Vibrations of Structures
Prof. Anirvan DasGupta
Department of Mechanical Engineering
Indian Institute of Technology – Kharagpur

Lecture - 20
Application of Modal Solution

In the previous lecture, we have discussed the modal analysis of beams. So performing modal analysis is nothing but solving Eigen value problem as we have seen and what we obtained, we obtained the circular Eigen frequencies or the natural frequencies of the beam. And we also obtain the Eigen functions, these Eigen functions they define or they describe the modes of vibration of the beam.

Now this Eigen functions they form a complete basis for the shape of the beam. So what we mean by a complete basis is something like this.

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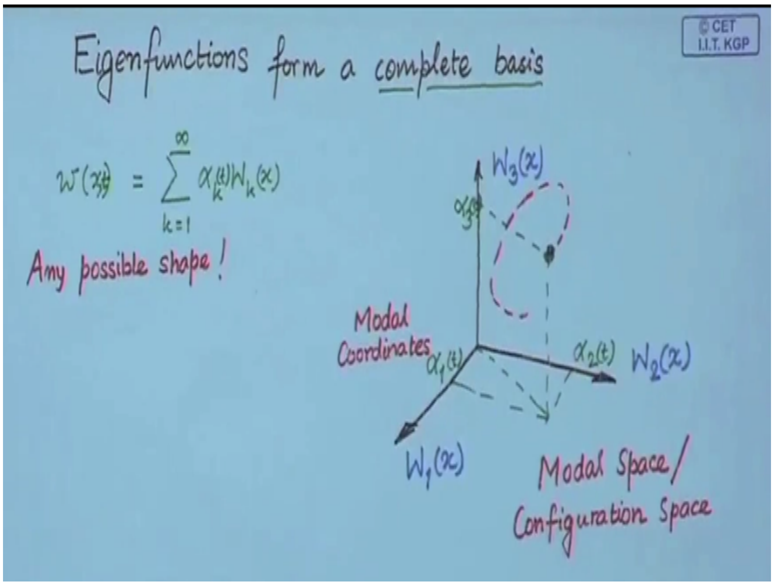

So what we mean by this? Suppose we have these Eigen functions, let us call them. Now there are infinitely many Eigen functions as we have seen. So here I can draw only three and with a little stretch of imagination, you can think of this as an infinite dimensional space with each axis labelled with one Eigen function. So any point in this infinite dimensional space with coordinates alpha 1, along W1, alpha 2 along W2, alpha 3 along W3, etc.

So this describes the shape of the beam. So any shape of the beam may be represented as a linear combination of these Eigen functions. Now when we say this forms a complete basis, it means that any possible shape of the beam can be represented using these Eigen functions. So any possible shape can be represented in this form, then we say that this form a complete basis.

Now so these Eigen functions, therefore give us good way of representing solutions, which may also be functions of time. So suppose if these are coordinates which may be in general functions of time, in that case the motion of this point in this space may be represented through this expansion where now these alpha case, they are functions of time. So they form what are known as modal coordinates.

So and this space is known as the modal space or the configuration space. Now this fact that any shape or any dynamical shape can be represented through this expansion allows us to solve the number of problems related to vibrations of beams. And this fact we have also seen in case of strings and this is true in general for continuous systems. So let us today look at two such examples.

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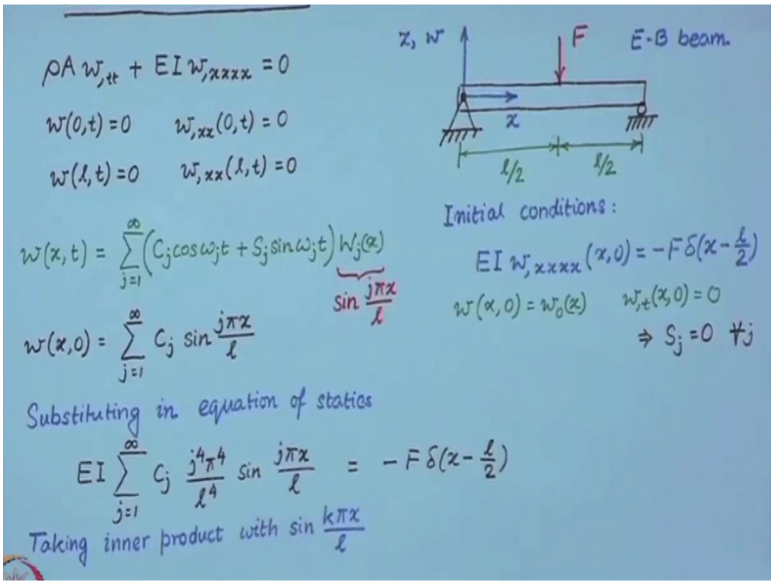

The first we look at an initial value problem for a beam. So the problem that we consider, we have a simply supported Euler–Bernoulli beam. This beam is initially loaded with force, let us say F at the middle, so there is a static force at the centre of the beam, which is F. So let me write down the equations. So the problem that we are going to address is like this, that initially this beam is deflected under the action of this constant force F, applied at the centre of the beam and a time T is equal to zero, this force is switched off.

So at the point this force is switched off, so the beam is going to spring back. So the equation of motion for the dynamics problem, so this is the equation of motion and the boundary conditions. So this is the deflection is zero, this is the bending moment is zero at x equal to zero and same way at x equal to l. Now the initial conditions, we need to determine the initial conditions, which is a deflected shape of this beam.

So that will be obtained by solving the equation of statics, so by solving this equation along with the boundary conditions that we have here. So if you solve this then you will obtain, and we assume that initially at time t equal to zero the velocity of the beam is zero. And this is the shape, so this we can substitute in here and integrate out to determine the initial shape of the beam. But we are not going to do it immediately.

We are going to now first write this the solution of this problem as an expansion in terms of the Eigen functions. You know that we can represent the solution in this form, where these are the Eigen functions and for a simply supported beam, Euler–Bernoulli beam the Eigen functions are given by sin of j Pi x over l. So now let us first solve this statics problem at time t equal to zero, therefore this is going to be the equation.

So at time t equal to zero this expansion, so this is, so Cj times sin of j Pi x over l. So at time t equal to zero this is the expansion so if you substitute in here, so when we substitute this in the equation of statics, this is what we obtain. Now from here, from this equation we are going to solve for these Cjs and this velocity condition will immediately tell you, so if you consider that this is the velocity condition which is zero.

So from here substituting in the velocity condition, this is going to tell us that all these Sj are zero, so we only have these Cjs' non zero. Now to solve this we follow the standard procedure we multiply both sides by sin k Pi x over l and integrate over the domain of the problem. So we take inner product, so because of the orthogonality this is going to filter out the k-th term.

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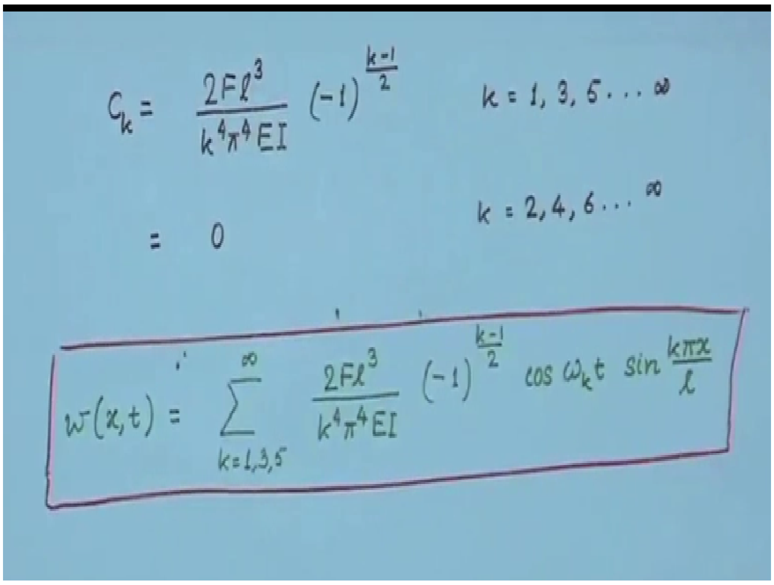

So if you do that and simplify then we obtain for odd values of k we have Ck as non zero given by this and for other, so for the even values of k, we have Ck equal to zero. So finally once we have this, we can write the final solution, so this is the final solution of the initial value problem. So this can be, this is the function of space and time, so one can animate the solution to determine the response of the beam.

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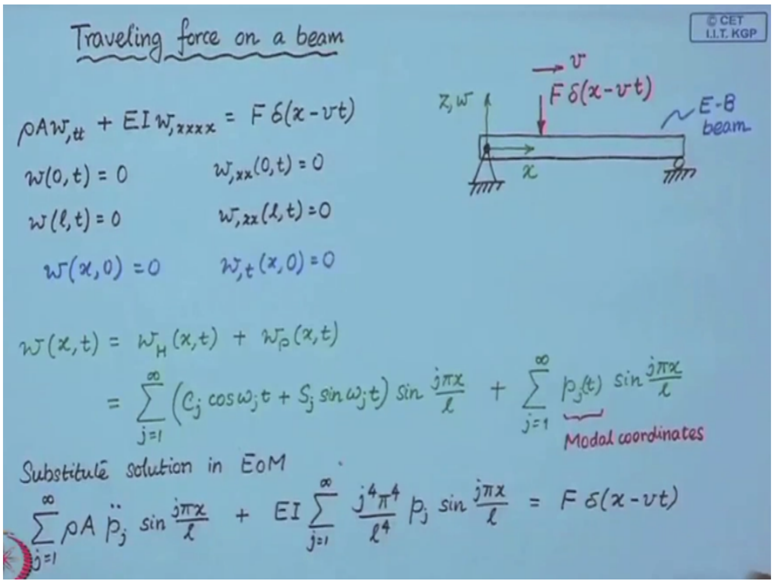

Next let us look at another problem, which is the problem of a travelling force. Now here we consider once again a simply supported Euler–Bernoulli beam carrying, so this is beam is carrying a force, which is travelling with a speed v. So this problem is important in case of, let us say, bridges on which you have travelling loads. This is simplified version, here we are considering a constant force travelling on Euler–Bernoulli beam.

So the equation of motion of this system, this long with the boundary conditions and we also consider initial conditions to be zero. So which means that the beam is undisturbed before the force enters the span of the beam. Now this is the forced vibration problem with a general forcing, so we can write down the solution of this problem as the homogeneous solution plus the particular solution.

Now we already know that this homogeneous solution can be expanded in terms of the Eigen functions and represented as in this form. This is a simply supported beam, therefore the Eigen functions are sin j Pi x over l plus, so this particular solution also we can expand in terms of this Eigen functions along with theses modal coordinates. So these coordinates capture the dynamics because of the forcing. So these coordinate capture the forced motion.

Now when we substitute this solution form in the equation of motion, so this is the homogeneous solution. So this is going to vanish, now this is going to now contribute, so this gives us, so this is what we obtain.

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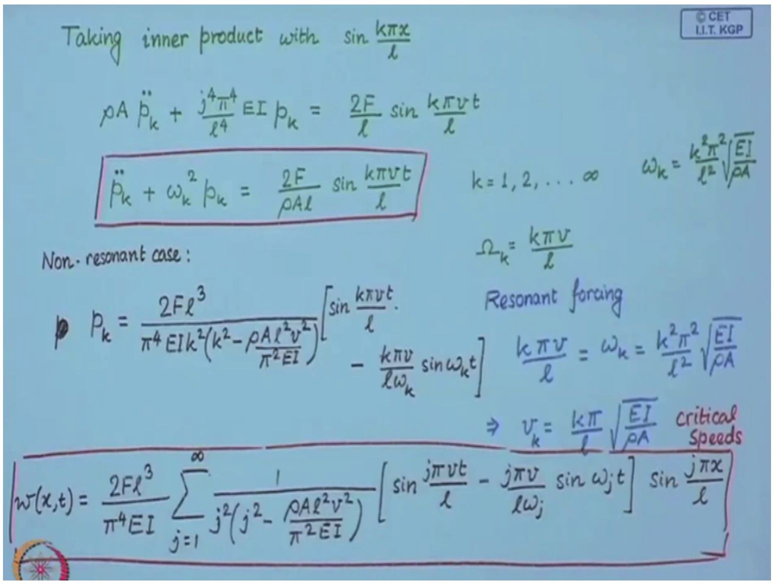

And if you simplify this by taking inner product on both sides with sin k Pi x over l. We can write this as, so using the orthogonality we multiply this equation with sin k Pi x over l and integrate over the domain of the beam, which is from zero to l. On account of orthogonality this, taking this inner product is going to filter out the k-th term in this expansion.

So on the right hand side, since we are multiplying this direct delta function with sin k Pi x over l and integrating so this x gets replaced by this vt. So then we can rewrite this, now I am dividing throughout by rho A, so this j power four Pi power four by l power four EI by rho A, that is nothing but square of the k-th circular natural frequency of the beam.

So this is the, this defines the dynamics of the k-th modal coordinate. And this can be written out for all the modes. So we have the dynamics of all these modes. So here of course, now this is a forced vibration problem for a discrete system. Now you see all these modal coordinates are decoupled. So they can be solved independently and we know the general solution of this system.

So we can easily write down the general solution of Pk and construct the solution of the beam. So now here you can note that it is a harmonic forcing. So the frequency of the harmonic forcing, let us name it omega, index with k so that is the circular frequency of the harmonic forcing. Now there can be velocities for which this harmonic forcing equals the natural circular frequency of the k-th mode.

So such a, in that case we will call this forcing as a resonant forcing. So for resonant forcing, for any k, suppose if you have this condition, and from here we know that this is, so we can find out the velocity for which the k-th mode is resonant. So let me index this as also with k. So this is the velocity which will send the k-th mode into resonant. Now for simplicity let us first assume that this is not resonant.

So for non resonant case, we can write the solution, so let me write the k-th coordinate. Now this solution also satisfies the initial conditions for the beam which is that it is undisturbed at t equal to zero. So the initial shape as well as the initial velocity of the beam is zero. So using this solution therefore, so this is our solution for the response of the beam under a force travelling at a non resonant speed. So which means the speeds do not take any of these values.

So if the speed of travel is one of these resonant speeds, that is also called critical speeds. So if you have a force travelling at a critical speed in that case, the solution of this gets modified, which can be easily written out and we have studied this case for the string. Now let us look at the solution at certain time instance for two speeds, one is a low speed and the other is a high speed transport over a beam.

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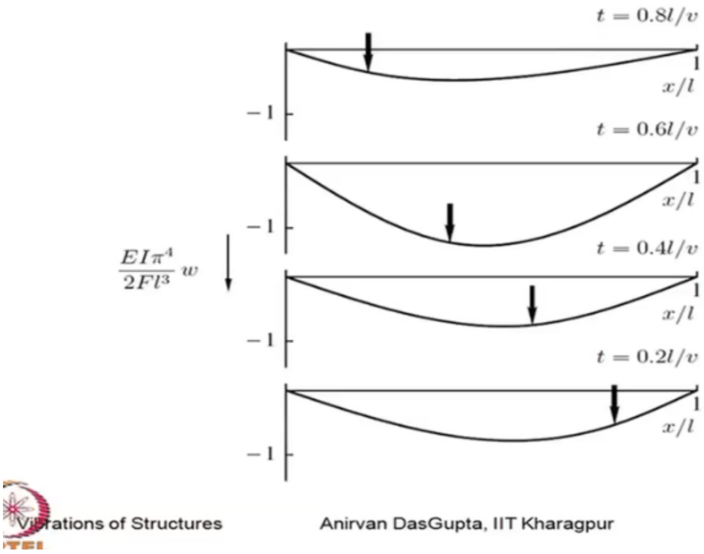

So this, this figure shows snapshots at certain time instance of a force travelling on a beam. So you can see how the beam deflects at various time instance. Now this is for the low speed.

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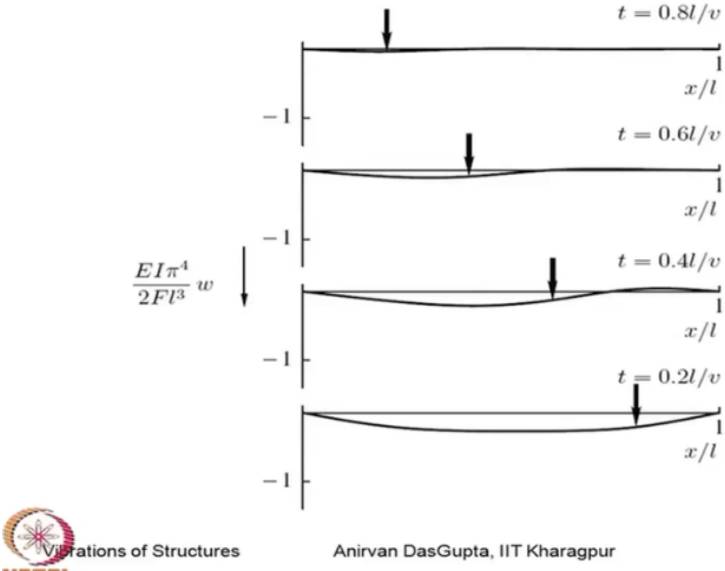

This figure shows the same thing when the speed is high. Now here you can see a difference from the previous figure, from the previous figure the deflection of the beam is completely on the negative side, below the equilibrium position. Now here at certain time instance, the beam goes above this equilibrium line.

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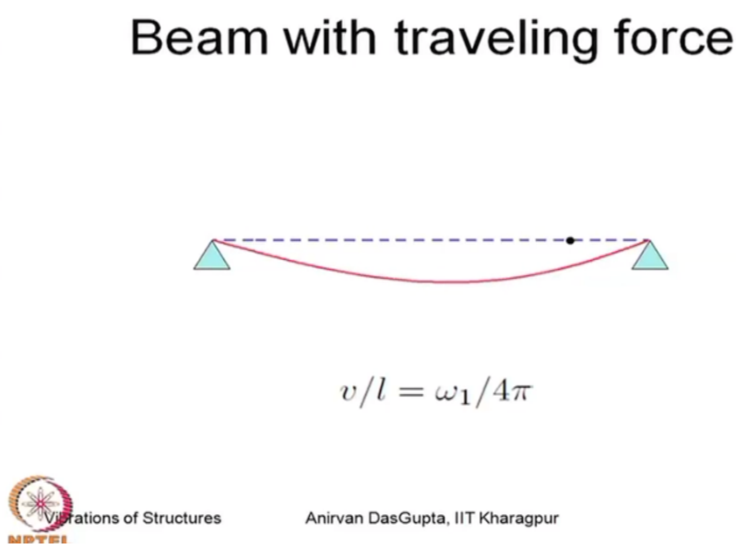

Now let us look at the solution through a simulation or animation. So here the velocity is, this is a low velocity that I have considered in this animation. So you can see that the beam is always on the negative side. This black spot indicates the location of the force. The force of course on the beam and this is of course exaggeration of the deflection.

Now there are two things that has to be remembered, when you see this animation. The first one is that this is a slow motion of what is happening. The second thing is that once this force leaves the beam the response of the beam is not shown in this animation. So this animation is looped and you see only the response of the beam when the force is on the beam and travelling on it at a constant speed, which is indicated below.

Now let us see what happens when we increase the travel speed. So here v is l times omega 1 into Pi over four. So this is a speed higher than the previous situation. Now you can see that here for example, the beam is going above the equilibrium line. Again, this is a slow motion of what is going to happen. One more thing to notice in this solution, which is different from the previous solution is that the deflection of the beam is much smaller in this case.

Since the force is travelling at a much higher speed, the beam gets less time for deflection, but this is what we observe when the force is within the span of the beam. So you can again see the disturbance propagating forward and it is reflecting back. Okay, so we have looked at these two examples related to application of the modal solution. Now before we close this discussion, let us quickly look at one of the important properties of the Eigen functions, which is orthogonality.

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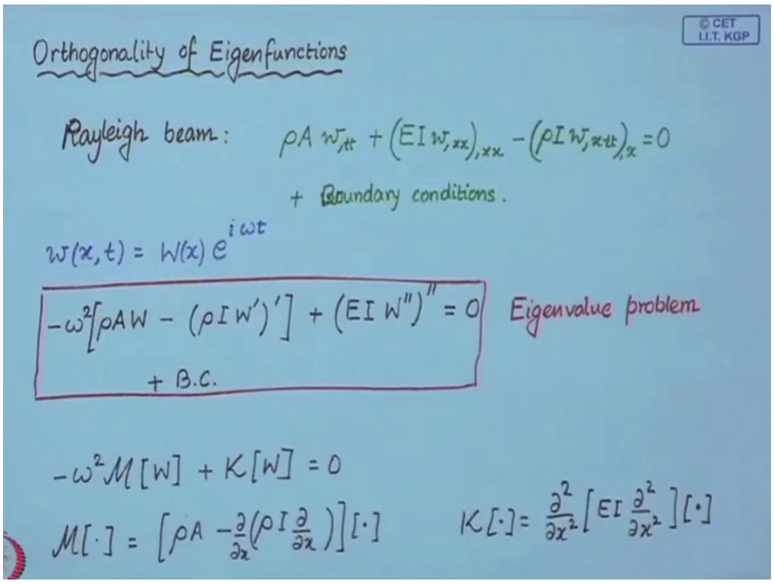

So we have been using this property in all our calculations, but let us now formally look at this property right from the equation of motion, so let us consider a Rayleigh beam. So for the Rayleigh beam, the equation of motion. So this is the equation of motion for a Rayleigh beam and along with this, of course we have these boundary conditions, let us say the boundary conditions. Suppose for example, we can have simply supported boundary conditions.

Now when we do modal analysis, we search for solutions of the form, which is complex and separated in space and time, so if you substitute this kind of solution in this equation of motion, we obtain this differential equation where the prime denotes derivative with respect to x. So this plus the boundary conditions, they complete the description of the Eigenvalue problem. Now for a moment, let us write this in a compact form, where this operator m and the operator k can represent in this form.

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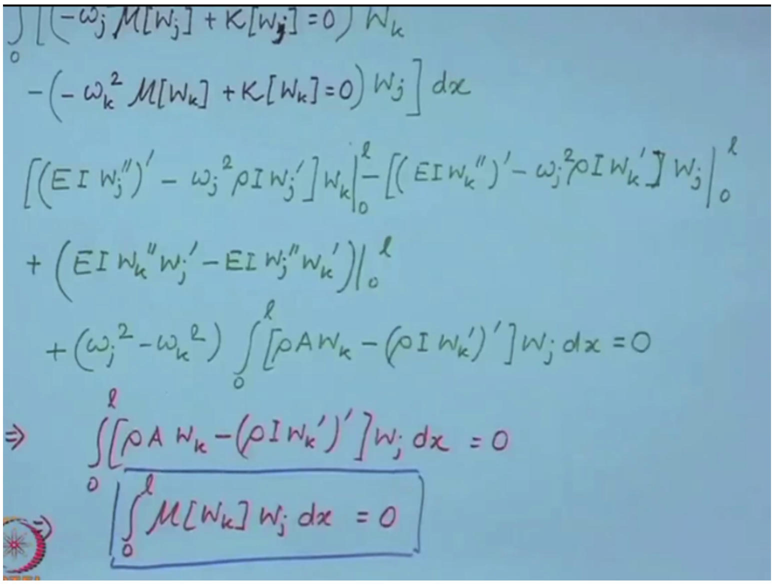

Now let us write this for the j-th mode, it is going to satisfy this equation for the k-th mode is going to satisfy this equation. Now what we do. We multiply the first equation by Wk, the second equation by Wj, and subtract one from the other and integrate over the domain of the problem. So if you do that and simplify, these simplification steps we have discussed also previously. So this is what we obtain.

Now if you use the boundary conditions of the problem, suppose simply supported uis then going to be zero at both 0 and l and the bending movement is also going to be zero at 0 and l, so this boundary terms, they all vanish, if you use the boundary conditions and what you are left with and this you can see. So this is the orthogonality condition for the beam. So you see that the Eigen functions are orthogonal with respect to this inertia operator.

So to summarize, we have today looked at some applications of the modal solution in solving initial value problem and solving forced vibration problem and we have also looked at the orthogonality condition of the Eigen functions. So with that we conclude this lecture.