Vibrations of Structures
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Lecture - 22
Topics in Beam Vibrations - I

Today we are going to take up some special topics in Vibrations of beams so, the first topic that we are going to discuss is that of a tensioned beam so what happens when uniform beam is subject to tensile loading? So this is important because when we discuss strings taut strings are in tension, but as we have discussed before that there are examples. For example, the guitar string which looks like more like a beam under very low tension.

And as you make it taut, it become string, so we are going to look at in this transition of behaviour from a beam like behaviour to a string like behaviour when you put beam under tensile loading so let us look out this example.

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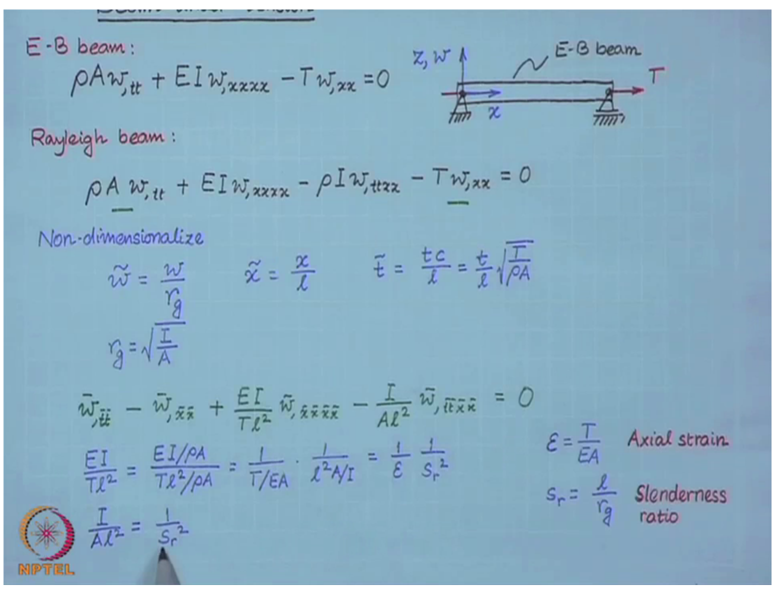

Let us consider this Euler Bernoulli beam which is subjected to a tensile loading say T, then the equation of motion so this is a Euler-Bernoulli beam so these two terms are from the beam equation now, when you have this tensile loading and then and then additionally you have so this is the equation of motion foreign Euler Bernoulli beam in case of so this is for an Euler Bernoulli beam.

In case of Rayleigh beam you have this additional term because of rotary inertia so this is for a Rayleigh beam which has this additional term because of the rotary inertia. Now since this can always fall back to the Euler Bernoulli beam if you drop this rotary inertia term so we can discuss the Rayleigh beam so let us first Non-dimensionalize the equation of motion. So if you do this non dimensionalization using this scheme.

So the special co-ordinate is non-dimensionalized with the length whereas the field variable is non-dimensionalized with this rg which is the radius of gyration which is defined as square root of the second moment of area divided by the area of cross section. Further, we non-dimensionalize time using the speed of propagation of transverse waves so if you use this non-dimensionalization scheme.

Then the non-dimensional equation of motion turns out to be in this form. Now here we make so now you see that these two terms are from the - from the string equation of motion whereas these two terms are from the flexural stiffness and the rotary inertia. Now if you look at these coefficients so these coefficients this can be written if you divide the numerator and denominator by rho A then I can recast this as where epsilon is always the axial strain in the beam.

So, T divided by EA so that is going to give us the strain because of this tensile loading and s r is defined as the length over the radius of gyration this is known as the slenderness ratio, so this is the slenderness ratio and this is the axial strain. Now so this coefficient is nothing but inverse of the axial strain and inverse of the square of the slenderness ratio similarly, this coefficient is 1 over slenderness ratio square.

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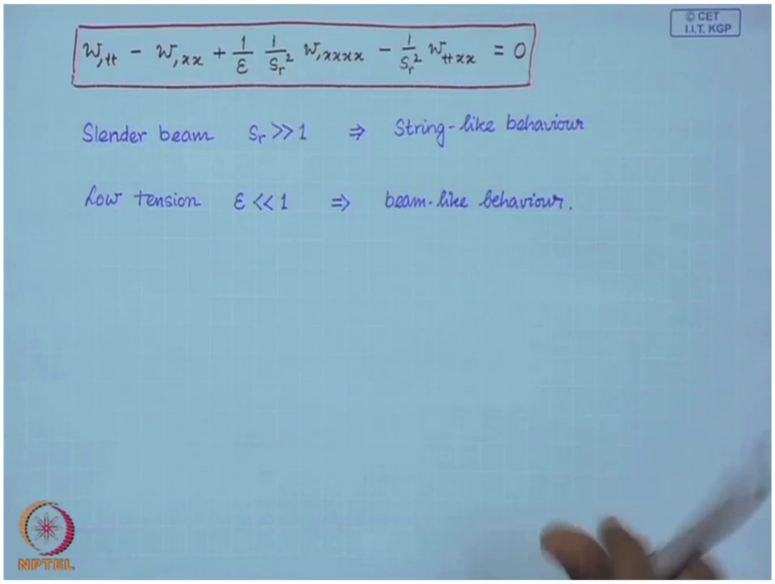

Now using these definitions then we can rewrite the equation of motion so, this is our equation in non-dimensional form and the parameters and the coefficients defined in a very special way so here I have of course drop the tilde symbol for simplicity, now let us look at these terms so this is the slenderness ratio which indicates how slender the beam is so it's the length over the radius of gyration.

So if the beam is very very slender then the slenderness ratio is very large because its length is much larger than the area property so when the beam is very very slender then this is very large and therefore these terms can be dropped in comparison to this term so in that case we expect the string like behaviour so in other words you can treat a beam under tension very slender beam under tension like a string.

On the other hand, if this strain is very small so the axial strain is very, very small so if the tension is small and the axial strain is very small in that case this term might become significant in that case you may have to include this flexural term in the equation of motion so this rotary inertia term therefore is negligible. In that case of slender beams whereas this term is negligible if the - if the beam is again slender in the tension is very high.

So the tension is very high and the beam is slender then these two terms can be dropped therefore it will have a string like behaviour or the system can be modelled as a string where as if tension is very small then this flexural term may be significant so this must be considered so we you tend to more like a beam like model. So this is precisely what we observe from this equation and elements like the guitar string which are extremely slender.

When they are put even though they have sufficient bending stiffness, but when they are subject to a large tense - tension in that case the behaviour is like a string so you can treat guitar string under tremendous amount of tension as the string so this is what we have observed. So another way to look at it is that the restoring force in a - the restoring force in a guitar string is more because of tension then because of its flexural rigidity so or flexural stiffness.

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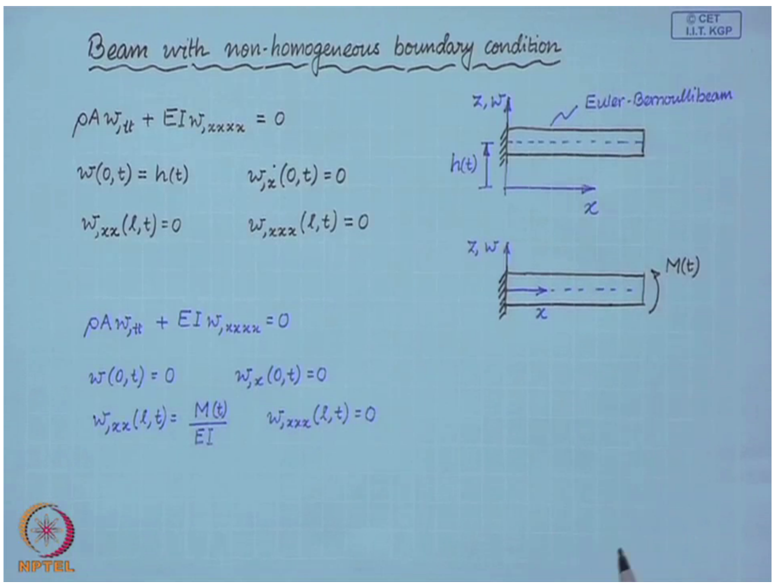

Next let us consider a beam - beam with non-homogeneous boundary condition, so in - in various situations you find beams which are excited at the boundary so, for example when you have an absorber attached to a vibrating structure an absorber made of a continuous system like a beam which may be attached to a vibrating surface or structure for the purpose of vibration absorption, so usually these beans are like cantilever beams.

So they are fixed on the structure and they vibrate because of base excitation, so let us consider this example of a cantilever beam, so this is the cantilever beam which is subjected to base excitation, so we can consider that this motion of the base is given by h of t so this is the time varying function which is specified let us say and we will consider that this is a Euler-Bernoulli beam.

So the equation of motion is given by this and the boundary conditions, so the displacement of this boundary suppose given by this function h of t at x equal to 0, we also have this condition that the slope is 0, whereas for the free end we have the bending moment as 0 and the shear force is 0, now if you have so this is one kind of non-homogeneous boundary condition you can have other kinds.

For example, you can have a cantilever beam once again, but let's say with the time-varying moment at the free end, so in this case we can write the equation of motion and boundary conditions, so once again at this free end now they are both 0, the displacement and the slope whereas the bending moment at condition at this free end maybe written like this and the shear force at the free end is 0.

So this is another kind of non-homogeneous boundary condition, where you have the force here it was the displacement here it was the force now when you have - when you have force it is possible to include it in the equation of motion, so we are going to first discuss this - this kind of system where we have excitation boundary excitation in terms of displacement when you have forces again it is possible very easily to include this force in the equation of motion.

And then we can treat it as system with homogeneous boundary condition now why do we need this because whenever we have solved the Eigenvalue problem we have considered homogeneous boundary conditions usually, so if you want to use this modal expansion theorem for solving the system in such cases you will require that the boundary conditions be homogenized first.

Otherwise, the modal expansions the Eigenfunctions do not satisfy will not satisfy the boundary conditions with time-varying functions, so we would like to have a method of homogenizing the boundary conditions for the problem, so let us look at this problem first so we have this cantilever beam with base excitation.

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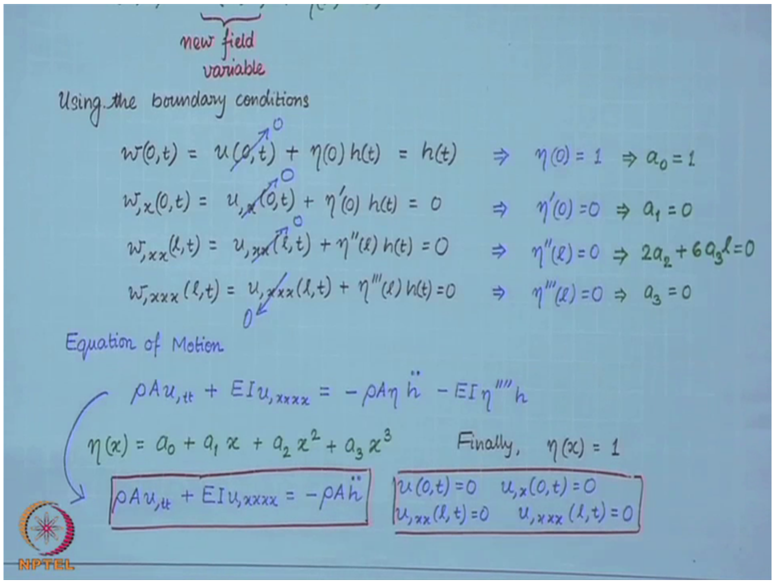

Now let us try to make transformation of variables so, let us say that our field variable w we want to convert or transform to another field variable let's say u x, t + an unknown function of x times h of t so, which is this function that we have as the forcing, now this structure is of course motivated by the structure of the boundary conditions, now if you consider this - this new so this is a new field variable and this function is at as yet unknown so this is an unknown function.

Now let us use this transform - this transformation in the equation in the - in the boundary conditions, so the displacement boundary condition so this must be h of t the slope condition so the prime here denotes derivative with respect to x that must be 0, now the bending moment condition is 0 and similarly, the shear force condition at x equal to l is also 0, now we would like to have so this is our new field variable.

So the equation of motion, so if you substitute this in the equation of motion that is equal to, so minus of rho A eta h double dot that comes from the inertia term and similarly, this comes from the flexural stiffness term, now we would like to have homogeneous boundary conditions for this equation of motion now the field variable is you and the right hand side is now we have in terms of the time varying function h and as yet unknown function eta.

So if you want to have homogeneous boundary conditions for u which means that this must be 0, so these are all 0, so this implies eta of 0 is 1, now we have to construct a function eta which satisfies these four conditions now to keep things simple we can consider a polynomial, in this form and if you now use these conditions so this will imply a 0 is 1, eta prime 0 is 0, so this will imply a 1 is 0, eta double prime l if that is to vanish then twice a 2 + 6 a3 l must vanish.

And, if the triple derivative of eta is to vanish then a 3 must be 0, so in that case a 2 is also 0, so therefore eta is nothing but 1, so this is a very simple example, so in that case finally, and therefore the equation of motion so what we obtain from here so this is the equation of motion and this is accompanied with accompanied by boundary conditions which are all homogeneous so these are the boundary conditions.

So we have now a system which is forced, but as homogeneous boundary conditions, if you perform therefore the modal analysis of the unforced system which is nothing but normal a cantilever beam then you can use the Eigenfunctions of the unforced to system to solve the forced vibration problem, so we can easily convert any in non-homogeneous problem with non-homogeneous boundary conditions to problems with homogeneous boundary conditions.

Next let us look at this important phenomenon of Damping so as we have discussed before as well we can have two kinds of damping in continuous systems one is the external damping which is more common or very easy to see and then there is an internal damping which is very actually hard to model and also very less understood this is because it depends on the material constitution the internal structure of the material.

Now these internal damping we are all considering throughout one dimensional elastic continuum so beam is also one dimensional elastic continuum so the source of this internal damping is usually friction between the layers of the molecular layers inside the structure or inside the material so when we are saying we are considering one dimensional continuum then there is no question of having layers.

So we must introduce this internal damping in phenomenal logical way, so to do that we follow the Kelvin voigt model so let us look at this model for internal damping.

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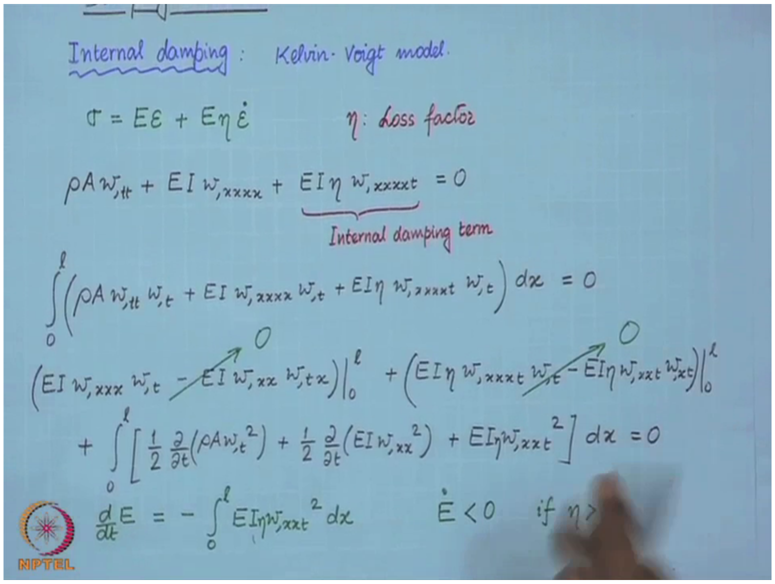

So first we are going to look at internal damping, so to model internal damping we use the Kelvin voigt model so this in order to implement this model we modify the constitutive relation of the material so we know we have used that for our beam model the Hooke's law given by the stress equals the Young's modulus times the axial strain, so this was the axial stress in the fibers of the beam.

Now in addition to this the stress because of the mechanical straining we add the second term which is dependent on the strain rate, so the stress not only depends on the amount of straining in the fibers but also the rate at which the fibers are being strained, here this eta is known as the Loss factor, it is written in a special way in order to match this term, so e times sum eta times epsilon dot.

So if you use this model to derive the equation of motion then you can easily see that the equation of motion, so these are the two terms that we already have in the Euler Bernoulli beam now because of this additional term you can expect that there is going to be, so this - this is the new term because of internal damping.

Now to see that this is really a damping term so we multiply this whole equation with the velocity del w del t and integrate over the domain of the beam that's the length of the beam so we obtain and integrate this and that must also be 0, now we integrate by parts this term two times so and similarly, this term also integrates by parts two times and the rest of the terms in the integral I can write this term as.

Similarly, after integrating by parts this term two times so this is what we are going to get, now if you use the boundary conditions say for example in a simply supported beam then at l the velocity must be 0 at and the bending moment must be 0, so this term will vanish similarly, this time will also vanish, so using the boundary condition you can show that these boundary terms will actually vanish.

So therefore we are left with this integral equal to 0, now you can easily identify that these terms these two terms are nothing but the total energy the rate of change of total energy half rho A del w del t whole square is the kinetic energy, this is the potential energy so this is the total energy and this term I am going to take on the other side.

Now if there is an eta as well so now E is positive, I is positive, if eta is also positive and this a squared quantity so this is the positive quantity so integrated over the length of the beam so this must be positive quantity if eta is positive and therefore the energy the rate of change of energy is negative which means that energy always reduces so if this loss factor is positive then this term is going to drain the mechanical energy.

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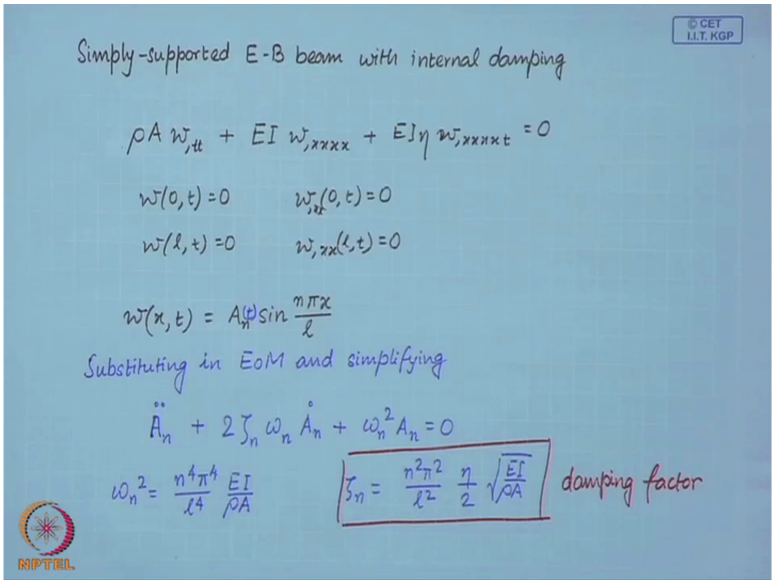

Now let us look at a Simply supported Euler Bernoulli beam with internal damping, so the moments are 0, now we have seen that for an undamped beam the modal vibration is given by, let us for a moment assume that this beam is vibrating in this nth mode so if I substitute this in the damped equation then I can rewrite if you substitute this in the equation of motion and simplify.

If you simplify then, so this can be in general function of time you can recast the equation of motion in this form where so this is what is interesting to observe so if you consider that the beam is vibrating in the n th mode than the equation of motion for that mode the - the modal co-ordinate the modal dynamics is governed by this equation where the natural frequencies as suspected but there is a damping factor which we find is proportional to n square so higher the mode higher will be the damping factor.

So which means that the higher modes get effectively damped because of internal damping. You can repeat this analysis with external damping and you can easily show and this we have seen before as well, in the case of strings and bars, that with external damping the lower modes are more effectively damped then the higher modes. Same conclusion can be drawn even for this beam vibration where we find that internal damping is effectively damping the higher modes.

And the external damping will effectively damped the lower modes. So to conclude today, what we have discussed we have considered two topics from beam vibrations three topics. So first we have considered the non-homogeneous we have considered non-homogeneous boundary conditions, we have considered internal damping in beams and at in the beginning.

We also looked at the beam under tension, so how the string and the beam behaviour gets delineated. So that was another topic that we looked at, so with that we conclude this lecture.