Vibrations of Structures
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Lecture - 23
Topics in Beam Vibrations - II

Today we are going to discuss an important problem in beam vibrations or even beam stability you can say which occurs when beam is subjected to an axial load, so you find for example a column in a column it is subjected to an axial force on the column and we know that I mean this column cannot with stand any amount of load there is a limit after which the column will buckle or get destroyed actually it might get destroyed, this is one kind of axial loading that can happen.

There is another kind of axial loading that occurs in case of let us say missiles so, in case of missiles you have a jet coming out from the rear and so there is an axial force from the rear on this but now the property of this axial force is that it will always maintained the angle which is there at the rear end of the missile so if the missile is flexible it is going to the force is going to maintain the angle.

In the case of columns, even if the column deflects the force for small deflection the force is still going to be vertical so there is a difference in these two axial forces so which we are going to now discuss.

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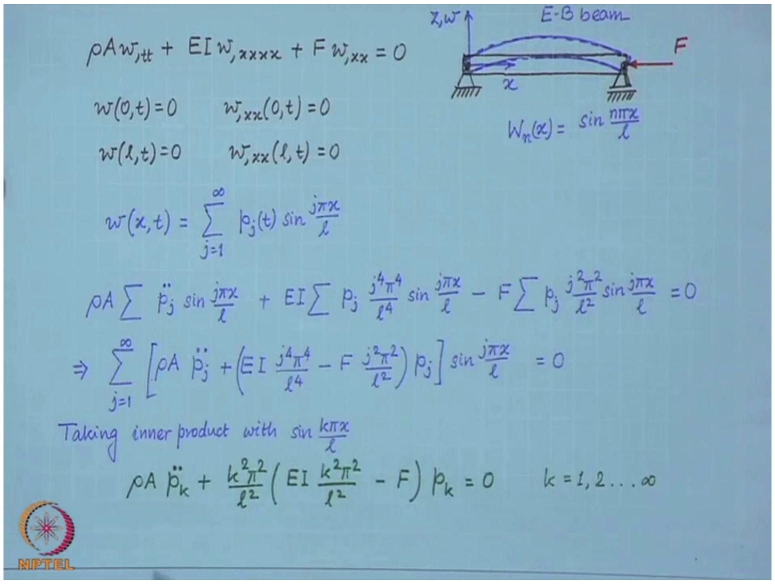

So when we say beam with axial force we are going to consider a column like loading so, let us consider simply supported Euler Bernoulli beam with axial force F, so we have this an axial force F and the beam can deflect in this manner the force retains its direction in this particular case, so the equation of motion for this beam with the axial force so these are the two terms for the Euler-Bernoulli beam.

And in addition because of this axial force which is now compressive we have this additional term so, in the case of strings for example the force is tensile so this is with negative sign now it is compressive so this is the term because of the axial force, now the boundary conditions so the deflection and the bending moment at 0 and l they must vanish. Now for this problem we can - we can use our modal expansion.

We know that the Eigenfunctions of a simply supported beam is given by the sin n pi x over l, so the solution of these dynamics with axial force let us construct that solution as an expansion in this form now if you substitute this in the equation of motion that must be 0. Now this can be simplified, now because of the orthogonality of the Eigenfunctions, we can use the orthogonality property to determine this co-ordinate p j, the dynamics of the modal coordinates p j.

So we multiply this for example with sin j pi x over l and integrate over the domain of the beam so, if you take this inner product so you have the dynamics of the modal coordinates even by this equation.

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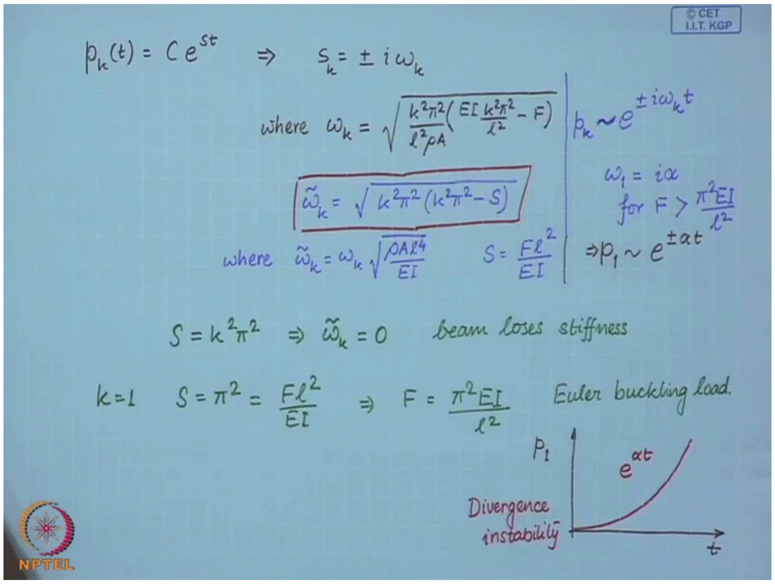

Now suppose we look for solutions of the form C times exponential s t, so if you substitute in here so, we will obtain solutions of s in terms of this omega k, where omega k is given by this expression, now we can make a little bit of simplification here by some redefinition we can write this, where so using this redefinitions we can recast this in - in this non-dimensional form the Eigenfrequency in the non-dimensional form.

Now here for so this S represents the load the axial force, so there is a value of this axial force when omega k goes to 0, so if S is k square pi square when omega k tilde goes to 0, which means that the system loses stiffness, now so this if you now substitute here so for the lowest mode so when k is 1 then S is pi square and from here we know that F l square, now this expression we know as the Euler buckling load, so this is the Euler buckling load.

Now so the way we have right at this Euler buckling load is through the dynamics, so let us first see this so when - when we are increasing S from 0 then this omega k is a real number, so you have solutions in the form exponential so when omega k is real you have solutions, so solutions have these terms so this times time, so the temporal variation of these solutions are like this, so exponential I omega k t, so which means these are harmonically varying.

Now when this goes to 0 omega k goes to 0 then it loses the temporal nature so this is neutral stability so you have reached the Euler buckling load for k equal to 1 for example, and the beam has lost its stiffness, now if the load is increased further beyond that points so beyond this value, if increases the load beyond this value than omega 1 will become imaginary. So let us say it becomes I alpha for F greater than - so for F greater than this buckling load.

Omega 1 becomes imaginary, in that case so you have the solution as an exponentially increasing and exponentially decaying terms, now at a finite time the exponential increasing solution is going to dominate, so even for a very small disturbance initial disturbance so the solution is going to have this nature this is known as divergence instability. So this kind of behaviour is known as divergence instability.

So when the axial load exceeds the Euler buckling load for the first mode than you have this divergence instability and the beam actually buckles, so let us look at the variation of this frequency with increasing load.

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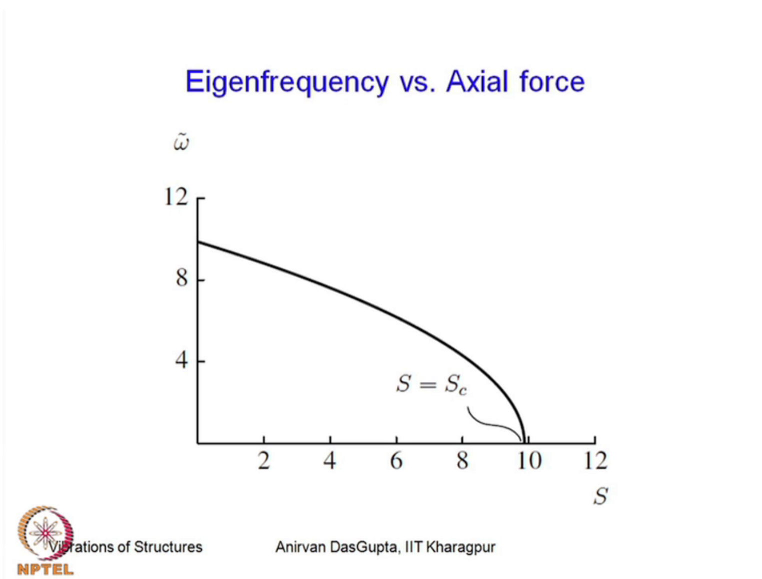

So here I have plotted this non-dimensional frequency circular frequency with the non-dimensional load so when the load is 0 you have this is the first circular natural frequency non-dimensional and as the load increases this falls and here at a certain load it goes to 0, the circular natural frequency goes to 0, so this is the buckling load and the critical load.

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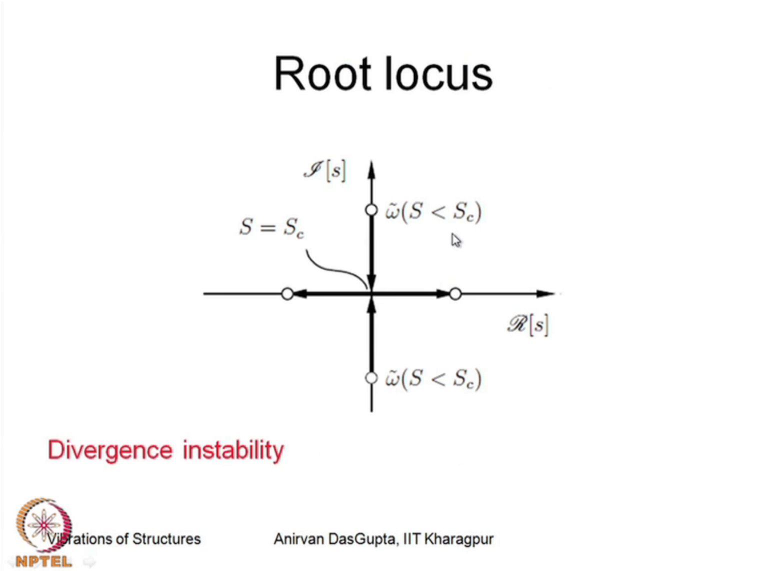

Now let us look at this picture in the root locus diagram, so s is the coefficient of the - in the exponential term in the solution so here I have plotted the real part of s and the imaginary part of s, so s is plus or minus I omega, so plus I omega tilde this is the minus I omega tilde, when the load is less than the critical load or the buckling load, as you increase the load these roots they go to 0, once they reach 0 then they move on the on this axis.

So here as we have seen so we have this plus minus alpha, now there is this route on the positive side which is going to lead to instability, so that is what we look here in the solution, so we have an exponentially increasing solution and which looks like this even with a very small disturbance it is going to exponentially diverge, so this is the divergence instability, so beam with axial force will show divergence instability if the force exceeds the Euler buckling load.

Now this can be used to estimate the buckling load of a column without destroying the column, if you if you want to estimate the buckling load of a certain column, so if you give this axial load and keep increasing it and when it reaches the buckling load this column is going to deflect have large deflection and very likely this column is going to get destroyed or spoiled you cannot use it.

But suppose you want to have if you want to device the procedure of non-destructive evaluation of the buckling load of the column experimentally so you can use this dynamic feature now you increase the load and look at the lowest Eigenfrequency and when it reduces and comes to very small value which you can tolerate.

You can estimate from there on use the expression of the variation of the Eigenfrequency with axial load to finally find out the intersection without actually doing the experiment without actually destroying the column so you extrapolate the curve but I am just shown you with load versus the natural frequency so you extrapolate the curve and you obtain the intersection with the x-axis which is going to give you the buckling load without spoiling the column.

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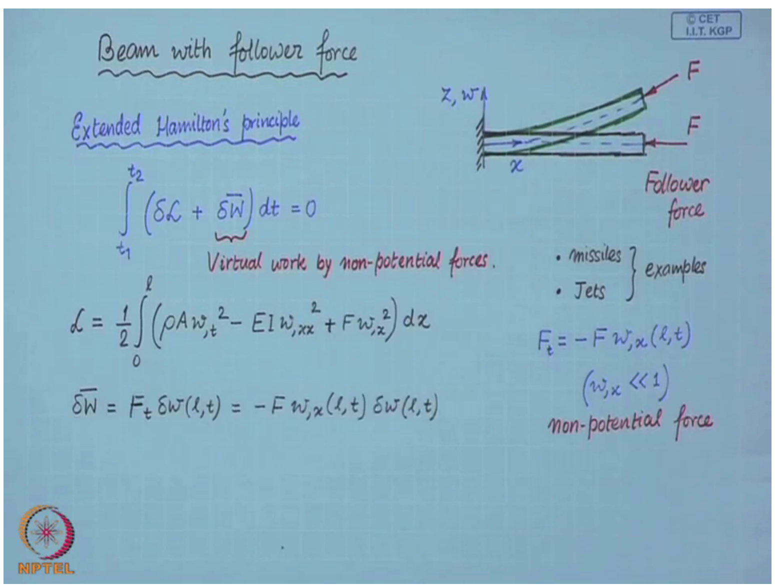

Next let us look at this example of a beam with a follower force now what is this follower force so let us consider a cantilever beam with a force at the free end this is an axial force once again, but then this force has this property that when the beam deflects so this is an exaggerated figure so when the beam deflects, the force is going to maintain the same angle as the central line of so it is always going to be tangent to that center line or the neutral fiber of the beam at the free end such a force is known as follower force.

So it follows the tangent to the neutral fiber or the central line of the beam now we need to so such examples are observed in missiles so this kind of dynamics is observed in missiles inflexible missiles if you say then also in fluid carrying jets, so when you have a pipe or a tube which has a jet at this end, so the water comes out and that gives force on the free end of this pipe and that is also a follower force so these are some examples.

Now we need to derive the equation of motion of this system one thing you can immediately see that this - this force has a transverse component which is given by, this is of course assuming that this is small so if you make this assumption that that the transfers component of this force at x equal to l, maybe written in this form now this force is a non-potential force which means that this does not have this cannot be derived from a potential it is not radiant of any potential.

So to derive the equation of this system if you want to use the variational principle to derive this equation which is safer way for this system because the dynamics is a little tricky as we will very soon see, so let us so since we have a non-potential force we have to use the Extended Hamilton's principle to derive the equation of motion for this system, so in Extended Hamilton's principle but it's says is this must be 0.

Now here this is the Virtual work done by non-potential forces, so in our case we have this as then non-potential force and we must include the virtual work done by this force, now the Lagrangian let us first write down the Lagrangian so this is one half so we have the kinetic minus potential energy, so that comes from the kinetic energy of beam element minus the potential energy due to flexure.

And then there is a potential energy because of this axial force so because of axial straining, here I have taken the sign of this F appropriately so that this indicates a compressive force, so if F here is positive then it is a compressive force, now so this is the Lagrangian of the full the beam, now the virtual work done by this non-potential force, so it’s given by this, so that turns out to be, so you have to note that this virtual work is done at the point x equal to l.

So now we put these things back in our Extended Hamilton's principle now so, if you do that.

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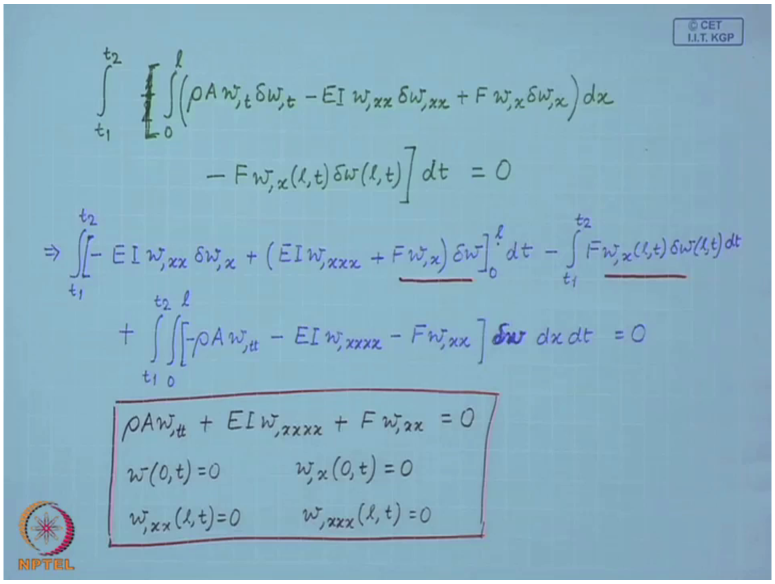

So we have so let me write out the variation directly so that is Extended Hamilton's principle, so we know that this virtual work already gives a boundary term in this - in this equation so now we integrate by parts this term with respect to time and if you use the arguments of variational calculus or the so at t1 and t2 the variation must vanish.

So I am not going to write the term the boundary term generated because of from this term at t1 t2 then this has to be integrated by parts with respect to space and similarly, this so if you carry that out so once when you integrate by parts this term two times so this is one term this is the other term with delta w and this term also gives you delta w, and then we have this already as the boundary term.

And then we have the integrant in this, so delta w taken out and that must vanish, now using the standard arguments that the variation over the domain and over the boundary must vanish independently, so we can easily see that from here we can obtain the equation of motion and then the boundary conditions considering that the beam is the cantilever beam and here if you look at these two terms now here so this term and so this term evaluated at l.

And this term is going to cancel and finally, what we have left with at x equal to l, so then the natural boundary conditions at x equal to l, so at x equal to 0, this is going to vanish at x equal to l, this must be 0 and the other term is comes from here which is the moment the bending moment. So one term at x equal to l comes from here, the other term comes from here, this term gets cancelled at x equal to l.

So this is the equation of motion and the boundary conditions for the beam for the cantilever beam with a follower force F. now it is interesting to note here you see the first of all the equation of motion is same as that with that of the beam with a normal axial load, the other thing is there is this boundary conditions are all homogeneous.

Now that is very surprising because there was a transverse force at this free end of the cantilever which is this, but then this term cancels the boundary contribution from the - from this dynamics, so - so finally all the boundary conditions are homogeneous and this is a very surprising thing in the case of the follower force beam with the follower force.

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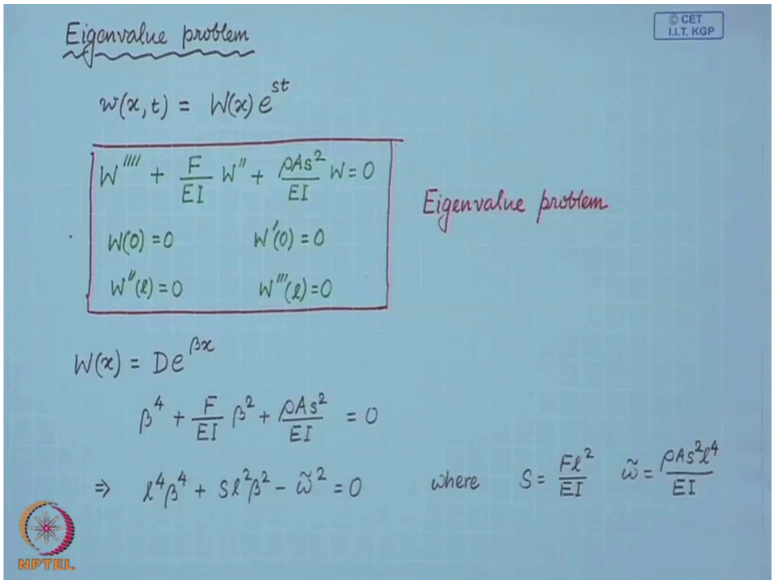

So let us now look at the Eigenvalue problem, so we look for a separable solution a complex solution of the form, let us say W x exponential s t, now when we substitute this in the equation of motion and we do the simplifications so that is the differential equation for the Eigenvalue problem and along with this we have so this describes the Eigenvalue problem. Now let us make some redefinitions here.

So - so first we let us search for solution of this W suppose we look for solutions of W in this form then substituting here we obtain, now we - we can write this as where we have use these definitions, so using these definitions we restructure our characteristic equation in this form so here we are going to solve for beta, so we can see that there will be two solutions of beta.

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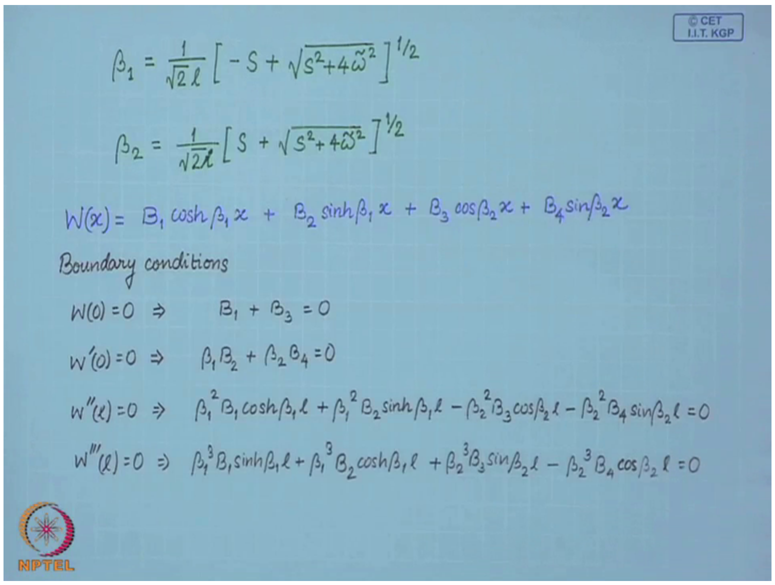

So this is one solution of beta, the second solution reads so I have written this beta 1 and beta 2 in such a way that I can write my solution of W as, so that is the solution for W general solution, now we must use the boundary conditions, so the boundary conditions as I had written down so the slope - the displacement slopes being 0, this implies, and the fourth condition, so these are the boundary conditions.

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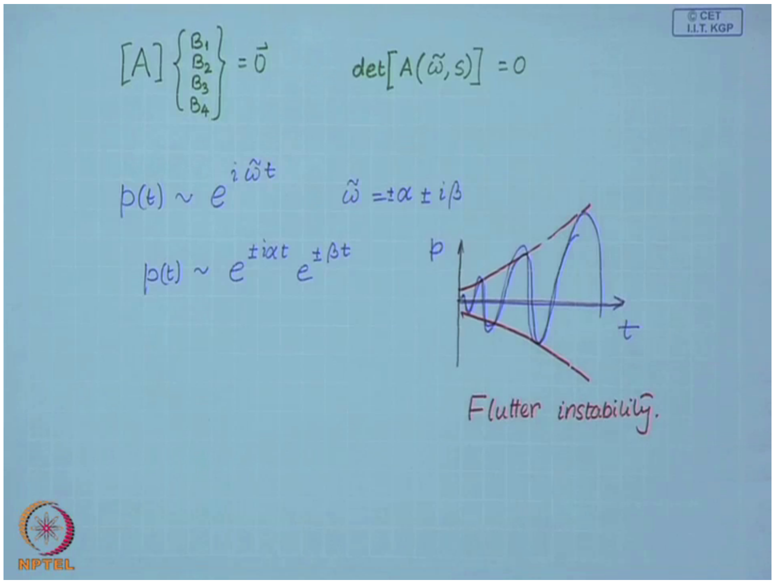

Now, so they can be recast as a matrix A times this is 0, so for non-trivial solutions of this coefficients the determinant of A and remember this A that A is the function of omega tilde and S the loading is must vanishing, so if you so you can solve for a given load S you can solve for the circular natural frequencies of the beam and this can be solved numerically.

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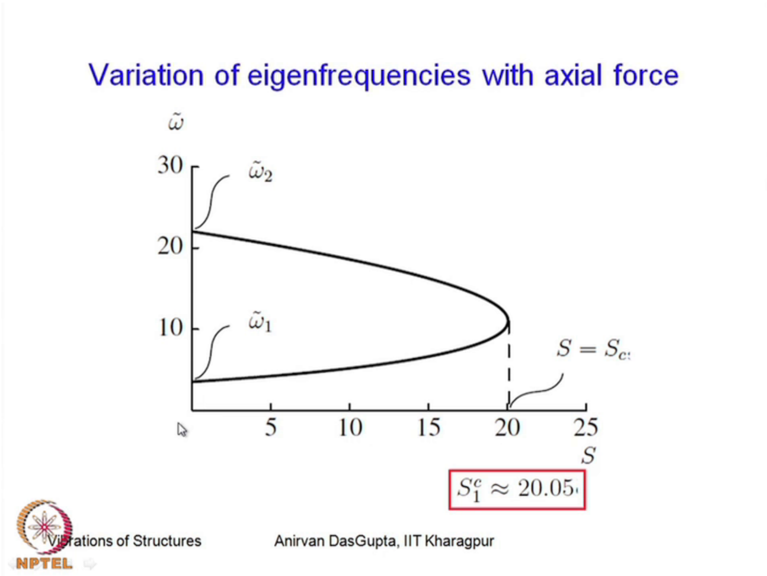

So here this shows the results of such a solutions so when the load is 0 the first two circular natural frequency non-dimensional are these two points and as the load is increased as you can see that omega 2 decreases while omega 1 increases and they collies at the certain value of load, now this is the value which is critical because after this they vanish, now to see what happens to these - these Eigen circular Eigenfrequencies.

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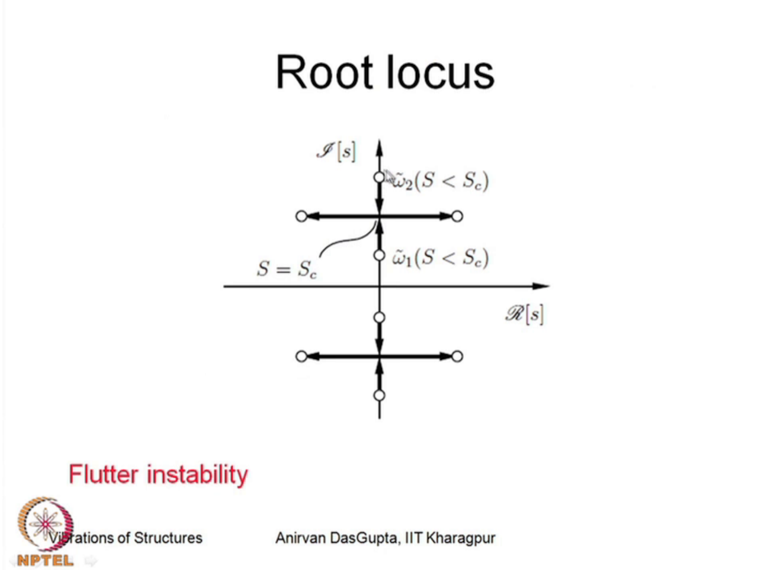

We look at the root locus which again we plot s the real part and the imaginary part of s, so remember that s is plus or minus I omega tilde, so when you have the - the load less than the follower force magnitude less than the critical value then these are the distributions of the circular natural frequencies plus or minus I omega 1 and plus and minus I omega 2, now as the load is increased these two come together and collies at a certain value which is the critical value of the load.

And then they move into the they - they have real parts as well so this branch this root has positive a real part similarly, this route has a positive real part. Now therefore you have solutions so these were the solutions now if omega becomes as you have seen in the root locus you have both plus minus alpha, plus minus I beta, so if you have if you consider this solution then you have solutions like this.

So you have exponential term along with a fluctuating component so which means the solution will look like, so this has an exponentially increasing envelope and so this is t versus the displacement, this kind of behaviour is known as Flutter and we observe flutter instability in beams with follower force, so there is an exponentially there is the fluctuation with exponentially varying amplitude.

So to summarize what we are discussed today, we have looked at the dynamics of beams with normal axial force and follower force so with that we conclude this lecture.