**Vibrations of Structures**
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**Lecture – 24**
**Dynamics of Curved Beams**

Discussions on vibrations of beams, we have till now looked at only straight beams. However, there are various places you find a curve beams. For example, in arches, in bridges, also we have rings which can be treated as beams, which are curved. So today we are going to initiate some discussions on the dynamics of curved beams. So today we are going to look at essentially the modelling aspects of a curved beam.

Now the first and the fundamental difference between a straight beam and a curved beam that we will see is, because of this curvature the axial motion or the circumferential motion is coupled to the transfers or the radial motion of the beam. So there is a coupling between these 2 directions, so we can no longer treat using 1 field variable for our deflection. So we must use 2 field variables.

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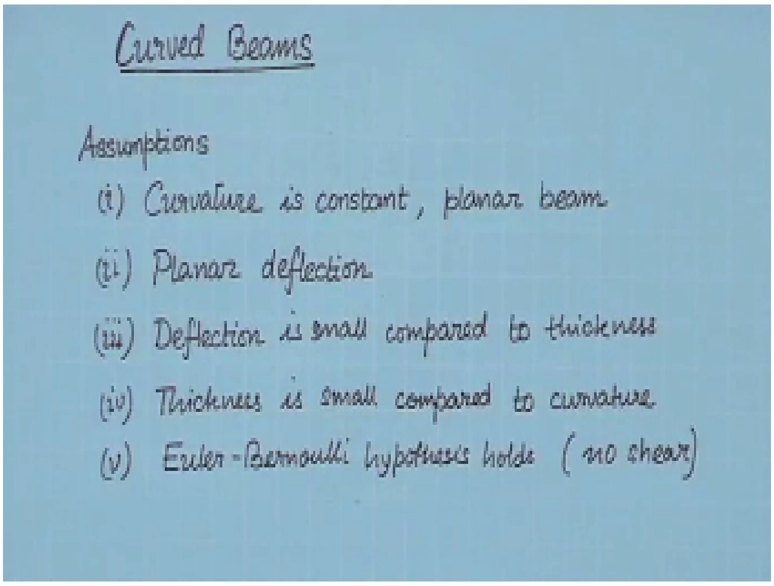


One for tracking the circumferential or the axial motion and the other is for the radial motion. So as with any beam theory, we are going to make some assumptions to simplify our modelling process. So the first assumption; so the first assumption, we make is that this whole the curvature of the beam is; so we will look at a very special situation, where the curvature of the beam is constant and the beam is planar.

So the second assumption we make is the deflection is also planar. Thirdly, we will assume that the deflection is small compared to the thickness; compared to the thickness of the beam, the deflection is small and we also assume that the thickness itself is small compared to the curvature. Finally, we assume that the Euler-Bernoulli hypothesis holds and along with that we say that there is no shear.

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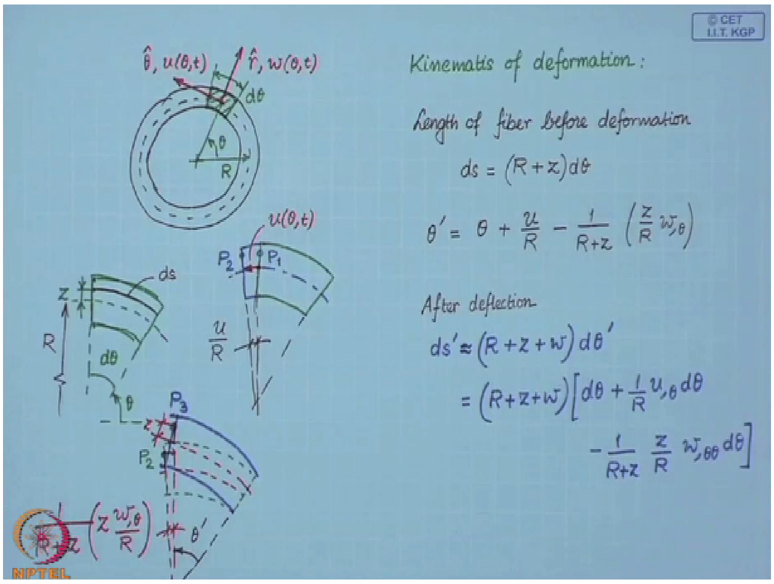


So under these assumptions, we are going to model a curve beam, so we are going to restrict ourselves to the case of a beam with constant thickness. So let us consider; so here I have drawn a ring really, but it could be a part of a ring. We consider a small element of this ring, so this is the radial direction and the field variable will be indicated by w theta, t and this is the theta direction, where we; so the radius of this circle the dashed circle, which we will consider as the neutral fiber that has a radius R, which is constant as we have assumed.

Now let us look at this little element, now we are going to look at the deformation kinematics of this little element, so let me draw; so when you consider the deformation kinematics, you can understand that, so this is at an angle theta and the small this is the small angle, d theta. Now initially the length of any fiber in this element at a height z, so the length of this fiber before deformation.

Let me indicate this by ds is, we are looking at a fiber, which is at a height z from this neutral fiber. So ds is the length of this fiber before any deformation is R + z times d theta. Now when this element deforms now you can imagine that you can consider this deformation in 2 steps; one is its axial elongation, so this moves from so any point, which was here moves here. So this angle is nothing but u over r.

So u is the deflection of this point in the circumferential direction, so as we have mentioned that this is u. The second deformation is that, you can take this is that this defects out radially, so let me draw that first, so here is P2 and now here is P3, so this point P2 moves to this point P3. Now here again, there is this angle which can now be written like this that, this is the slope of the central line of this element.

When there is deflection in the radial direction or the transverse direction and which is captured by this field variable w. So, del w del theta one over r, so that is nothing but that slope of the central line, because w is the deflection of the neural axis, so that multiplied by z, z is the distance of this point from this neutral fiber, so this time z is the deflection, so this is nothing but the angle for small deflections.

We know that this is the angle times the radius gives the linear deflection of point P, so it goes from P2 to P3. So essentially this linear distance is what is being measured by this quantity and when you divide this by R + z, that gives us this small angle. So therefore now when you combine these 2 deflections then the total angle, so this went; this line went from here to here and now this line has travel back.

So this angle; let us say is theta prime, then theta prime is given as theta + u/R because of this motion, circumferential motion and minus this angle. So that is theta prime. So therefore after deflection, ds prime, if we call it ds prime then that is; that is approximately; this is R + z + w; is the transverse or the radial deflection so that is the new radius times d theta prime, so we have to calculate d theta prime, so that turns out to be d theta +; since r is the constant.

So del u d theta times d theta minus; so that, so this is the expression of ds prime.

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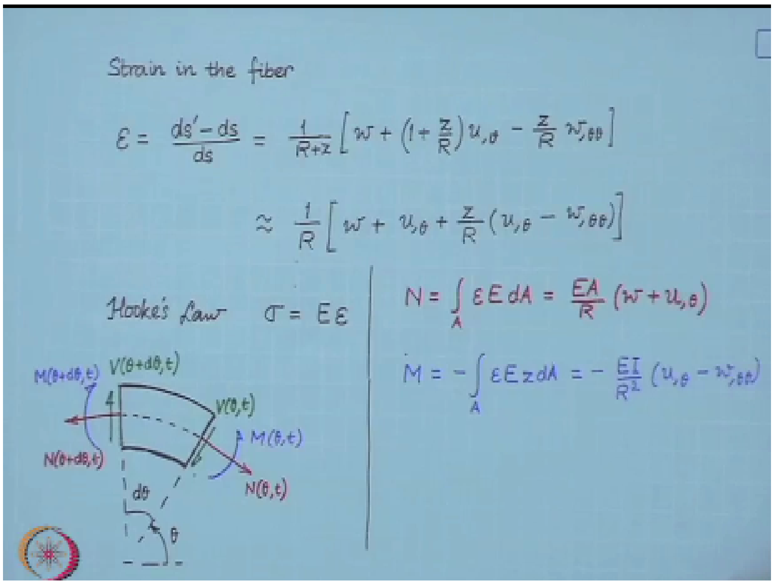


And therefore, if we calculate the strain now, in that fiber at height z from the neutral plane that can be calculated as ds prime minus ds, over ds and if you do this calculation, it turns out to be; and which can be approximated by considering that z over R is much, much smaller than 1, so you can take this R and write this as 1+z over R and take into the numerator and you leave out terms which are quadratic z over R.

So then it can be simplified, so that is the strain in the fiber. Now using this strain, we can use Hooks law to write the stress as Young's modulus times the fiber strain so that is going to give as the stress in the fiber. Now these stresses can be integrated over the cross sectional area of the beam to obtain the force resultants. So let us calculate the various force and moment resultant.

So let me first draw out this free body diagram of; so this is the little element and we have the shear force on this phase. The bending moment and the circumferential forces, now let me calculate this stress resultants, so N is nothing but integral over the area, stress times the area and if you substitute these expressions and calculate this it turns out to be; since there is a z turn here.

So when you integrate over the area since this is already measured from the neutral fiber, so this term will vanish. So you are left with this, so this is the expression of the normal force on the phase. Then we have this bending moment, which is once again we calculate as we did for the beam. Now because of this additional z, this becomes z square and that makes it newer function.

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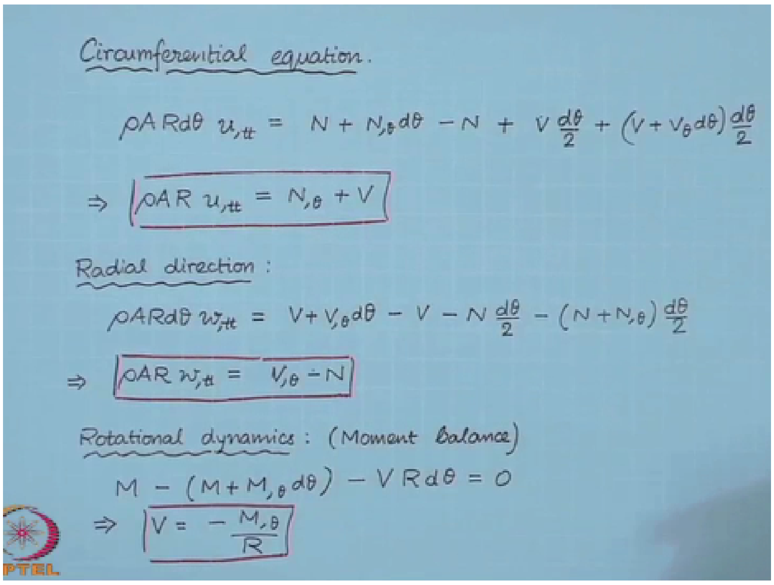


So whereas this becomes an odd function that cancels off, so you are left with contributions only from this term, so that is the bending moment. Now using and the shear forces, as I have mentioned that we did not consider shear, so it is infinitely rigid in shear so that will come out from the equations of equilibrium. So let us now start writing the equations of equilibrium, so first we will write the circumferential equation.

So that reads, so rho A is the mass per unit length, so Rd theta is the small length of that element, so this is mass of the little element times u is the circumferential motion or the field variable, so the acceleration and that must be equal to the forces in the circumferential direction, so let us look at refer to this figure once again. So in the circumferential direction we have, so N theta + d theta can be written as N + del N del theta to d theta.

And this minus, so this in to cosine of this small angle, so that is small, so that is taken as 1minus N, because of this; + because of the shear force, for example here it is this into sin of this small angle which is d theta over 2, so that is almost d theta over 2 and +, we have this again. So therefore if you divide by d theta and drop terms smaller than the first order then you can easily write this equation.

So this is the equation of motion for the circumferential; in the circumferential direction. Next we look at the radial direction, so once again; rho A R d theta is the mas of the little element times the acceleration in the transverse or the radial direction must be = the summation of all forces in the radial direction. So this is what we have. So you can write this as so v, then there is this normal force, which is towards the centre and so this negative.

And there is a projection, so these are the forces in the radial direction so that implies, so this is the equation of motion in the radial direction. Next we will look at the rotational dynamics of this element; we will neglect the rotational inertia of this element, so in that case this equation actually boils down to only moment balance about the centre of mass of this element which can be written as, so M is this moment.

And we have 2 contributions from these shear forces about the centre of mass so that can be combine and written as, so this is R theta over 2 and this is also R theta over 2 and they produce moment in the same direction, so this V times R d theta, so that must be =0, so that implies, so that is what we are obtained from moment balance. Now we are going to combine these equations, so essentially we are going to eliminate this V and also replace this N.

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Equations of motion

$$\rho A R\, u_{,tt} - \frac{EA}{R}\left(w_{,\theta} + u_{,\theta\theta}\right) - \frac{EI}{R^3}\left(u_{,\theta\theta} - w_{,\theta\theta\theta}\right) = 0$$

$$\rho A R\, w_{,tt} + \frac{EA}{R}\left(w + u_{,\theta}\right) - \frac{EI}{R^3}\left(u_{,\theta\theta\theta} - w_{,\theta\theta\theta\theta}\right) = 0$$

So if you do that, then you get the equations of motion. so these are the equations of motion for the beam with constant curvature R.

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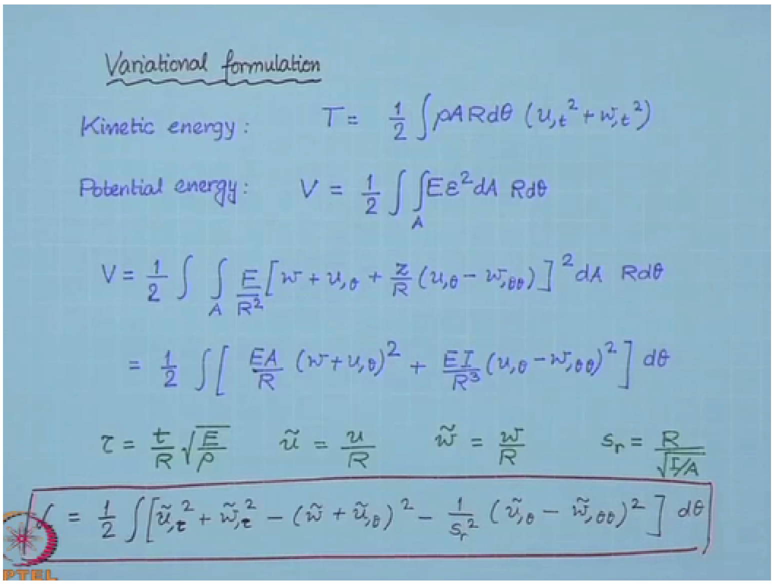


Now next we also look at the variational formulation for this, this we will need when we do an approximate calculations using for example Ritz method. So let us look at the variational formulation so we start with by writing the kinetic energy, one half the mass of the little element times the velocity square.

And this integrated over the full beam, so that is the kinetic energy. Similarly, now we derive the potential energy, which is nothing but half, now we know from theory of elasticity that the energy per unit volume is stress times the strain, so one half stress times the strain for linear theory so stress is Young's modulus times the strain times the strain and so this per unit volume. So we integrate first over the area and then over the length of the beam.

Now we will substitute the expression of this epsilon and that turns out to be so then V, so this bracketed term is the strain epsilon, so that squared dA and this; and there was one, one over R in the strain expressions so that becomes 1 over R square. Now if you squared this, these terms, so this will give w + del u d theta whole square and that when integrated over the area, so these terms will have nothing to do with the area.

So it is the area itself so we have; so that is the contribution from the square of this term, then there is the square of this terms, so you will have z square and you have R power 4 and here there is a R, so that will 1 over Rq and z square integrated over the area that is going to be the second moment of the area, so the square of the second term is going to give us; then there is a third term 2 times this into this and that is linear in z.

And when you integrate over the area, since z is measured already from the neutral axis or neutral plane so that term vanishes, so these are the terms in the potential energy expression. Now to move on further, before we move further, let us make a little bit of simplifications using certain redefinition. So let me redefined time, like this is; so this is the non-dimensional time tau.

The non-dimensional circumferential displacement and a non-dimensional transverse or radial displacement w tilde and I will also define the slenderness ratio, which will be; I will defined like this. So, the radius of curvature of the beam divided by the radius of gyration of cross section, so that reflects the slenderness of the beam. Now if you use these expressions then the Lagrangian can be written as; so integrated over the length of the beam.

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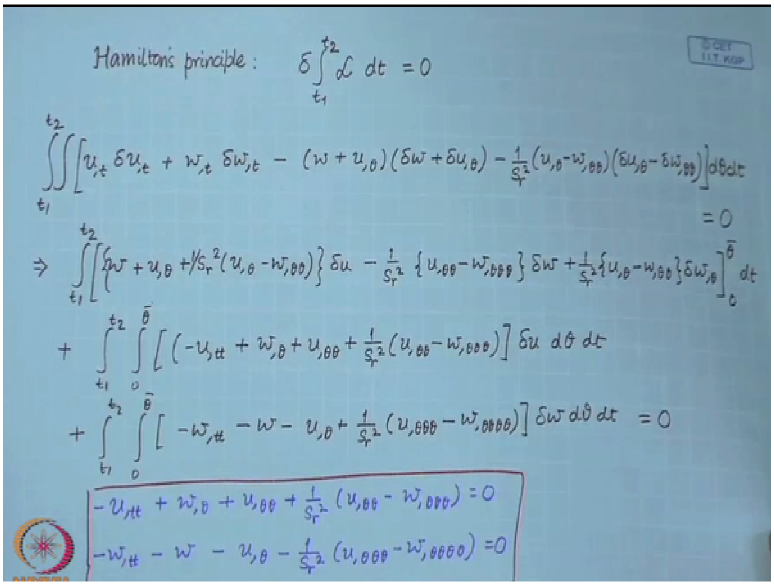


So this is tau, so that is the Lagrangian of the system. Now we use the Hamilton's principle to derive the equation of motion which says that this must vanish, the variation of this Lagrangian of this action integral must be 0. So if you do that, then using the expression of the Lagrangian that we have just now derived, I will drop that in these calculations now. So this must vanish, now this has to be integrated by parts with respect to time, these 2 terms.

And here we have to integrate by parts with respect to theta in these 3 terms. So if you do that, then finally when you get the boundary terms and the variation over the domain, so this is; this integral so here you have the limits over the domain of the beam, so it is 0 to some angles, let say theta bar and +; so this is what you will obtain. Now from here, it is easy to see that this using standard arguments, we can see that this integral must vanish.

And similarly this integral must vanish delta u and delta w being independence, so we have the equations of motion. So these are the 2 equations of motion, which we have derived earlier as well in a slightly different form. Now you can once again see the coupling between the circumferential and the radial directions. Now here since this is non- dimensional these equations are non- dimensionalised, you can look at the contribution of these terms in the equation.

Now if the beam is very, very slender, which means slenderness ratio is very high, then these terms will become insignificant. In that case the equations will get simplifies, so you have only 3 terms in each equation, which you can then try to solve. But if the beam is not slow slender in that case these terms will also contribute in the dynamics. Now let us look at the boundary conditions.

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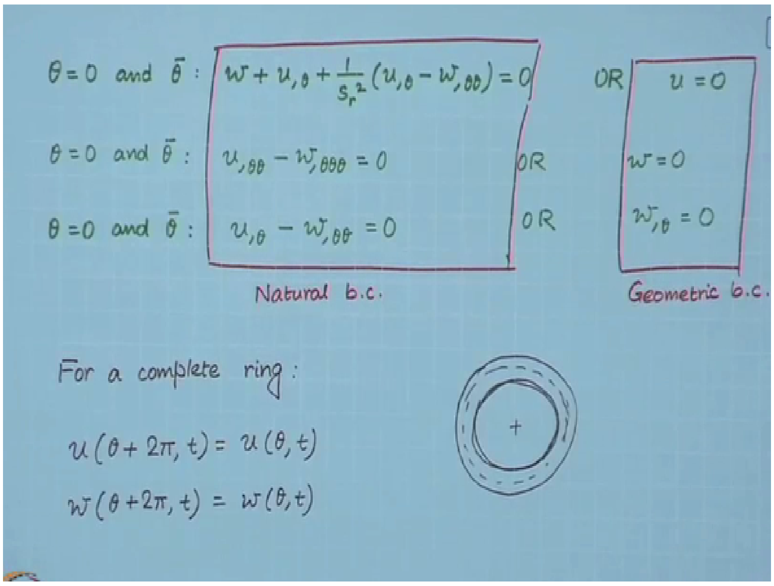


So there obtained from, so from boundary terms we have a theta =0 and theta bar, so we must have this=0, or u must be =0, either the circumferential motion is restricted or which is the geometric condition and this is the natural boundary condition. Similarly, again at theta =0 and theta bar, so this R; so this 2 boundary conditions follow from these boundary term, so this is the first boundary term.

And this is the second boundary term corresponding to displacements, so delta u and delta w. There is a third boundary term, which is in terms of the slope, so once again at theta = 0 and the theta bar, del u del theta, so this must be =0 or the angle must be =0. So you have these boundary conditions, so these are the geometric boundary conditions, whereas these boundary conditions are the natural boundary conditions.

So now in case of complete ring for, so when there is a complete ring the; in that case you do not have boundaries like this, so what you have is? you have this periodicity conditions, so which means that u at theta + 2 phi must be u at theta at all time and similarly and all that follows from these periodicity conditions, so everything is going to be a periodic. So this displacement, slope, bending moment, shear force etc, so they are going to satisfy the periodicity conditions.

So for a complete ring, we have these 2 conditions. So let us recapitulate what we have discussed today. We have today discussed the; initiated some discussions on the dynamics of the curved beams, we have considered in a particular beams of constant curvature and we have derived the equations of motion using both Newtonian as well as the variational formulation.

And this is very interesting and peculiar about this curved beams that the transverse and circumferential motions now are coupled they; in the straight beams we have only the transverse motion, we can treat the transverse and the axial motion separately. They are decoupled, but this curvature in the case of curved beams couples the circumferential and the transverse dynamics that is what we have seen through the equations of motion. So with that, we conclude this lecture.