Vibrations of Structures
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Lecture - 27
Dynamics of Membranes

Today, we are going to initiate some discussions on Vibrations of membranes, so today first we are going to look at the dynamic modelling of membranes, now to begin with what are the membranes? A membrane is the two dimensional elastic continuum which cannot resist or transmit bending moment, now when we discussed about strings the string was a one dimensional elastic continuum which does not resist bending moment.

But now this is membrane is a two dimensional elastic continuum but then there are some fundamental differences between string and a membrane which means that membrane is not just a two dimensional extension of string, so let us first in order to start our dynamic modelling let us first enumerate our assumptions that we make for the modelling mathematical modelling of membranes.

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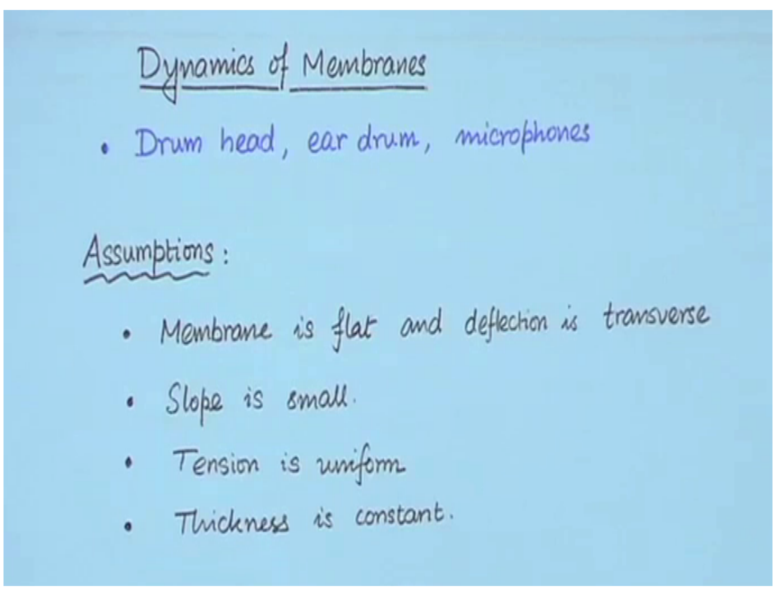

So first of all we where do we find membranes we find membranes in ear drums, and drum heads and in certain kinds of microphones etc. now we study membranes not just to study drum heads or ear drum or microphones, but to understand this two dimensional elastic continuum which is simplest of its kind because it does not transmit bending moment, now in order to model membranes.

We will assume that the membrane is flat and the deflection is purely transverse, so when you have let us say an inflated membrane let say balloon in that case the dynamics is more complicated because the deflection can no longer be considered to be just transverse there is a coupling between the transverse and the in plane modes. So we are going to make this assumption that our membrane that we analyze is flat.

So that only we have to consider the transverse deflection of the membrane. The second assumption that we make is that the slopes are small, so if the deflection is so you have certain deflection then by small slope you can imagine that the maximum deflection divided by the characteristic dimension of the membrane. So suppose I have a membrane of let us say this size than the transfers deflection should be much much smaller than this dimension or this dimension of the membrane.

So slopes should be small, we assume that the tension does not so the tension is uniform and it does not change with the deflection so it is constant it does not change with the deflection and finally, we also assume that the thickness does not change. So under these assumptions we are going to now look at the modelling of the membrane.

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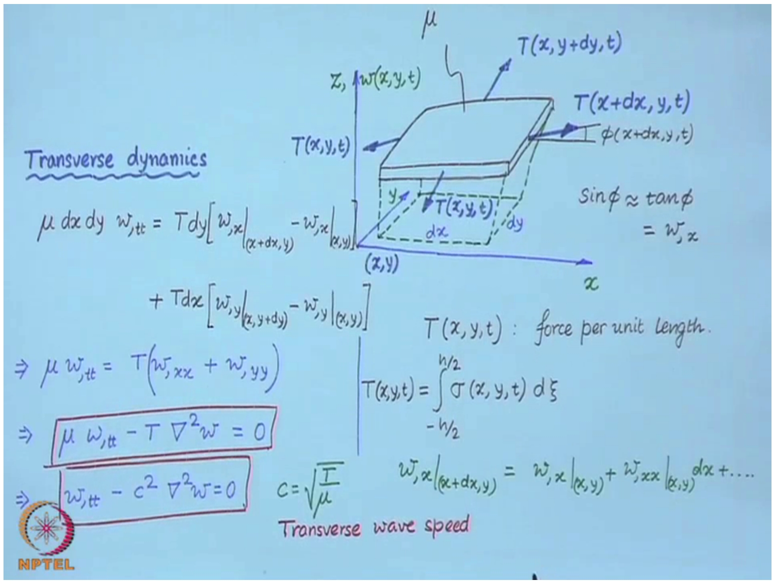

So let me first draw an infinitesimal element of this membrane, so here we have I have considered an element of this membrane which was initially flat on the x y plane, so we have in order to locate the membrane we consider this field variable w which is a function of now x y and time. So here we have these forces, so I have taken a little element of the membrane and I will draw the free body diagram.

So if this point is x y, I will denote this as the force on this edge as T so here I have so this is dx and this is dy similarly, the force on this edge is T x y t, so these are the forces acting on this little element now here this T is the force per unit length so you can think in this way that there will be stresses on this cut. So if you integrate this stress over the thickness then you obtain this T, which is force per unit length.

So stress integrated over the thickness of the membrane is force per unit length so some kind of resultant stress resultant, using this we are now going to write down the equation of motion, so let me consider this little element so the transverse dynamics so let mu be the mass per unit area so mu is the mass per unit area. So the area of this element may be written as mu times so dx into dy.

So that is the mass of this little element times the acceleration in the transverse direction that must be equal to the net force in the transverse direction, so I can write so this is - so this T is the force per unit length of the cut now, because it has been indicated over the thickness so it is equal to the force per unit length of the cut. So for example the force because of this on this face is T times dy because this length of this space is dy.

So that is the force but it must have so I must take a component which is the in the transverse direction so that component, so if this angle let us say is pi then I must have sin, so I must take sin of pi, to find out the component of this force in the transverse direction now sin of pi is approximately equal to tan of pi and that is equal to del w del x. So to find out the transverse force because of this I will use del w del x at x + dx and y.

And of course at time t, and on this side it would be del w del x at x y and similarly, from these two edges so this T dx is the force so since we are assuming that the tension in the membrane is uniform so it's essentially isotropic is homogeneous so everywhere its T, though I have written it like this to indicates that this vector might be different so magnitude wise it is same. So here it is del w del y is in this direction it is del w del y.

So this must be the equation of this must lead us to the equation of transverse dynamics, so if you divide throughout by dx dy and make some simplifications, so this so del w del x at x + dx so let me write this down so del w del x x + d x, y written as and so on. So I can have an expansion the Taylor series expansion of del w del x at x + d x, y in terms of del w del at x, y and del w del x and its higher derivative at x, y.

So using this expansion here I can therefore finally, write this as, so from this will contribute a term del square w del x square and this is going to contribute the term del square w del y square, so that implies, now this is the Laplacian of w and this can also be written as. Where this c is - is once again looking at the structure of this equation and the dimension of this quantity so this is - this has a dimension of speed, so this is the speed of transverse waves in the membrane.

So, this is the transverse waves speed in the membrane, so this is the structure of equation for transverse vibration of membrane, now the you see that here we have double derivative in x as well as y, so we would require boundary conditions at on this edges so let us look at the boundary conditions.

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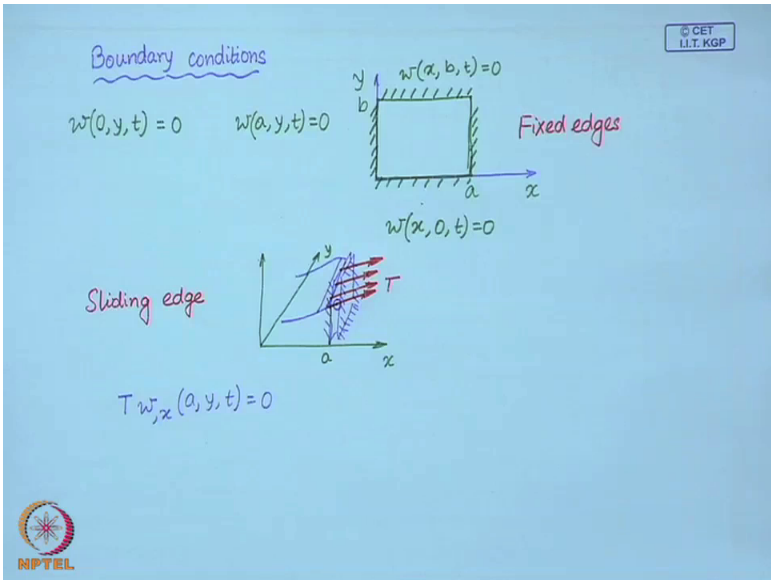

So if you have let us say a rectangular membrane, please note one more thing that we have derive the equation considering the Cartesian co-ordinate system we will look at the general situation in a moment, so if you have this x y plane and a flat rectangular membrane then if this is fixed on this edges then we can easily write the boundary conditions, so at x equal to zero that could mean this face and for all of y and for all time this is fixed.

Similarly, on this face we would have w at x equal to a and all of y and for all time must be zero, on this edge for all of x and y equal to zero this must be zero and on this edge, so when the edges are fixed then you have zero displacement, there is another possibility where you can have Sliding edge, so in a sliding edge the membrane can, so it is very similar to the sliding edge of a string.

So if you have an edge of a membrane which can slide, in that case the condition at this boundary let us stay at x equal to a, it is very similar to that of the string so tension times the force per unit length times the slope that must be zero, so this is the force per unit length at this edge so the whole thing is, so this is the - the tension in the - in the membrane the force per unit length.

So we must take the transverse component of this force so that is determined by taking the multiplying it with the derivative of w with respect to x as we have seen, so that must be zero for a sliding boundary.

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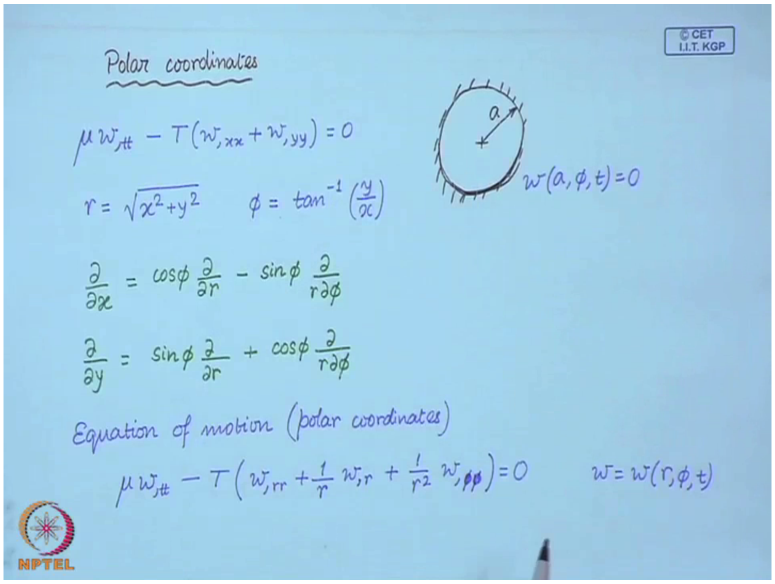

Now when we go to the Polar coordinates, we use polar coordinates when we have a circular membrane, so suppose you have a circular membrane of radius a, then first of all the equation of motion for this membrane, so we have the equation of motion for a rectangular membrane written out in this form, so we consider we can derive the equation for a circular membrane from this by simple co-ordinate transformation.

So if you consider that the radius radial co-ordinate r is square root of x square + y square and pi is tan inverse y over x, so using this you can derive these operators del del x in terms of derivative with respect to the radial and the angular co-ordinate, so if you use these transformed operators then finally you can get the equation of motion in the polar coordinates so these are obtained - this is obtained as.

Where now w the field variable is the function of r pi and t and for a fixed boundary we have so the boundary condition is quite simple, now this choice of coordinates actually is guided by the boundary, so for simple boundaries like rectangular membrane or circular number the choices is very clear but for other shapes, some of which we are going to discuss later on we will see how they are dealt with so here we have discussed about right now about Newtonian formulation.

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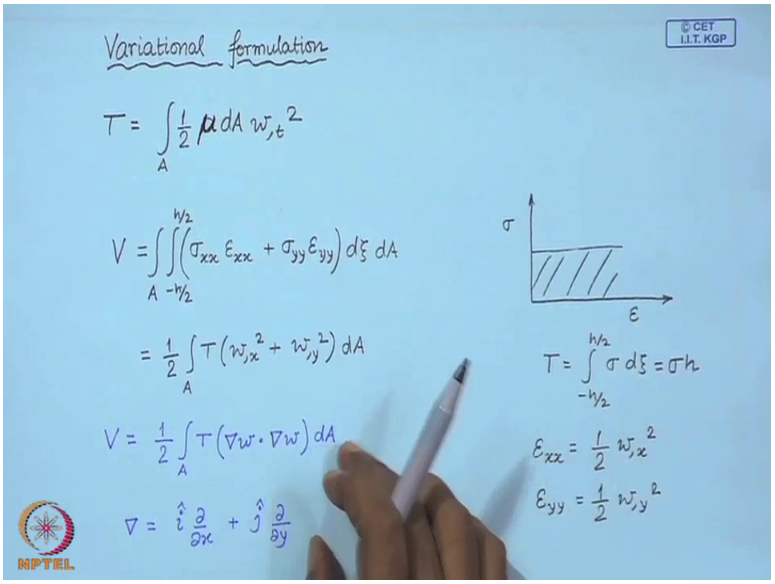

Let us also look at the Variational formation, so the variational formulation for the dynamics of membrane once again I mean to motivate so the variational formulation is quite powerful not only it gives us the equation of motion, but also the boundary conditions and even more important than that it gives us methods by which we can solve the or discretized the dynamics of continuous systems for example we have looked at Ritz method.

So for that reason we must also discuss the variational formulation for the membranes now to begin with we write down the kinetic energy expression of membrane so if w is our field variable then half, so we have considered mu as per unit area, so mu times dA is the mass of a little element of the membrane, so half mass times velocity square and this when we integrate over the area of the membrane we are going to get the total kinetic energy of the membrane.

Next we would like to find out the potential energy when the membrane deflects, so to determine the potential energy we have consider that the force per unit length the tension in the member the stretch in the membrane so that remains almost constant it does not change so if you think about it in this way that the stress and the strain, if the stress in a material because if stress remains fixed only then the force per unit length which is determined by the integrating the stress.

So that must also be constant because T is fixed so the energy per unit volume is given by the area under this, so it must be this sigma in the x times epsilon in the x + sigma in the y times epsilon in the y, so this is per unit volume, so you must integrate this over the volume so first over the thickness and then for the area of the membrane so xi is along the thickness direction.

Now this stress so this is all taken almost uniform over the thickness so this turns out to be sigma in to h so sigma is T over h and in both direction is the same, now the strain we have seen from the case of the strain the strain is half of del w del x whole square, so in the other direction as well we must have, now so therefore finally when you substitute all these expressions in here, so this is what you will get.

Now this can be written as in a co-ordinate independent manner we can write this, where this delta is for the Cartesian, so this is the gradient of this field variable the magnitude square of the - of that vector the gradient of the field variable, now this is the potential energy

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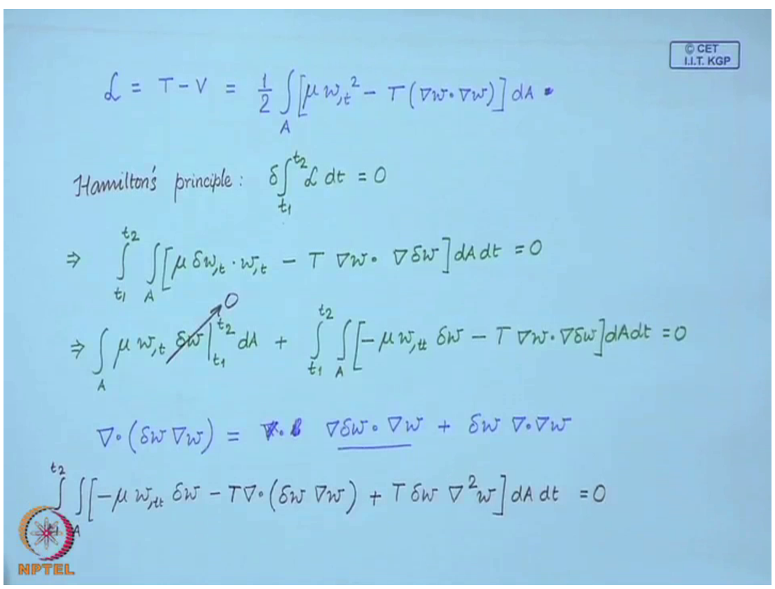

So finally our Lagrangian, so that - so that is the Lagrangian and from Hamilton's principle we can - we can write so this must very – vanish, so the variation of the action integral which is the integral over time of the Lagrangian between two time points so that variation must vanish, so if you consider this Lagrangian and take the variation.

Now here I have used this property that the variational operator commutes with the gradient operator, now this term I will integrate by parts with respect to time once, now for this term let me, so this term in order to simplify this term let us look at this identity, so I am taking the divergence of a vector so this is the vector and this is a scalar so the scalar multiplied by a vector and we know that this turns out to be.

So this is the gradient of the scalar dot the vector and so this is the divergence of this vector, now so therefore I can use this here to rewrite this left hand side now this term goes to zero because at t1 and t2 the - there cannot be any variation, so what I am then left with is this so this goes on this side, so this comes with positive and this is the Laplacian, so this must vanish.

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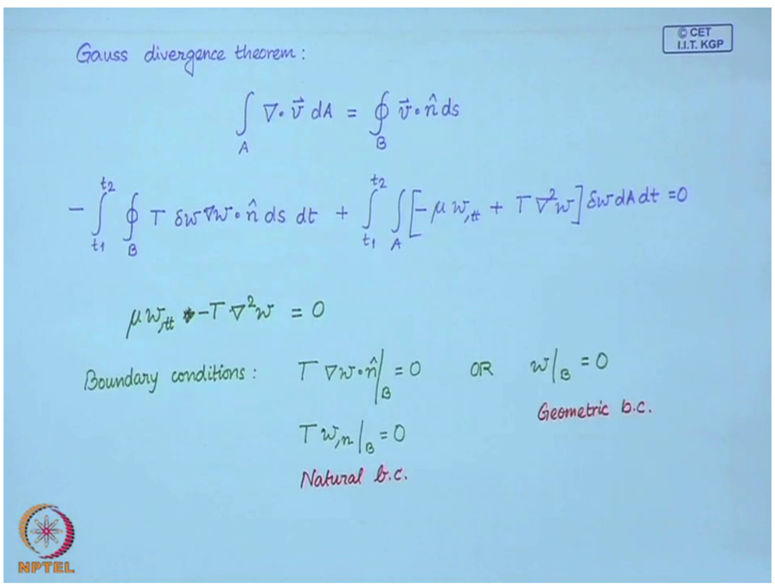

So then now here I have the divergence of vector quantity so here I can use the Gauss divergence theorem, so which says that so the divergence of any vector over this area integrated must be equal to this vector dot the normal to the boundary so integrated over the boundary, so if you this here, then we have boundary term arising out of this which so this integral over the area of this term that gives us this boundary integral dot the normal to the boundary.

Now once again we apply the argument that the variation over the boundary and over the domain are independent so therefore we obtained directly from here the equation of transverse dynamics that we had obtained before, now we have from this boundary term we obtained the boundary conditions, so we must have, so this calculated at the boundary of course, this is sometimes written as, so it's the normal derivatives.

So gradient dot the unit normal at the boundary is the derivative along the normal to the boundary, so this is a this boundary condition is a natural boundary condition whereas this is the a Geometric boundary condition now let us consider this example.

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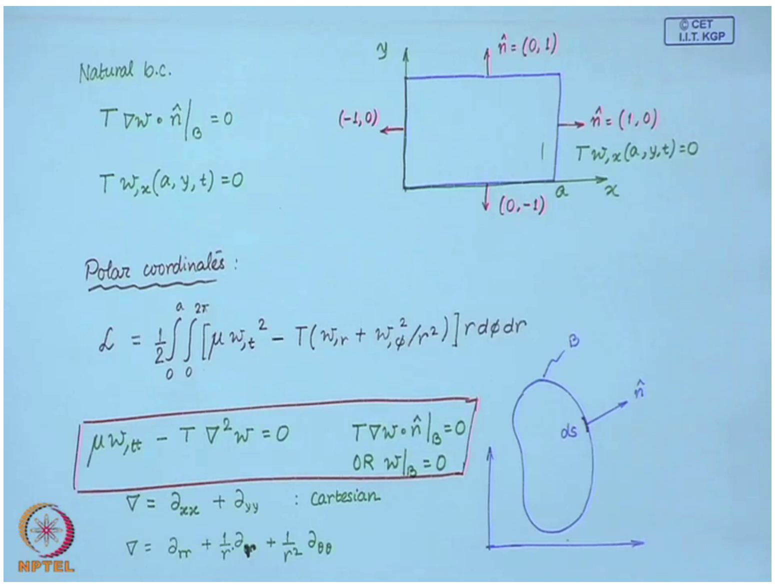

So if you have for example a rectangular membrane, so the unit normal at the boundaries, let us say this, so let us understand this the natural boundary condition which says, so for example at this boundary at the right boundary at x equal to a, so n is 1 0, so therefore so at the right boundary you will have this condition, if it is dynamic - natural boundary condition.

If the boundary is fixed, then of course you have the geometric boundary condition which says that the displacement is zero. now just briefly let us look at the - the polar coordinates, so the Lagrangian in the polar coordinates may be written as, so in the case of let us say circular membrane you have the kinetic energy and the potential energy and of course there is a one half.

Now so in the general case suppose you have an arbitrary boundary so here, so if you have an arbitrary membrane then you can so this is the boundary, you can represent the unit normal ds and then the boundary conditions for an arbitrary boundary can be written in terms of more general coordinate free notation.

So finally you have in the coordinates free notation the equation of motion so corresponding to for example the Cartesian coordinates this is nothing but and for the polar coordinates, so this is the operator and the boundary conditions in can be written like this, so this is the general representation of the equation of motion of a flat membrane which is undergoing transverse vibrations.

So to conclude, we looked at the dynamics of membranes both from the Newtonian as well as from the variational perspective in the following lectures, we are going to study the vibrations of membranes with circular and rectangular shapes so with that we conclude this lecture.