Vibrations of Structures
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Lecture – 21
Approximate Methods

In the previous lectures, we have been discussing about the model analysis of beams and what we observe that even for simple configurations of beams or simple beam models, you can have fairly complicated eigenvalue problem, which we have to solve in order to accomplish model analysis. So, one would be interested in knowing if they are approximate methods which can quickly tell us, give an estimate of the eigenvalue frequencies.

And the modes of vibration of a continuous system and for example for beams. Now in our previous lectures, we have discussed some of these methods which are used for approximately performing the model analysis and as we have discussed that these methods can be improved to, improved the accuracy of analysis. So today we are going to look at some of these approximate methods applied to beams.

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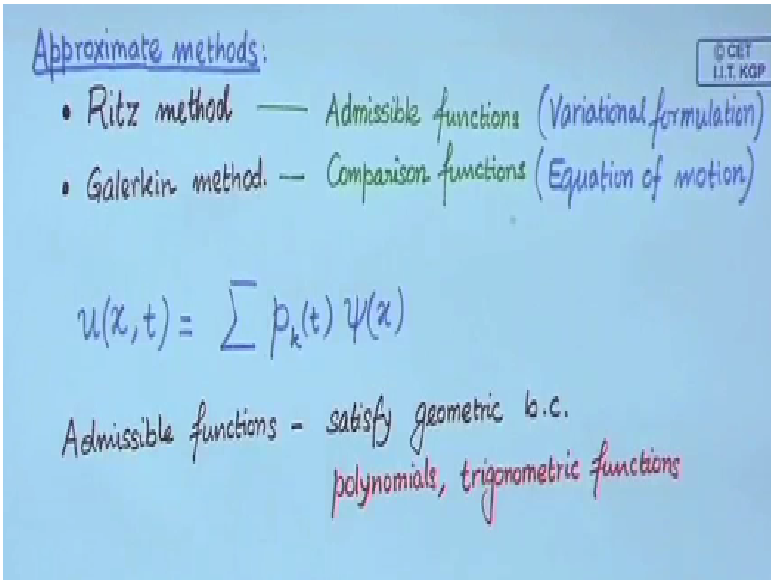

So the first example that, so let me first enumerate the various methods that we have discussed that we will use also in the case of beam. So for example, we have use the Ritz method, so these are all, so the Ritz method we have also looked at; now in the Ritz method, what we need? We need admissible functions, so we expand the solution in terms of this admissible functions.

On the other hand, for the, in the Galerkin method, we use comparison functions. So suppose we have a field variable, u that we expand in terms of these spatial functions which in the case of Ritz method, these are admissible functions, on the other hand in the case of Galerkin method, these are comparisons functions and then in the Ritz method, we use the variation formulation.

So we substitute or replace our field variable in directly in the vibrational formulation of the problem with this expansion. When the Galerkin method, we do, we work with the equation of motion. So these have its own advantages and disadvantages, for example in the Ritz method, it is sometime tricky to consider non potential or non-conservative forces, while it is much easier with the Galerkin method.

On the other hand, for Galerkin method, this comparison functions they have to satisfy all the boundary conditions of the problem, which are more difficult to construct, while this admissible functions must satisfy only the geometric boundary conditions. So this is an advantage of admissible functions and these can be very easily constructed using polynomials or trigonometric functions or other such elementary functions.

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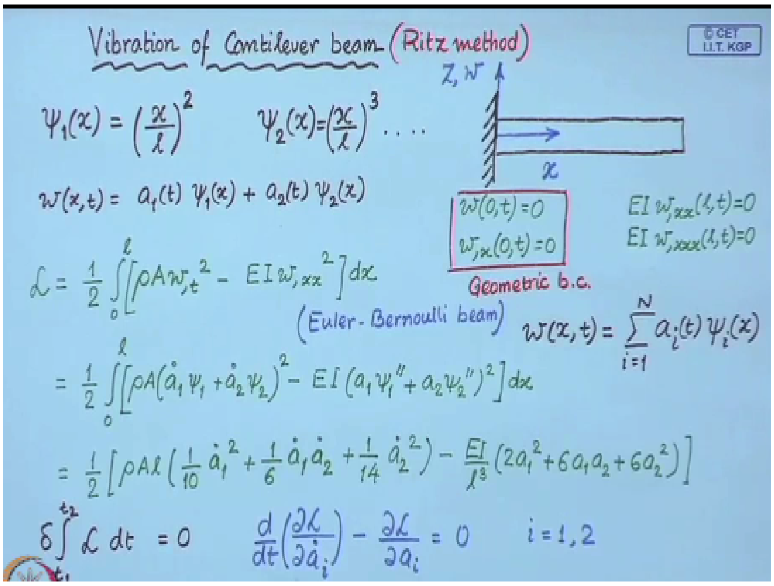

Now, so today we are going to look at these application of the Ritz method for certain problems in beams. So the first problem, is that of vibrations of a Cantilever beam, so let us consider this Cantilever beam, now the boundary conditions for this beam, we have discussed this before, we have the displacement at this point to be 0 and the slope also is 0 for all time.

On the other hand, on this free hand of the Cantilever beam, we have the bending moment to be 0.

And we also have the shear force at this free end to be 0. Now so the, since we are applying Ritz method and in the Ritz method, we must we use admissible function which must satisfy the geometric boundary conditions, now these are the geometric boundary conditions. So whatever admissible functions we choose, they must satisfy these conditions. So let us consider admissible functions.

So if I consider a function like this, so remember that we are going to use this, we are going to use an expansion like this, so we must choose our admissible functions, which must satisfy the geometric boundary conditions of the problem. So if we consider this to be, let say linear in x, then at x=0, this is satisfied, so psi 1, 0 must be 0, but when we look at this boundary condition, which is a slope condition, so del del x of psi 1 at x=0, must also be 0.

But if we choose the function like this, then this boundary condition will not be satisfied. So from these considerations, one can easily come to the conclusion that this must be the function, one of the functions that can be used as an admissible function. Then we can use the higher powers of etc, so let us first begin with only a 2 term expansion, so which means, so first use, we will first use this 2 term expansion.

So a1 and a2 are the 2 temporal coordinates. Next we introduce this expansion in the Lagrangian, which reads, so this is the Lagrangian of Euler-Bernoulli beam. So let us consider an Euler-Bernoulli beam the Lagrangian is given like this, now we substitute this expansion in here and what we obtained if you and if you simplify this further, so you substitute these expression of psi 1 and psi 2 and perform the space integration.

That means integration over x, these are polynomials, they can be integrated out easily the final result. So this is the Lagrangian that you have, now this is the Lagrangian of a discretised system with coordinates a1 and a2. Now we can write down Hamilton's principle for this. So this will give us the equations or equations of motion and you know that this is going to lead to the Euler–Lagrange equations which, so these are the equations for the 2 coordinates a1 and a2.

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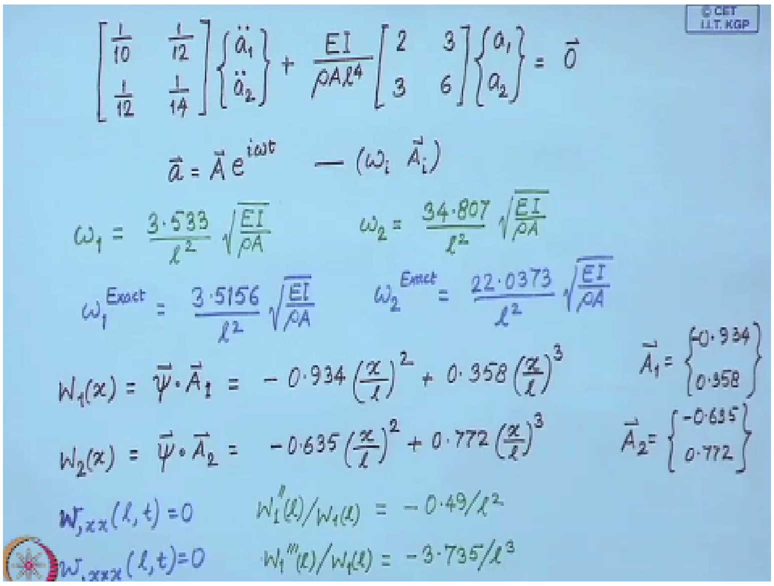

And when you derive these 2 equation, they are of the form, so these are; I mean this is the discretised equation for the Cantilever beam. Now we perform the standard model analysis for this discretised system and we can calculate the Eigen frequencies, the circular Eigen frequencies and the modes of this Eigen vectors, which can be used to determine the modes of vibrations.

So let us first look at the Eigen frequencies, so when you do this calculation, so this is the first circular Eigen frequency of the Cantilever beam, calculated from this discretised equation and the second one is obtained like this. Now if you do the exact calculation, so which we have discussed before, so this is the exact this turns out to be; now you can make a comparison.

So while these fundamental circular Eigen frequency compares very well with the exact, the second circular Eigen frequency this is on the higher side, so as we have discussed before, this Ritz method gives us an upper bound on the Eigen frequency. So the what; when we calculate by this approximate method we are going to get this omega 2 and what this tells us is the actual Eigen frequency is less than this value.

Similarly, her also you can see the actual Eigen frequency is less than this value, so this is an upper bound property of this Eigen frequencies calculated from the Ritz method. Now let us look at the Eigen functions, so when we substitute here, we are going calculate omega and A, so the Eigen pairs. So we are going to get this Eigen vectors and using this Eigen vectors, we actually construct our Eigen functions using the expansion that we have used.

So we do a dot product, so the first Eigen vector that we get with corresponding to mega 1, which is A1, so if you dot product with the vector of the admissible functions, so you get the first Eigen function and this turns out to be; so this A1 vector r is actually, so this was the A1 vector, we take the dot product with the psi vector. Similarly, the A2 vector was actually this, so this is our Eigen function.

Second Eigen function. Now as we discussed that these admissible functions do not satisfy, I mean they are not required to satisfy the natural boundary conditions which in our case of this Cantilever beam, these are the movements and the shear force and the bending moment being 0 at x=l. So let us look at, so what we have is; this must be 0, but since, so this was the bending moment condition, this was the shear force condition.

Now let us see how well these Eigen functions satisfy these conditions. So if you calculate, for example, w1 double prime at l, that turns out and divide this by w1 at l, that turns out to be and similarly so we are trying out with the first Eigen function, we take double derivative of that and see how close to 0, this is. Now as you can see with increasing length of the beam, this is going to go to 0, quiet fast and similarly for the shear force.

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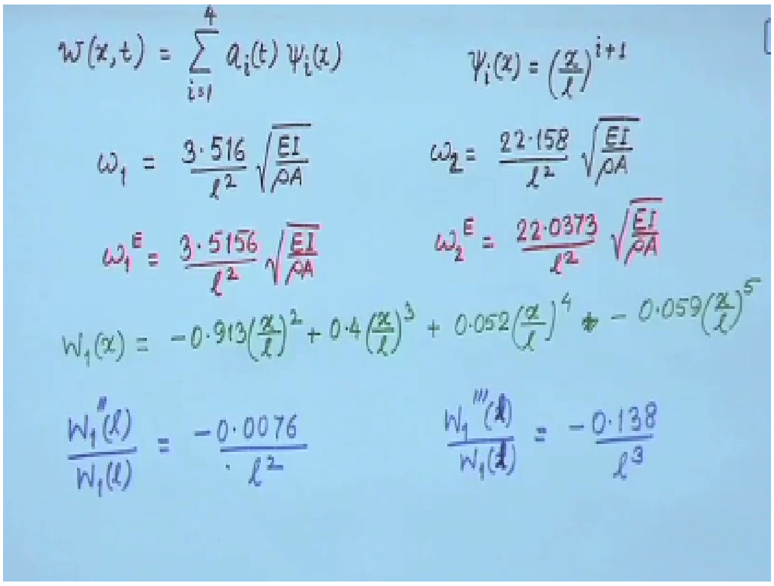

Similarly, you can do for w2, the second Eigen function, but since our; we are more confident about our first Eigen functions so here I have taken this example of the first Eigen function. Now but I mean this may or may not be satisfactory for the purpose, so what we can try out is we can increase the number of terms in our expansion. So in the second example, I have considered with 4 terms in the expansion.

And I have taken the admissible functions in this form, so psi 1 is the square of x over l, so I have gone up to 4 terms in this expansion and when I calculate the; following the procedure that we just discussed, if you calculate this Eigen frequencies, so this turns out to be and remember the exact was, so this one, so now you can see that with 4 term expansion, we have comparative close to the exact solution.

And now once again if you calculate the first Eigen function this turns out to be, similarly you can calculate the second Eigen function and third and fourth and now we are focussing on the first again let me calculate this ratio, which will tell us, how far the natural boundary conditions are satisfied at the free end, so these are at l. So now you can see with increase in the number of terms in the expansion.

Even the natural boundary conditions at the free end, which are the binding moment and the shear force, they are also going to 0 quiet rapidly. So as you increase the number of terms in the expansion, you are going to get accurate solutions of the Eigen frequencies as well as the Eigen functions will also get more and more accurate and they are going to automatically satisfy, they will tend to satisfy the natural boundary conditions which you have neglected while doing this expansion.

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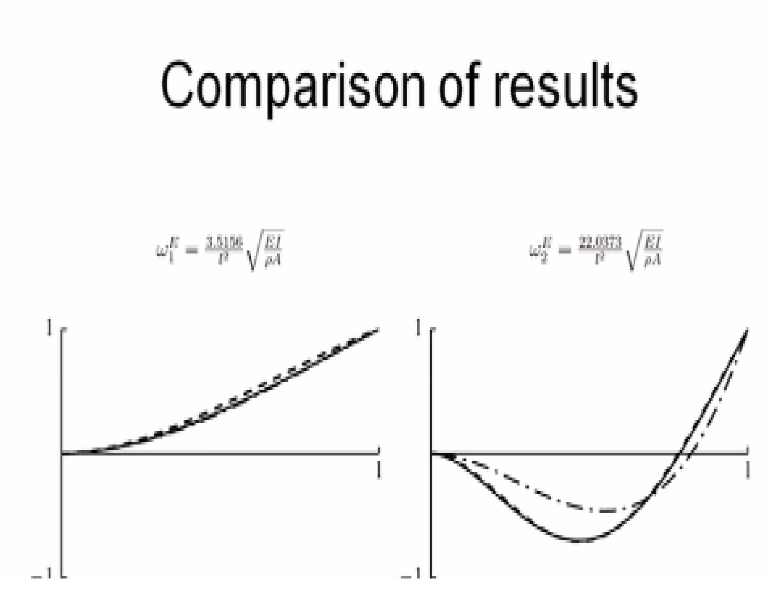

Now let us look at these Eigen functions, which I have plotted here, so this is the first Eigen function, the solid line is the exact and this chain dotted line is with, with 2 term expansion and with 4 term expansion you have this dashed line so you can see that the Eigen functions they also tend to go close to the exact Eigen functions, which we have discussed in one of our previous lectures.

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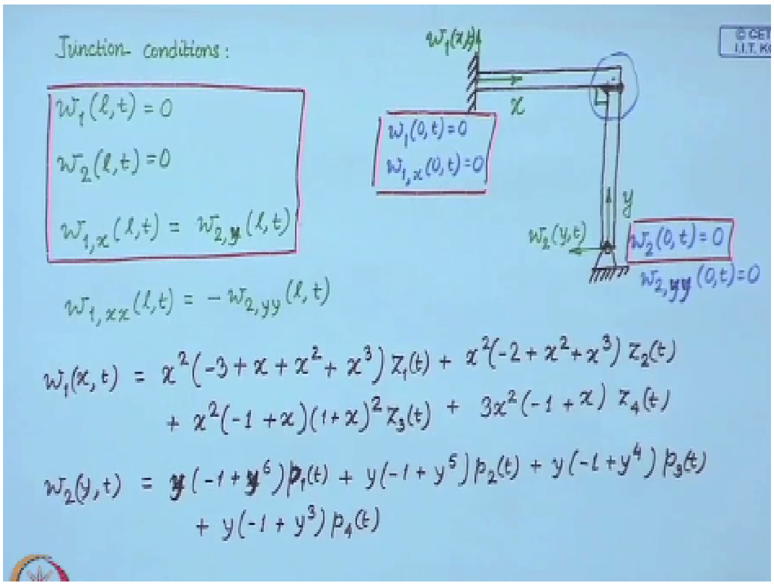

Now let us go over to a second example, this example is of a plain frame, so let us look at this plain frame. So we have this plain frame constructed out of 2 beams, which are welded at this point, so for simplicity, we consider that the lengths of both these beams is the same, so we have l and l. Now here we have built in end of this frame and here it is a pin, pin end. Now these are essentially 2 beams, which have a junction.

So we must; we can treat them like that, so let us consider that the coordinate here is x and the displacement in this direction for this beam horizontal beam is represented by w1and this coordinate is y and the field variable for this vertical beam is w2. Now we intend to determine the Eigen frequencies and modes of vibration of this frame. Let us first write down the boundary conditions.

So at this built in end, so these are the boundary conditions at the built in end add the pin support, we have displacement as 0 and the bending moment and the coordinate is y, so this is 0. Now along with these boundary conditions, we also have this junction. So what are the conditions at this junction? So the first condition if we say, consider this beam the horizontal beam then there cannot be any vertical displacement of this beam at this point.

Assuming that there is no axial or this beam is axially rigid, so there is no axial displacement at this point. In that case, this point of the horizontal beam cannot have any displacement. Similarly by similar reasoning, for this vertical beam cannot have any displacement in the horizontal direction. Now since this point is welded, so these 2 beams are welded at 90 degree.

So under deflection as well as this angle has to be maintained, which means, so this is the slope condition. These 2 slopes they must maintain a certain relation. The second condition is on the bending moment, so this is course y, so there must be an equilibrium, so from those considerations, we can obtain this bending moment condition at this junction. So now we have all the conditions required for this plain frame.

Now let us identify the geometric boundary condition, so here, so these are the geometric boundary conditions for the problem. So we must satisfy, so when we are following the Ritz method, we have to satisfy these geometric boundary conditions and the others, the natural boundary conditions are not so much essential. So let us now consider this expansion. I will write out this expansion, which has been constructed using polynomials, etc.

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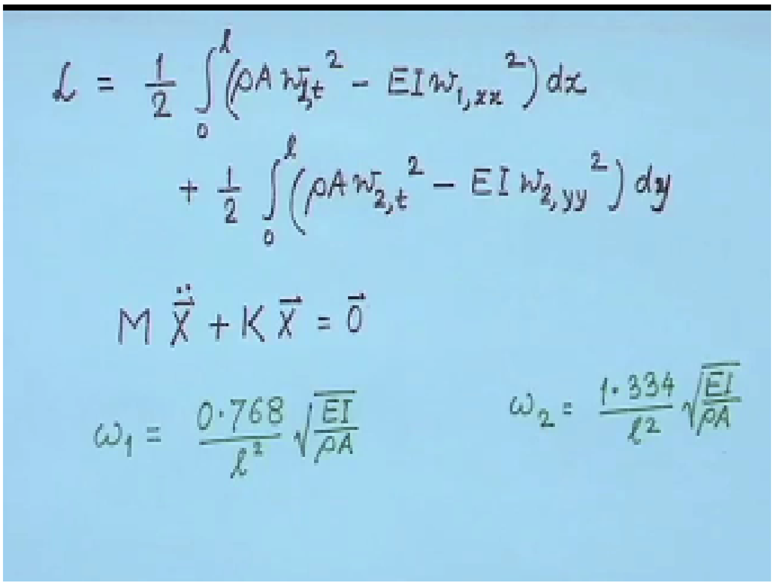

So there can be various ways of constructing this expansion, these individual polynomials which satisfy the geometric boundary conditions, so these are etc. So I have considered these expansions, but I mean, they can be these admissible functions but these functions can be construct in various other ways. Now using this expansion, these 2 expansions of the field variables, we write out the Lagrangian.

So we have written out the Lagrangian for this individual beams and when we substitute these expansions in this Lagrangian and integrate out the space part, so here we integrate over x, here we integrate over y and we obtained the discretised Lagrangian from where finally as we saw in the previous example, we are going to get the discretised equation in this form. Now these are the matrices; the mass matrices and the stiffness matrices; stiffness matrix.

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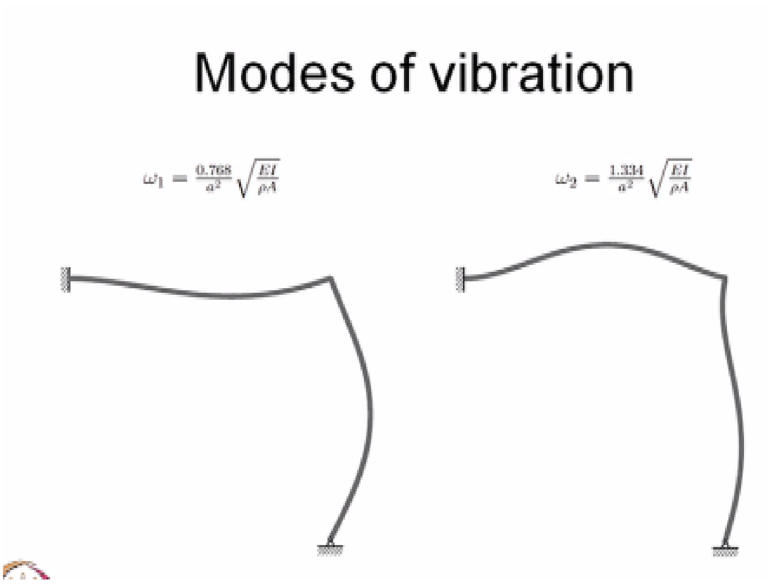

And we again perform the model analysis for this discretised system and if you do that then the result for the first 2 modes, so these are the first 2 circular Eigen frequencies of the system which has been calculated using this Lagrangian and expansion that I discussed just now. So this figure shows the first 2 modes of vibration of this plain frame, so you can see that, so this is the fundamental frequency and the corresponding Eigen mode of vibration.

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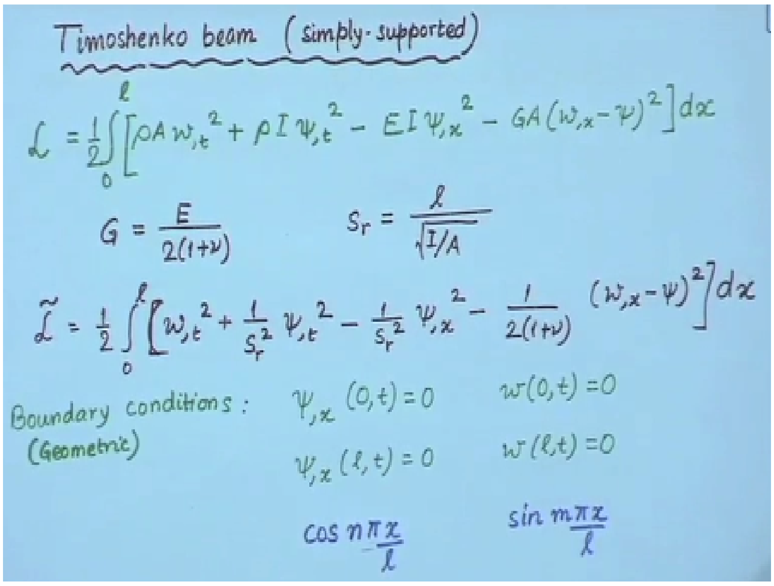

So you can see that this angle of 90 degrees being maintained in both these cases, since we have chosen our admissible function which satisfy these geometric boundary conditions already, so this was an example of a plain frame. Next we look at this Timoshenko beam, which is little more sophisticated model for beam, which considers also the shear deformation of the beam. So we will consider a simply supported Timoshenko beam.

Now if you recall the Lagrangian of the Timoshenko beam is given by this expression. Now in order to simplify this, we use the definition of the shear modulus and we also define what is known as the slenderness ratio. In that case, the Lagrangian gets simplified. So with these definition, and we can take out the material constants out and simplify, so this actually is a L tilde.

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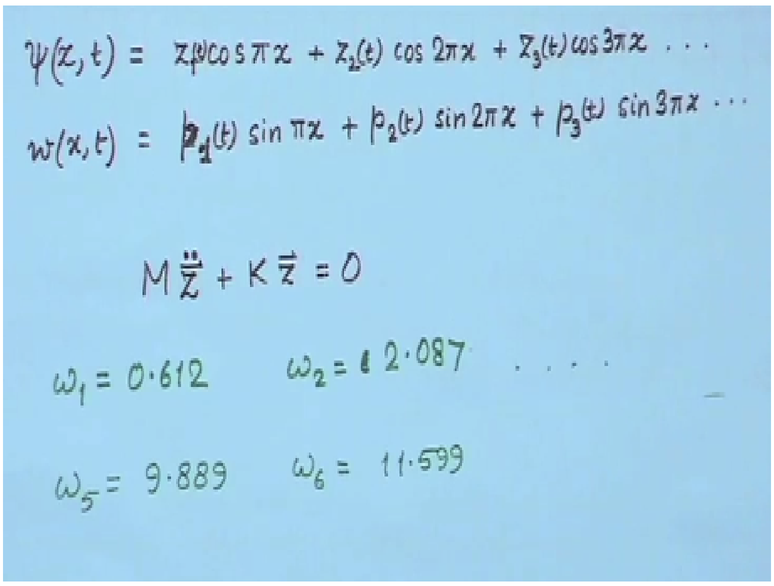

Now we can, let us look at the boundary conditions of a simply supported team Timoshenko beam, so the boundary conditions are obtained through variations, you can write them as, so these are the geometric, which we need to satisfy when we are performing the Ritz analysis. So in order to satisfy these boundary conditions, we can choose, we can expand these field variables.

For example, psi can be expanded as, etc and w, can be similarly expanded, etc. So you can see that these boundary conditions actually can be satisfied, so psi for example, can be satisfied by using an expansion like this, whereas so if the length is l, then one can use the expansion like this. So using these admissible functions, we have expanded the field variables and finally after applying the variations etc, we will obtain the discretized equations of motion.

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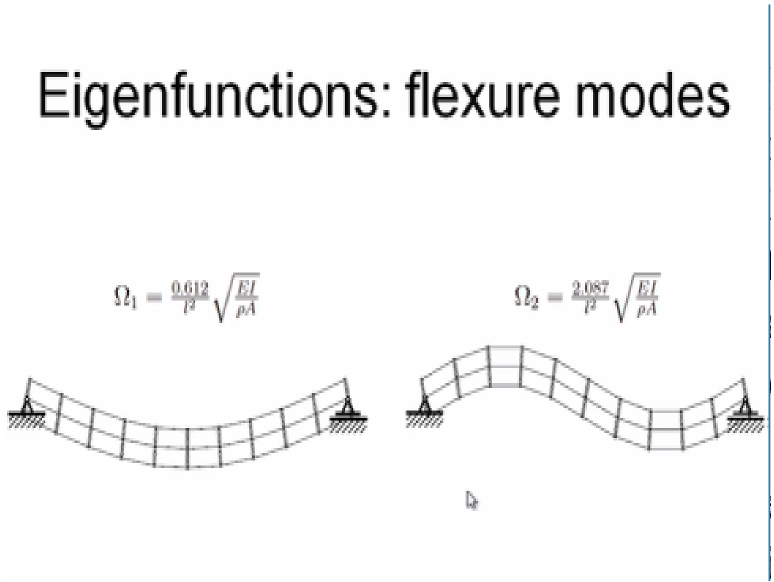

Now if you perform the model analysis of this discretised equations, then the first non dimensionalised circular Eigen frequency is obtained as 0.612, second one is obtained as 2.087 and so on. The fifth is obtained as 9.889, the sixth is obtained as, now there is a reason why I am writing 1,2 and then 5, 6, there are of course other circular Eigen frequencies in this range but let us look at the Eigen functions, which are shown here.

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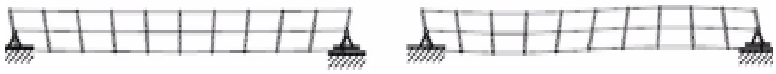
Eigenfunctions: shear modes

$\Omega_5 = \frac{9.889}{l^2}\sqrt{\frac{EI}{\rho A}}$ $\quad$ $\Omega_6 = \frac{11.599}{l^2}\sqrt{\frac{EI}{\rho A}}$



So this is the first Eigen function and you can see the first circular Eigen frequency, so you can see the mode of vibration in the first mode for the Timoshenko beam and similarly this is the second circular Eigen frequency and the second mode of vibration. So these 2 look very similar to a normal beam. Now let us look at this fifth and the sixth, now here there is hardly any transfers displacement, it is very small, not visible in this figure.

These are actually the shear modes of the Timoshenko beam and these frequencies are substantially higher. So what we have looked at in this lecture today, we have discussed about the approximate methods for model analysis for discretisation (()) (55:14) we can discretise the equation of motion of the beam and we have use the Ritz method for discretisation.

One can also use the Galerkin method in a similar manner, the only thing is in the Galerkin method, since we use comparison function, so they are little more cumbersome for calculations as to construct. On the other hand, the Ritz method, we have seen the admissible functions are very easy to compute and if you increase the number of turns in a expansion, then you can also satisfy the;

Or you can also make this natural boundary conditions, which are neglected while constructing the admissible functions, so you can make this natural boundary condition also to be 0, so the satisfaction of the natural boundary condition, so with increase in number of terms, you can satisfy these natural boundary conditions better. So with that we conclude this lecture.