Vibrations of Structures
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Lecture - 19
Modal Analysis of Beams

Today, we are going to discuss the modal analysis of beams. So, in the previous lectures we have been discussing about beam modals, mathematical models of beams. So, today we are going to discuss the modal analysis which means that determining the eigen frequencies and the modes of vibrations of beams. So, we will begin with the simple case of a uniform Euler–Bernoulli beam.

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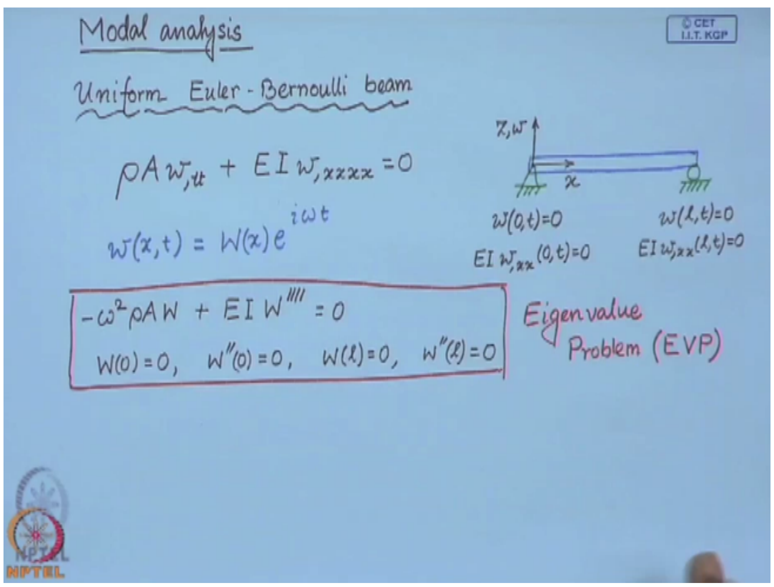

So. as you know that the uniform Euler–Bernoulli beam is governed by a differential equation like this and along with this we will have boundary conditions. Now, as we have discussed in our previous class there will be in all four boundary conditions. So, suppose if we consider a simply supported Euler–Bernoulli beam then the boundary conditions, so this is the displacement equal to zero condition.

This is the bending moment equal to zero condition. Similar at x equal to l we have displacement zero and bending movement zero. So, let us consider this kind of a uniform simply supported Euler–Bernoulli beam. So as we have discussed before we search for a solution of this type which is separated in space and time complex solution and if you substitute this in the equation of motion the differential equation then upon some simplifications this must be zero where this prime denotes the derivative with respect to x.

So, this is the differential equation along with this corresponding to these boundary conditions we will have –so we have a differential equation, ordinary differential equation in fourth order, in w which is the space function and we have these four boundary conditions. Which defines what is known as the eigen value problem abbreviated as EVP. So, this is the eigen value problem that we now have to solve. Now to solve this we consider a solution

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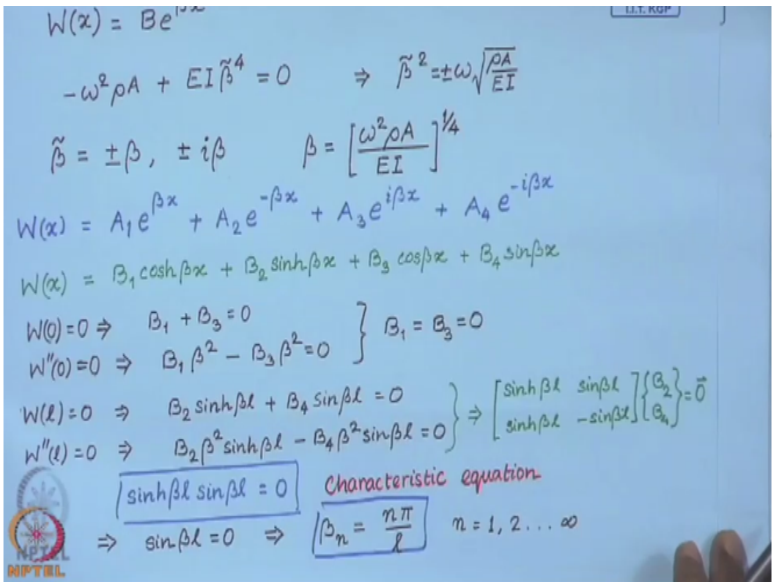

Of this form. So, let this be a solution of this differential equation so if you substitute this in the differential equal of the eigen value problem and that implies. So, I have substituted this in this differential equation and made some simplifications to obtain this which implies that this beta tilde square so which is an unknown constant so that is related to the circular eigen frequency. So, this can have two signs plus or minus.

And therefore this beta tilde will have four solutions which we will represent as plus minus beta and plus minus i beta where beta is –so beta fourth root of this quantity. And this beta tilde has these four solutions. Now corresponding to these four solutions therefore the general solution of this function W x can be written as –so in terms of this exponential function we can write the general solution. So where these A1, A2, A3, A4 are arbitrary constants possibly complex.

Now, you can also write this. So, by combining terms you can also write it like in terms of hyperbolic and trigonometric functions so here again B1, B2, B3, B4 are now real constants. Now, we have to determine or we have to I mean this solution, this general solution must satisfy these boundaries conditions. So, if you support let us say in the first boundary condition. So, you will get –if you use the second boundary condition then this is get differentiated two times.

All of this term will differentiated two times so they will be cos hyperbola either beta square, cos hyperbola beta square, sin hyperbola minus beta square cos beta minus beta square sin beta. So, if you put x as zero. So these are the two conditions in terms of B1 and B3 and if you can easily infer from here that so B1 and B3 must vanish. We are considering boundary conditions of simply supported Euler–Bernoulli beam. So, B1 and B3 they vanish.

Now, using the other two conditions. So, these two conditions are –so these are the two equations that remain and if we want to have non trivial solutions of B2, B4. So if we want to have none trivial solutions of B2 and B4. So, this I can write. So this is what we have. So if we want to have none trivial solutions of B2 and B4 then the determinant of this matrix must vanish and that will give us the condition.

So, determinant of this is –so it is minus two times so that must be zero. So vanishing of the determinant for none trivial solutions of B2, B4 this vector leads us to this determinant equal to zero and this is the characteristic equation for the simply supported Euler–Bernoulli beam. Now, this is sin hyperbola is zero only at beta equal to zero which will give us the trivial solution. So now this so far non zero value of beta because if beta is zero then you see the frequency has to be zero. So, for this to be zero we must have sin beta l must be zero and that implies.

Now this beta l, sin of beta l equal to zero that can happen at infinitely many values of beta which can be indexed. So these are the values of beta n for which all these boundary conditions will be satisfied. Now, you see beta n, beta is given by this expression which has this omega. Therefore, using these two - so beta square is and that must be as we have obtained.

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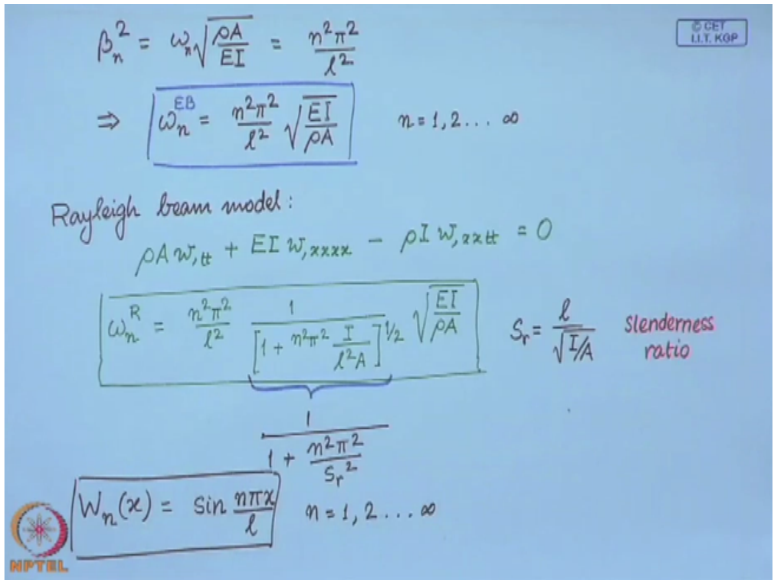

So this is indexed so omega n must now also get indexed. So we have now the circular natural frequencies of a simply supported Euler–Bernoulli beam. So this is for the Euler–Bernoulli. Now if you look at the differential equation of the Rayleigh beam model then the differential equation of the Rayleigh beam model reads. So, this is the differential equation of a uniform Rayleigh beam.

So this is the total inertia term and if you follow the steps similar to what we have just now done then you will get the circular natural frequency of the Rayleigh beam as –so this if for the Rayleigh beam. So we have this additional factor in the Rayleigh beam model, in the circular natural frequency of the Rayleigh beam. So, now let us analyze this. Now if you define so we have this factor in the denominator square of this factor.

Now what is this factor? So this is the length of the beam. The second moment of the area about the neutral axis I over the area of cross section of the beam. Now this I over A is the Radius of Gyration square. So under root of that is a radius of Gyration. Now length over the Radius of Gyration can be defined as the slenderness ratio. So this length over the Radius of Gyration is the definition of slenderness ratio. So it indicates how slenderness the beam is.

So if slenderness ratio is very high which means the beam is actually very slenderness. So the length is much larger then it is dimension of its cross sectional area. Now this appears in the square of this appears in the denominator of this term so you see if the beam is very slender which means slenderness ration is very high then for lower modes this term is negligible. Say n equal to one and if the slenderness ration is for example 10.

Suppose if the slenderness ratio is 10 in that case this factor is much, much smaller than 1 in that case this ration is almost one. So, which means for various slender beams the circular natural frequency of the Rayleigh beam model should match with the Euler–Bernoulli beam. But then this term bring in the difference for higher modes. So, for higher modes the frequencies are going to differ.

So, as we go to higher and higher modes the frequencies, the circular natural frequencies are going to be different. So, let us look at the comparison of this

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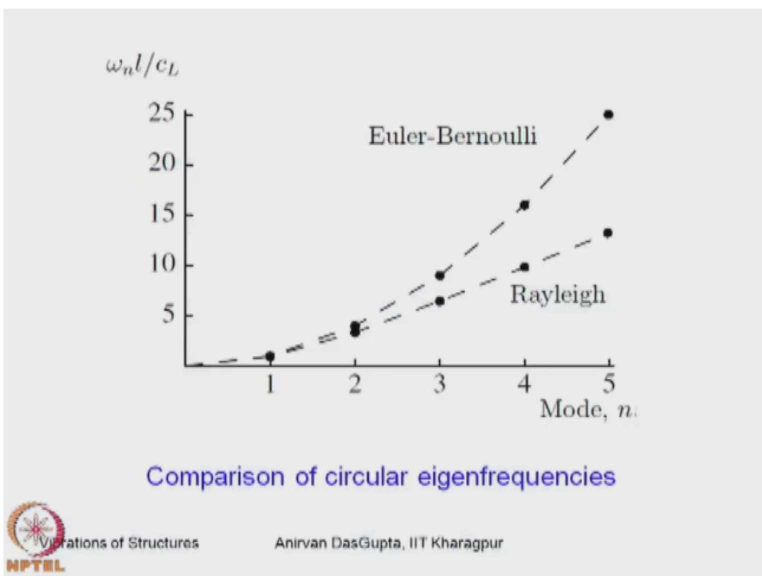

Circular natural frequencies of the Euler–Bernoulli and Rayleigh beam models. So as you can see at lower modes they are very close but as you go to higher and higher modes the difference is appreciable. Now, let us understand why this additional term that comes in our circular natural frequency I mean, see the source of this term is obviously the rotational inertia. So, that we can very easily see if you look at the differential equation of the Rayleigh beam then we can very easily see that that source this term is the rotational inertia.

So, which means the rotational inertia term is effective for higher modes. Now to understand this let us see what happens in the higher modes after we look at the eigen functions. Now, if you look at the eigen functions. If you do the analysis for the Euler–Bernoulli or Rayleigh beam then so the eigen functions are obtained so we have these expressions of beta n we will put in here and solve for B2, B4 then you will find that B2 must vanish so only B4 can exist. So therefore, the eigen functions will all be sin functions.

So, these are the eigen functions and these eigen functions are same for the string. So let us look at these eigen functions.

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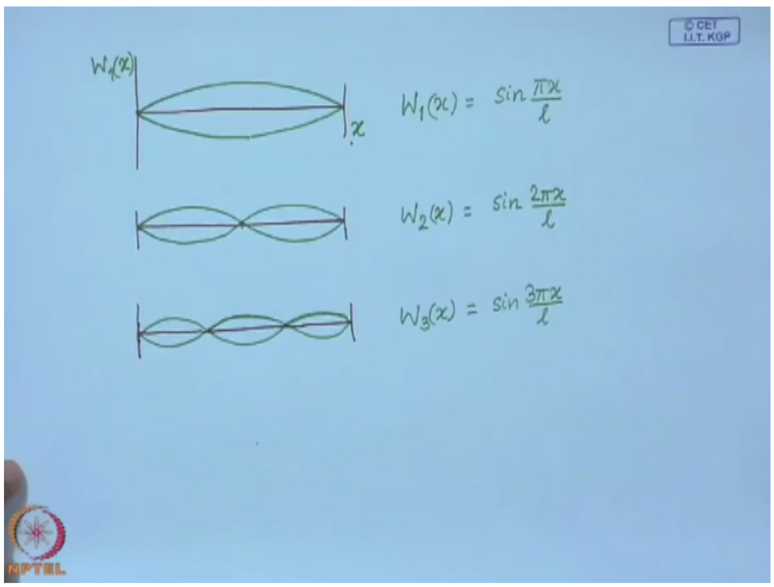

So, we have a simply supported Euler–Bernoulli or Rayleigh beam. So this is the eigen functions of the third mode so the mode shape is defined by these eigen functions. So you have two nodes here, one node here and no node in the first mode. So this is same as that of a string. Now let us see what happens as we go to higher modes. So if you calculate the curvature then the curvature is increasing as you go to higher and higher modes the curvature is increasing.

And this rotational inertia terms so more is the curvature, more is the rotation of this element. So, as you go to higher and higher modes this term it bring in increased effect of this term because of higher curvature and of course because of higher frequency. So, as you go higher modes this term becomes substantial and that then starts affecting the eigen frequencies which is not there for the lower modes.

Now you see if n is very high suppose n tends to infinity then you can easily see that this, the frequencies become proportional to n. So, if n becomes much, much larger than one then this becomes proportional to n. So, this term becomes much, much greater than one. So we can drop this one so square root of this will be n. So, here there is an n square so the Rayleigh beam circular natural frequencies will become proportional to n as you go to higher and higher modes.

Whereas for the Euler–Bernoulli beam it increases as n square as you can see here. Now let us discuss

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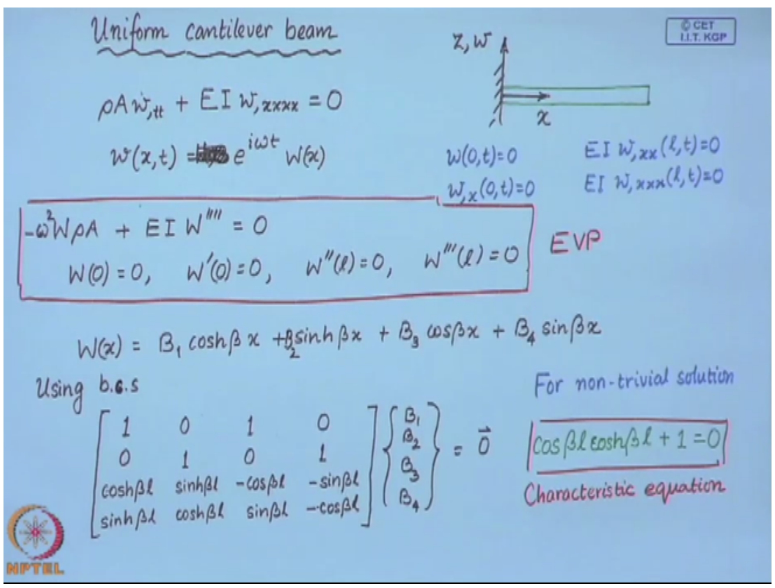

A Uniform cantilever beam. This is another kind of boundary condition which is quite common. Now in this case the boundary conditions are given by –so on the built in end we have displacement zero and the slope also zero. So, both of them are geometric boundary conditions on this end. On this end we have the bending moment and the sheer force as zero so that gives us –So the equation of motion it remains the same.

So, we are considering a Euler–Bernoulli beam with these boundaries conditions. Once again we search for separable solutions and so we have a space part and a time part. So when we substitute so the differential equation of the eigen value problem remain the same. Now, in this case of the

–so it was the differential equation of the eigen value problem along with the boundary conditions which we have now.

So, that is the eigen value problem for the cantilever beam. So, the general solution is like this and when we use this boundary conditions we obtain the following four conditions. So, these conditions can be simplified and written in this form. So, for non-trivial solutions of B1, B2, B3, B4, the determinant of this matrix must vanish and this leads us to –so if you calculate the determinant of this matrix and this is the characteristic equation for the cantilever beam.

Now, these are all transcendental equations so you have to solve them numerically. Now, if you plot these functions after some rearrangement if you plot this functions then we can have a graphical approximation for the solutions as shown in this picture.

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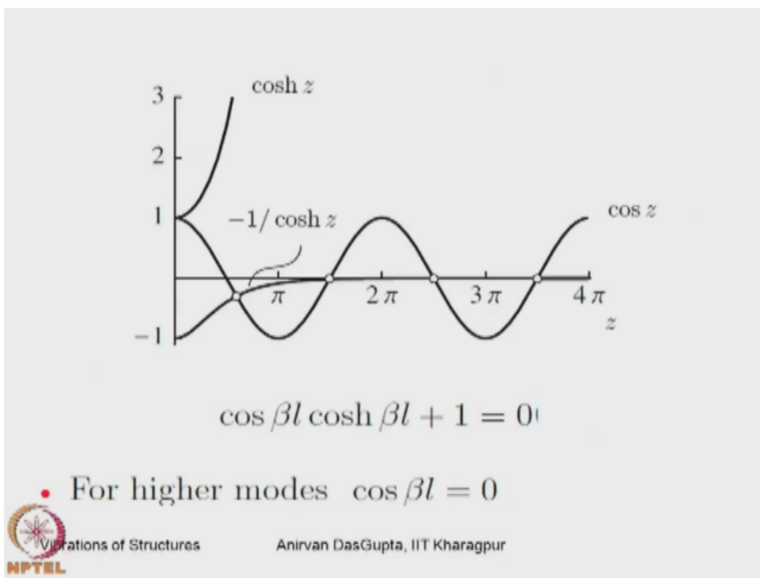

So, what we have done is we have written this cos beta l. So this is a characteristic equation we have written this as cos beta l is equal to minus one over cos hyperbola beta l and we have plotted this cos beta l. So, cos z, this is cos z and minus one over cos hyperbolic z. So, these two functions have been plotted and the intersections of these two functions are marked by this, unfilled circles.

So, these are going to give us the circular natural frequencies for our cantilever beams. So, let us see what we have done.

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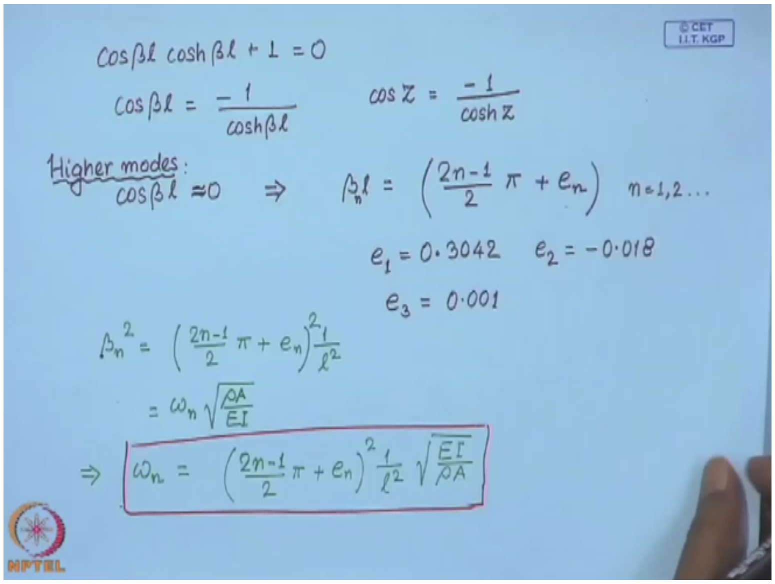

So, essentially we have plotted these two functions and wherever they met that gives us the solution of z which is the solution of beta l. Now, let us look at this figure once again now you can see that for higher, so each of them as I indicated they will tell us the circular natural frequency. So for the higher modes, so higher frequencies you see this 1 over cos hyperbola z it falls of very fast to almost zero.

So, these points can be written as zeros of this function cos z is equal to zero. So for higher modes you can reduce this characteristic equation to this simplified characteristic equation cos sin beta l equal to zero. Therefore, I can write this beta. So beta l has approximately this. Now what I will do is in order to get to this I will add here beta get index with n. So this is some error so this beta into l will deviate from the zeros of cos beta l equal to zero by these numbers e n.

So, when I have added this e n the small error then I can claim that this is a solution of this and for e n let's say e 1 is if you calculate the first solution of this characteristic equation then it will deviate from this value by this amount. If you calculate the for the second mode so beta two l then it deviates from this value. If you calculate for the third mode, then it deviates from this value by this amount.

So, as you see this is rapidly falling this e n the correction is rapidly diminishing. So, for higher modes it suffices to use 2 n minus one over two times pi as beta n into l. So beta n square is therefore and as you know this beta we have looked at this so beta n square is nothing but rho A over EI times omega square root of that times omega n so this implies. So this is the expression of the circular natural frequency of a cantilever beam.

Now, corresponding these circle natural frequencies natural frequencies one can calculate the eigen function from the general solution that we discussed. So, these are obtained as

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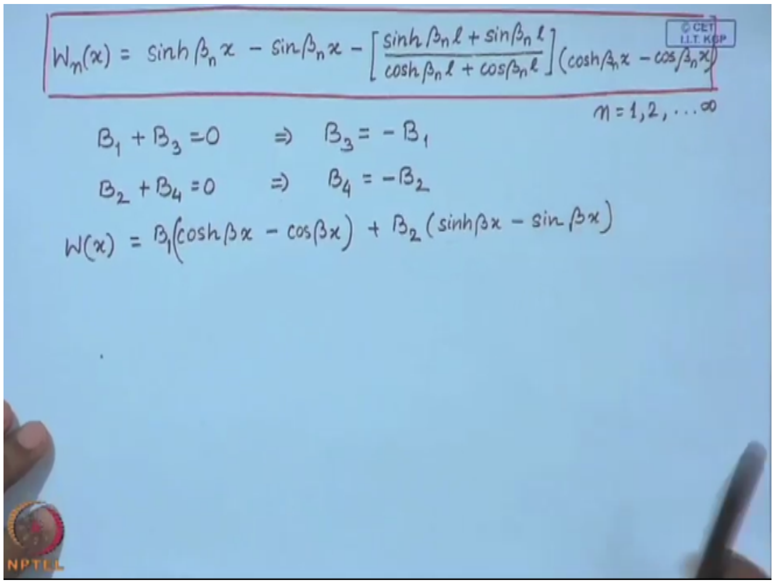

Now, let me just indicate how this has been obtained so this is what we had now the first condition that we have from here is that. So that implies B3 is minus B1 similarly from the second condition. Now we have used this third condition so therefore with these two conditions we can write so that is proportional to this and we have used the boundary conditions from there as you can see we have obtained this eigen functions.

So, this term appears here so we have considered B2 as one and at x equal to l we have used the boundary condition to obtain this factor, B1. So, these is the eigen functions again these are indexed. So let us once look at

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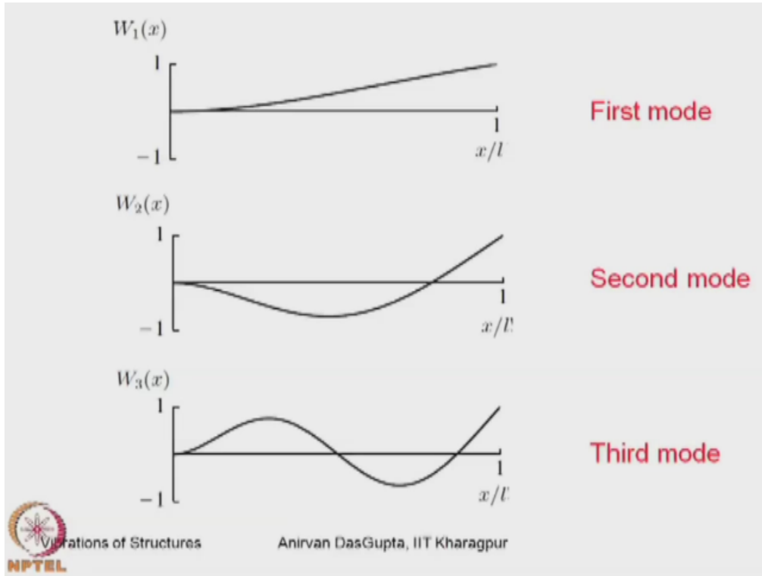

These eigen functions. So this is the first mode characterized by this eigen function. This is the second mode. This is the third mode so you can see in the third mode there are two nodes. In the second mode there is only one node and no nodes in the first mode and the slope here is zero. The displacement and slope as we expect. So, we have discussed the model analysis of themes which is essentially again solving an eigen value problem.

So, we have looked at these two examples of simply supported Euler–Bernoulli beam and we have also looked at a simply supported Rayleigh beam and we have understood the role of rotatory initiator on the effect of rotatory initiator on the circular natural frequencies of the Rayleigh beam. And finally we have looked at this Euler–Bernoulli Cantilever beam. So with that I conclude this lecture.