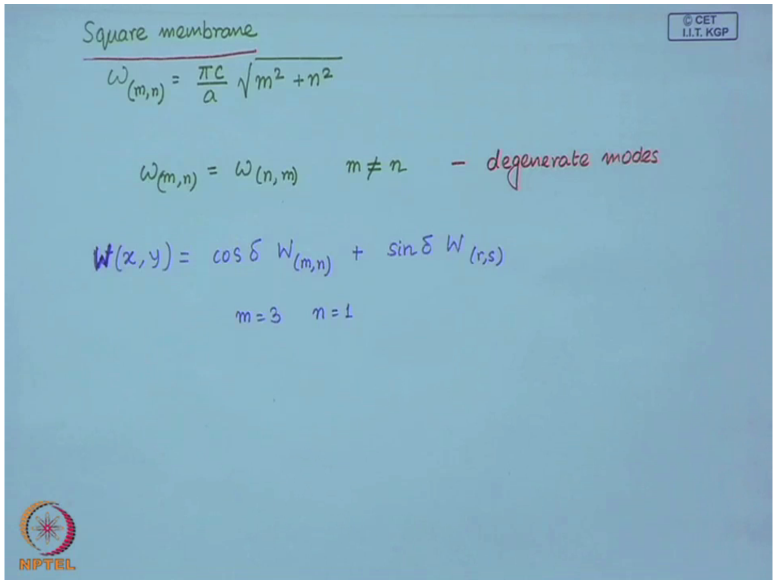

You see the, so with side equal to a, in that case this is m square over a square, n square over a square, that comes out so square root of m square + n square, so you will find that omega m n is equal to omega n, m where m not equal to n, and then these are degenerate modes, so all modes with m not equal to n are degenerate modes, for the square membrane so for the square membrane for m not equal to n these are all degenerate modes.

So once again let us have a look at this picture so here we have this combination, so the combination I have written like this, so I have written the combination like this, for the 3 1 mode, so let us look once again at this picture so here when delta equal to 0, so you will find it is only w 3 1 and here its value pi over 25. So you see the combination of 3 1 and 1 3, this is still higher value of delta.

So this combination changes at pi over four this combination of 3 1 and 1 3 gives as a nodal curve which is like this, and this changes for if you increase further to this but rotated than this rotated and then this rotated which means the nodal lines will be horizontal for delta equal to pi over 2. So you see when a square membrane vibrates in this combination then you will see the nodal patterns nodal line patterns changing continuously through these shapes.

So you will not see a steady nodal line picture on the membrane so the nodal lines will be kind of unsteady so this is something very interesting that you can observe in an experiment. So to summarize we have looked at the modal vibrations of rectangular membrane and we have observed very interesting property which is the modal degeneracy some of these aspects we will discuss further in our next lecture, so with that I conclude this lecture.