Vibrations of Structures

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Lecture - 18
Beam Models – II

Today, we are going to continue our discussions on beam models that we had started in the last class, so in the last lecture we had looked at two beam models under the assumption that the shear strain in the beam is negligible or in other words that the beam is infinitely stiff in shear. Now this assumption is very good if the beam is slender which means that its transverse dimensions are much much smaller than the - than the length of the beam.

So under such assumptions such an assumption this shear may be neglected, so that the Euler Bernoulli hypothesis in which says that the plane section perpendicular to the neutral fibers before deformation also remain plane and perpendicular to the neutral fibers after deformation. However, when the beam becomes thick or its slenderness ratio it goes on reducing so it becomes more and more thick in the transverse direction then in the length direction.

In that case, shear becomes important, so today we are going to look at a model which incorporates this shear in a very interesting manner. So - so in the previous lecture what we saw that the shear force was introduced in the equilibrium equations it was not a quantity derived from the deformation of the beam in terms of the material properties. But today, we are going to look at this - this more advanced beam model which is - which goes by the name of Timoshenko beam model so we are going to look into this modelling of the Timoshenko beam.

So the assumption that we made in our previous lecture that the - the slopes are small the material is linearly elastic homogeneous and isotropic all those assumptions still hold accept that we now we are going to introduce shear in our model, so let us look at this Timoshenko beam model.

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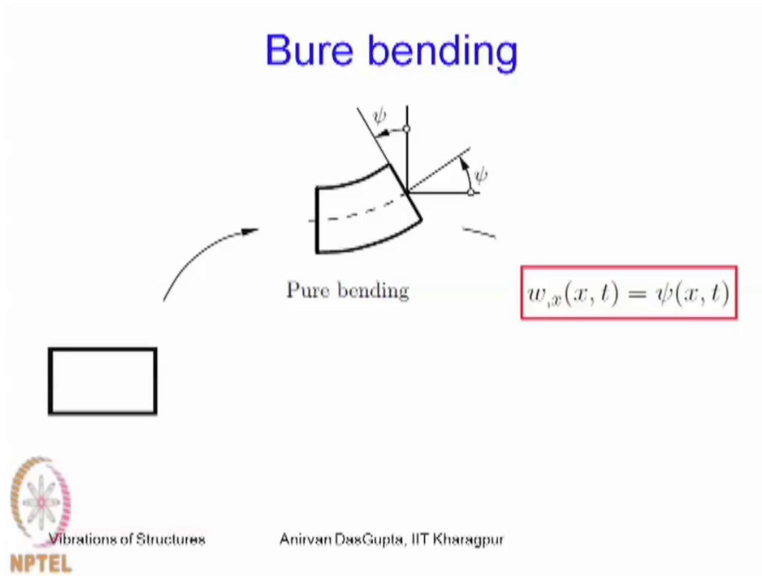

So here I have introduced what is the basic difference between the model that we discussed in the previous lecture and what we are going to discuss today, so here you can see an undeformed element of the beam which undergoes flexure or pure bending so this is what we have discussed in our last lecture.

So this is the neutral fiber or the neutral axis and the cross section of the - the cross section maintain this orthogonality with the neutral fiber so this is known as Bure bending, now in this case the slope is nothing but this angle psi that I have indicated so when a beam undergoes pure bending then the slope of the neutral fiber remember that we are always tracking the slope of the neutral or the deflection and slope of the neutral fiber so this slope is this angle.

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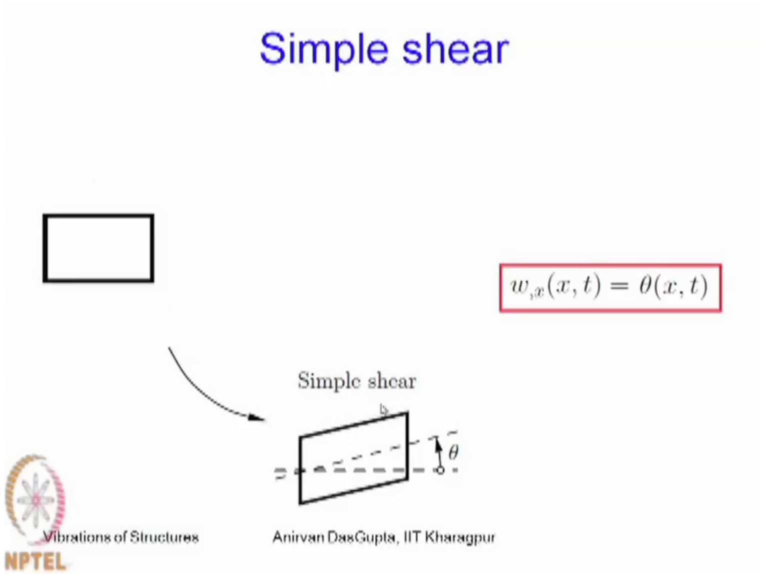

Now let us look at this case of Simple shear, when the element undergoes a simple shear so here this is - this simple shear is a - is a volume preserving deformation so this angle theta is the shear strain this is the angle that is the shear strain and as you can see here that the slope of the neutral fibers or the axis is nothing but this shear strain.

So therefore when we put these two things together which means the beam undergoes both flexure and shear so it is under bure - it is under bending and shear.

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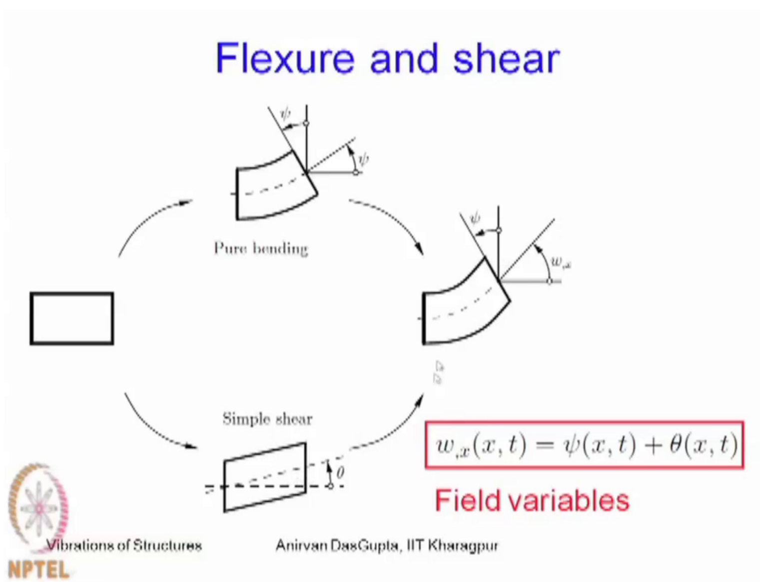

So the total picture looks like this, so this is the pure bending picture this is a simple shear picture and the net result is shown here now you can see that this shear because of this shear the angle does not change as you can see here so this was vertical so this remains vertical, so which means that this angle will not change the angle measured from the vertical this angle remain psi.

Whereas the slope of the neutral axis changes and this is what we have so the net slope of the neutral axis del w del x is the angle because of pure bending and the shear strength so that is the total angle, so we are representing the slope of the neutral fibers in terms of this flexure angle and shear strain so we can consider our field variables when we - when we denote or represent the deflection of the beam we can use any two of them.

So what we will do is we will choose w and psi as our field variables.

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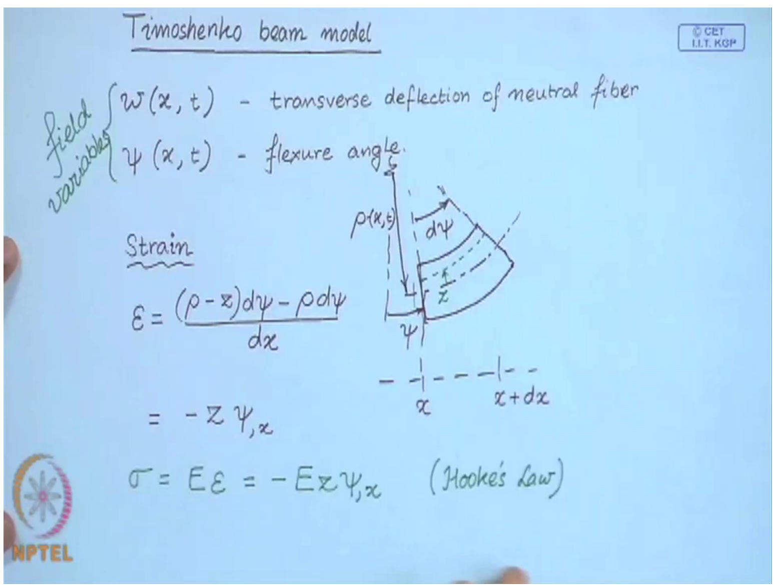

So this is the transverse deflection and this is the flexure angle, so these are the two field variables that we will use. Now let us then start representing our deformation in terms of these field variables. So the first thing is the strain so the fibers strain transverse to the in the - in the longitudinal direction of the beam. So if you once again consider fibers at height z from the neutral plane or the neutral axis.

Then following our discussions in the previous lecture, we can write the length of this fiber so if - if rho is the radius of curvature then rho minus z which is this radius times this angle is d psi. So this additional angle so we are considering an element between x and x + dx so the strain can be written as the length of the current length of this fiber minus its initial length divided by so that is the length of the element so this can be written as minus z del psi del x.

So this is in terms of the angle by which this section has rotated so that is the strain in this fiber so if z is positive and del psi del x is positive then you can see that this strain is compressive. Then we can write stress using Hooke's law so that then becomes so here now with this expression of stress we follow the steps that we carried out in our last lecture.

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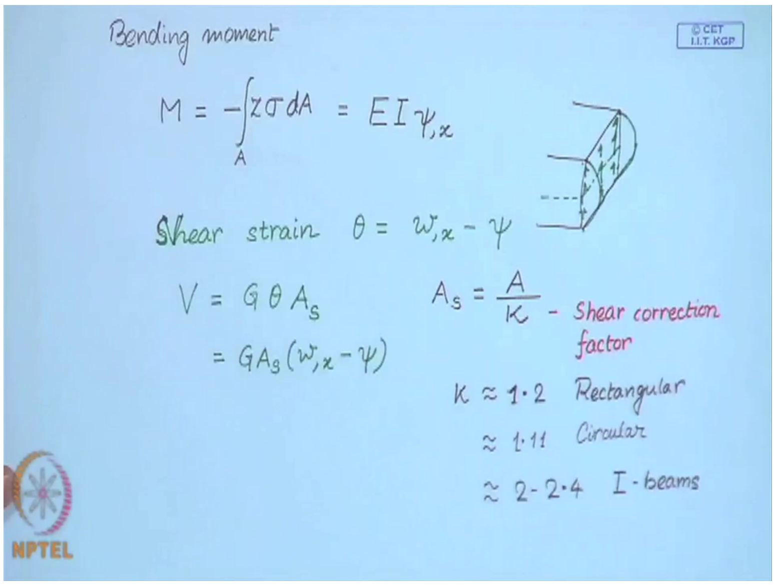

We determined the bending moment negative of sigma times dA this is the force so arm cross so z cross the force that integrated over the area and then if you substitute this expression of sigma then you can easily see the way we done before so the bending moment turns out to be EI del psi del x, now the next thing is to determine the shear force. Now in the - in the models that we looked at in the previous lectures shear force was introduced in the equilibrium equations considering that the beam element is infinitely stiff in shear.

But today since we have introduced the - the shear strain, so we know that the shear strain given by theta is nothing but del w del x minus psi, so we can represent the shear strain in terms of our field variables w and psi. Then I can write the - the shear stress acting on, so the shear stress is G times the shear strain where this is the rigidity modulus times the area of the cross section that is going to give us the total shear stress.

Now as you know that in a beam what we learnt in mechanics that shear stress is not uniformly distributed over the cross section so just multiplying it with total area of cross section is actually going to overestimate the shear stress. So the shear stress will actually lower than just a product of the shear force will be less than just a product of the shear stress and the area of cross section, so what we do to remedy this problem and to keep the formulation very simple.

We introduce what is known as shear correction factor we calculate a corrected area where A s is the actually area of cross section of the beam divided by a factor kappa where kappa is known as the shear correction factor. So as you know that say for example in the case of a rectangular a beam with a rectangular cross section the - the shear stress distribution is parabolic.

So it is parabolic it is maximum at the neutral fiber and reduces and goes to zero at the top and bottom fibers, because there is no shear stress on the top and the bottom surfaces, so the distribution of shear stress in a rectangular beam with rectangular cross section is parabolic. Now to take care of this - of this fact we introduced the shear correction factor and calculate a reduced area and multiply this area with shear stress to determine shear force.

So we want to keep the expression of this shear force simple so for that reason we have introduced this shear correction factor. Now this shear correction factor typically these are - take on values of about 1.2 for rectangular cross section, so approximately 1.11 for circular cross section and it’s from 2 to 2.4 for beams with I cross section I beams, so these are some typical values that are used in various situations.

Now here we now have a very simple expression of the shear force acting on the cross section so therefore we can write this as, so that is the expression of our shear force.

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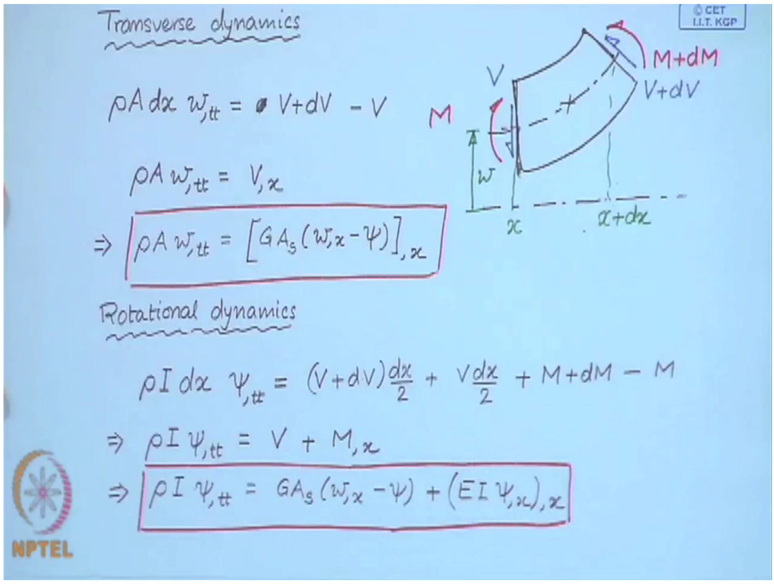

Now so we have these forces and moments let us look at the free body diagram of this beam element, so this is an exaggerated view so this is the transverse deflection at x at time t, here we also have this shear force and the bending moment, so if you write down the Transverse dynamics for this element.

So if rho is the density of this material A is the area of the cross section then rho times A is the mass per unit length times dx which is length of this element is the mass of this little element multiplied by the acceleration and that must be equal to here again considering the slopes are small we can write this as. So the force in the transverse direction so V + dV cosine of this angle so that is approximately one minus V times cosine of this angle which is also very small.

So this is what we have so dividing throughout by dx, now we know that this expression of V, so we have this expression of V so introducing this expression here so this is our equation of transverse dynamics for this element. Now the next thing is the Rotational dynamics, rho so times the second moment of area about the neutral axis, so this as we have discussed previously as well this is the moment of inertia about the neutral axis per unit length.

So this times the length of the element would give us the moment of inertia about the neutral axis which is perpendicular to the plane of the paper, so this times the angular acceleration now this angle is the rotational angle is psi. So this is the angular acceleration and that must be equal to all the moments about the so - so - so this gives us so this on the right hand side we must write all the moments about the center of mass of this element.

So we have because of this shear force and because of the moments bending moment. So we obtain dividing over by dx and considering the dV dx of smaller order then we obtain the equation of motion. Now once again substituting this expressions of shear force and the bending moment in here, the bending moment is E times I times del psi del x and this is what we obtain as the equation of motion of the rotational dynamics.

So these two equations now you see you have two field variables w and psi, so we need two equations of motion so these are the two equations of motion in w and psi. Now they are coupled partial differential equations now for so here in general this area of cross section can be a function of x so similarly, this moment of inertia can be a function of - the second moment of area can be a function of x for a beam with varying cross section.

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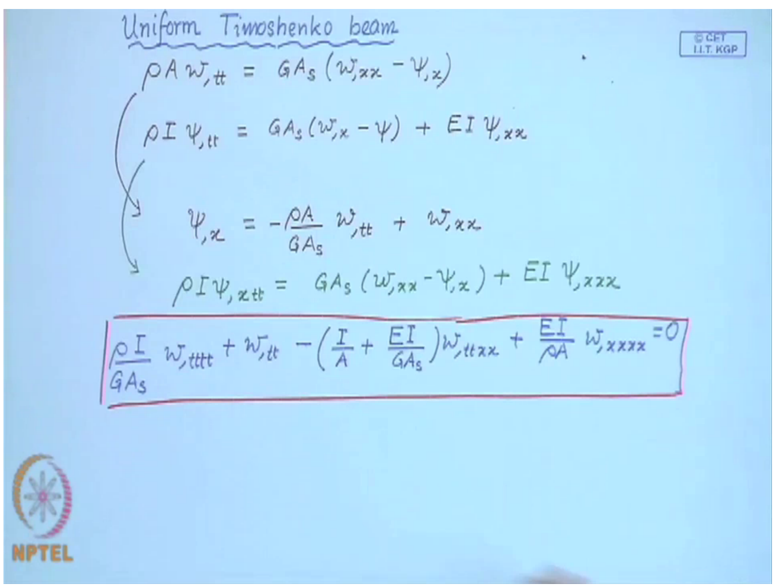

For a - for a beam of uniform cross section this equation will of course simplify, so this is the equation of motion of the transverse dynamics, so this is for a uniform Timoshenko beam, now here it is possible to eliminate psi between these two equations to obtain a single equation and w and to do that so here if you differentiate this equation with respect to x. So you will have del psi del x in all this in this equation and del psi del x can be solved from here.

So from here I can write and differentiating this situation with respect to x, now we are going to substitute eliminate del psi del x in this equation using this expression, so if you do that and simplify so this is what you are going to get. Now this equation is in terms of the field variable w but then notice that here you have fourth derivative with respect to time, this has fourth derivative with respect to space.

So to solve this - this equation you need to have three initial conditions and boundary conditions the number of - so you will - you will need four initial conditions and four boundary conditions, so and that usually you do not have I mean four initial conditions you definitely do not have. The reason is that so the reason that this equation though this is correct but when you solve this - this requires a little care.

Because this is going to generate four constants of integration for the time integration and four constants of integration for the spatial integration, now to solve for this - this eight unknowns which you actually do not have so what you will have to do is the general solution of this has to be once again substituted in this equation to find out additional constraints. So when you solve when you - if you want to solve this then you have to be little careful.

You have to do some additional steps, usually this is used to determine what we discussed in - in one of our previous lectures and what we are going to also discuss this is used to determine the dispersion relation for the Timoshenko beam so that is the equation of motion. Now as we have discussed previously we - we also would like to have variational approach for - for deriving the equation of motion of Timoshenko beam the reason being this gives us our boundary conditions directly which have not discussed as yet.

So we will see look at the boundary conditions from the variational principle and also I mean in the variational principle gives us very powerful way of approximately solving the system as we have seen before.

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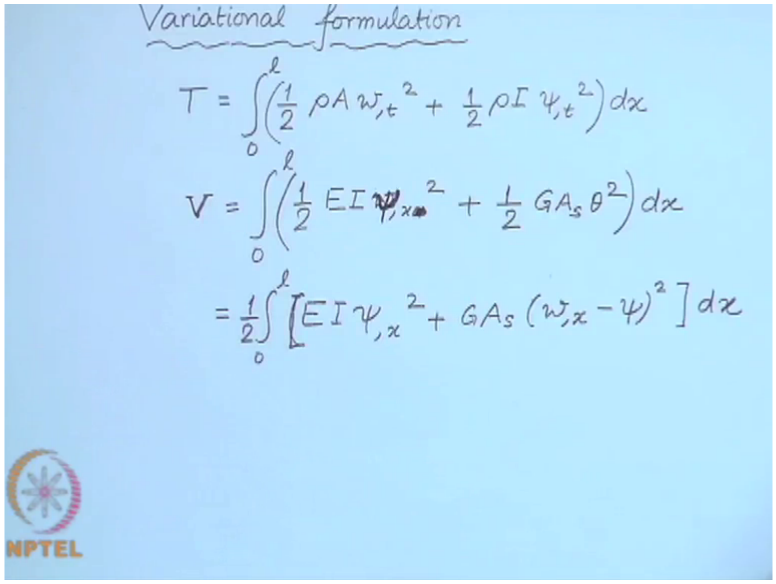

So let us look at the variational formulation of for the Timoshenko beam, so in the variational formulation we write down the kinetic and potential energy expressions, so the kinetic energy is one half mass per unit length times the velocity square, so this is the translational kinetic energy + 1 half rho times I is the moment of inertia per unit length times the angular velocity square.

Now this is per unit length this is kinetic energy per unit length so multiplied by a small length of the beam and integrating over the length of the beam will give us the total kinetic energy of the beam. Now the potential energy so here I am using the same symbol V for potential energy so that is given by one half EI the second derivative of w with respect to x square therefore so this is the potential energy due to flexure, so this we have written in terms of psi a single derivative.

So psi single derivative and that is - that is the expression that we have for the stress in the stress and strain. So the product of those two will give us the potential energy so this is per unit length over the area we have already integrated and in addition to this term we have the potential energy because of the shear the shearing of the element which reads this and these are all per unit length over the area we have already integrated.

So this times the small length and integrated over the length of the beam, so that is the potential energy expression. Now this theta, we have expressed theta in terms of our field variables w and psi, so this is our potential energy expression - expression in terms of w and psi and this is a kinetic energy again in terms of w and psi.

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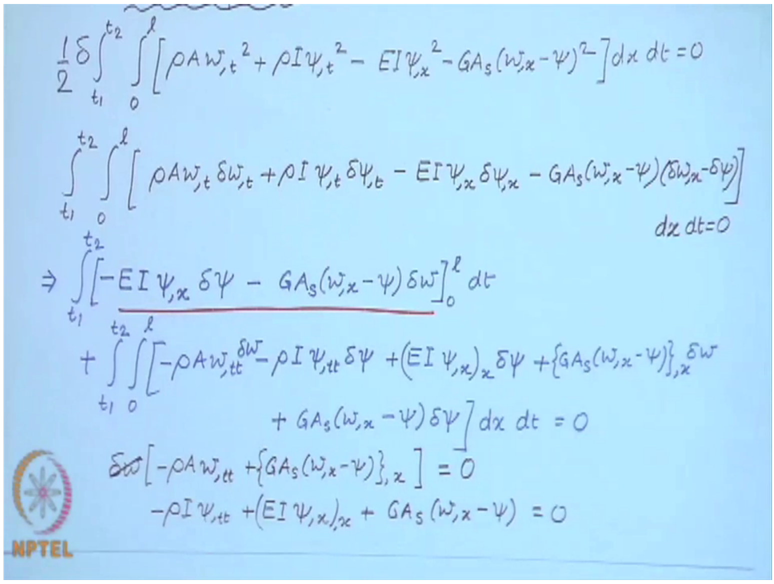

Now we apply the Hamilton's principle which says, so the variation of the kinetic minus the potential this must vanish, so if you take this variation so this is the variation of this kinetic energy translational kinetic energy, so this must vanish now we want to take out this variation delta w out common from here. So in order to do that we have to integrate this parts with respect to time and this by parts again with respect to time.

And these steps we have discussed many times before, so those terms at t1 and t2 must vanish now these terms we have to, for example this term and this term we have to integrate by parts with respect to space. So when we do that so because of this term you have this so these are the boundary terms so this term gets differentiated with respect to x.

So this must vanish now again using the- the conditions of vanishing of these variations of the boundary and variation over the domain we can again invoke the same arguments and obtain the equation of the motion so we have to collect terms of delta w so from here the coefficients of delta w. So we can say that for arbitrary variation w if this integral has to vanish then this must be 0 similarly, the coefficient of delta psi.

So these are the two equations of motion that we have obtained before as well, now from here we now look at the boundary conditions so let us look at the boundary conditions which we have obtained from this boundary terms.

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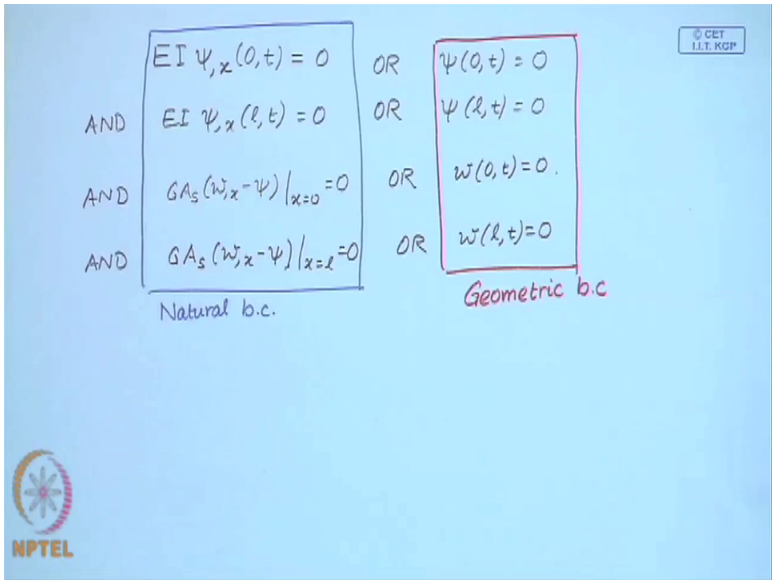

Now we have two field variables and corresponding to this two we have the for example for this, so this equal to 0 or psi so these are the connected by or these are conjugate so this is the angle and this is the bending moment and at l so these are the possible boundary conditions obtained from here and from here and similarly at x equal to l, so these are the possible boundary conditions that we have that we can usually have there are other possibilities.

Now as we have seen before as well so this set of boundary conditions, these are the Geometric while this set these are the Natural boundary conditions, so these are conditions on the bending moment or the shear force etc. on these are on the boundary the angle the rotation or the transverse deflection and these are conjugate these are one is the conjugate of the other so this the deflection is a conjugate of the shear force.

Similarly, this angle rotation angle is conjugate of the bending moments, so they come in or and so these are the possible boundary conditions we have, now this Timoshenko beam is still has it makes an assumption that the cross section which was plane before deflection that remains plane though it need not remain perpendicular to the - to the neutral fibers. So - so this is kind of limitation of Timoshenko beam.

But then you can very easily relax this by introducing more complex functions for your deflection field, so let us review and see what can be done.

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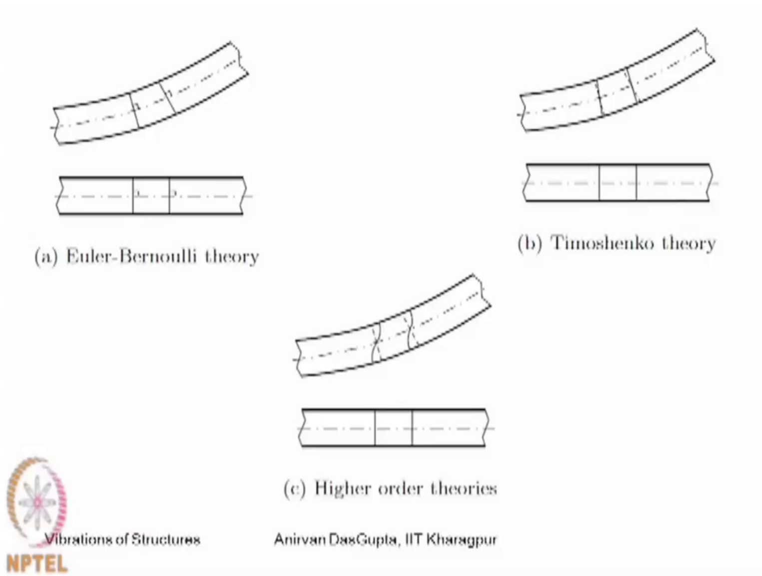

So in this figure I have shown this Euler Bernoulli theory, so what it says is that plane sections it will remain plane as well as perpendicular to the neutral plane neutral axis so this must be perpendicular here and still remains perpendicular, in the Timoshenko theory this was plane section still remains plane but now it need not be perpendicular to the neutral axis. Now you can then think of higher order theories in which plane section need not at all remain plane.

So there is warping of the cross section, so if you want to introduce or want to have higher order theories for beam for the deflection of beams or vibration of beams then you have to relax this flatness condition of the cross section.

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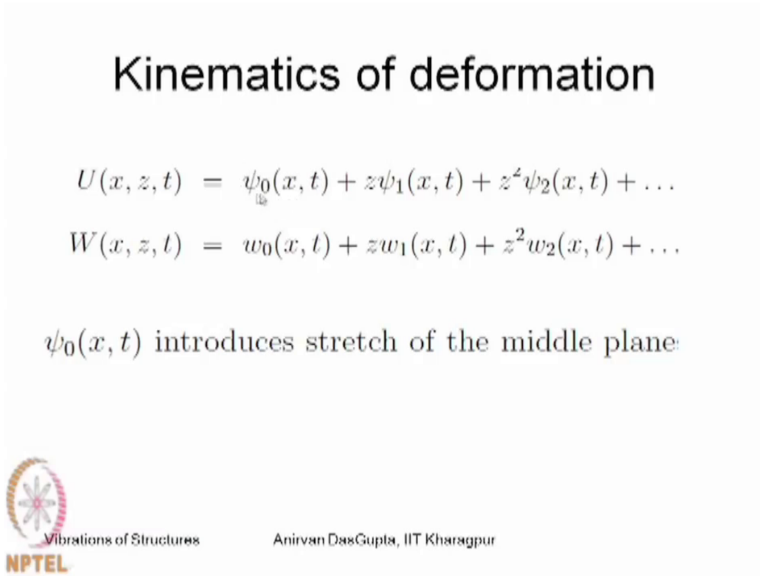

So - so we introduce the warping of cross section so to do that here you can see the kinematics of deformation can be written out in this way so where U is the deflection in the axial direction so it is being represented in terms of these functions psi 0, psi 1, psi 2 etc. and expanded in terms of this variable z which is the location of this fiber from the of this point from the neutral axis. Similarly, these transverse deflection is also expanded as a - as a serious as powers of z where you introduce these field variables w 0, w 1, w 2.

Now here this size zero introduces the stretch of the middle plane.

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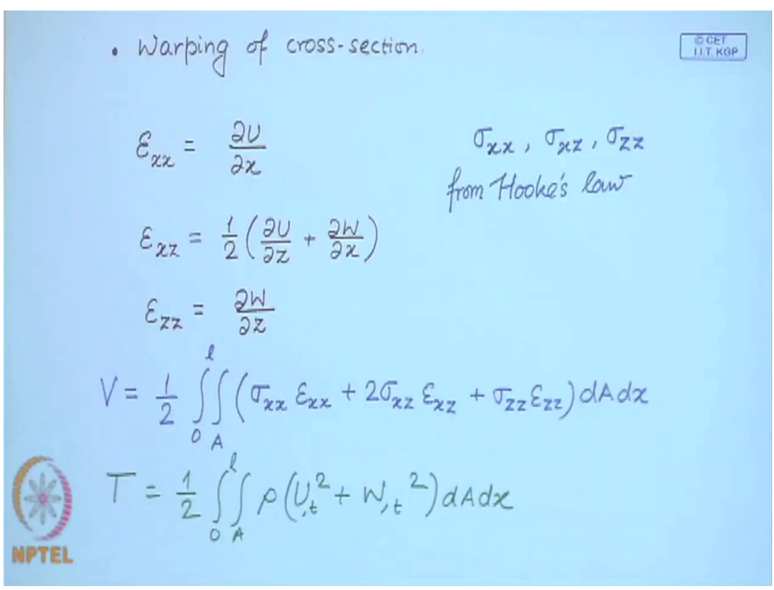

So using this kind of this deformation field then you can write down the strains in terms of this fields for example the axial strain can be written as say del U del x, the shear strain can be written as one half del U del z + del w del x and the strain in the z can be written as del w del z, now with these expressions of strain you can write down the potential energy, for example where this sigma x x, sigma x z and sigma z z are determined from the Hooke's law.

So once we have this we can write down the potential energy similarly kinetic energy can be written as one half, now once you have this expression then you follow the variational principle to derive the equation of motion, so you can use this so by taking different kinds of expansion for the field for our field variable U and W you can device higher order theories for beams.

So what we have looked at in today's lecture we have looked at the Timoshenko model which uses the or introduces us the shear in the beam which is and this theory is valid for even a thick beams and we have briefly looked at how we can device higher order beam theories so we conclude our lecture here.