Vibrations of Structures
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Lecture - 17
Beam Models – I

Vibrations of beams now before we discuss about vibrations of beams, we will discuss the beam models the way to model beams as and as a one-dimensional as a - as an elastic continuum in one dimension, so what are beams so you already have a lot of idea about beams because this - this kind of structural element is so ubiquitous and you find it everywhere and you have studied about beams in mechanics.

So you all have some idea about the mathematical some mathematical aspects of beams so how do we define a beam, because beam is also one dimensional elastic continuum and we have also discussed about strings which is also a one dimensional elastic continuum so how do we distinguish between string and a beam, now a beam is - is a one dimensional elastic continuum which can resist or transmit bending moment and shear.

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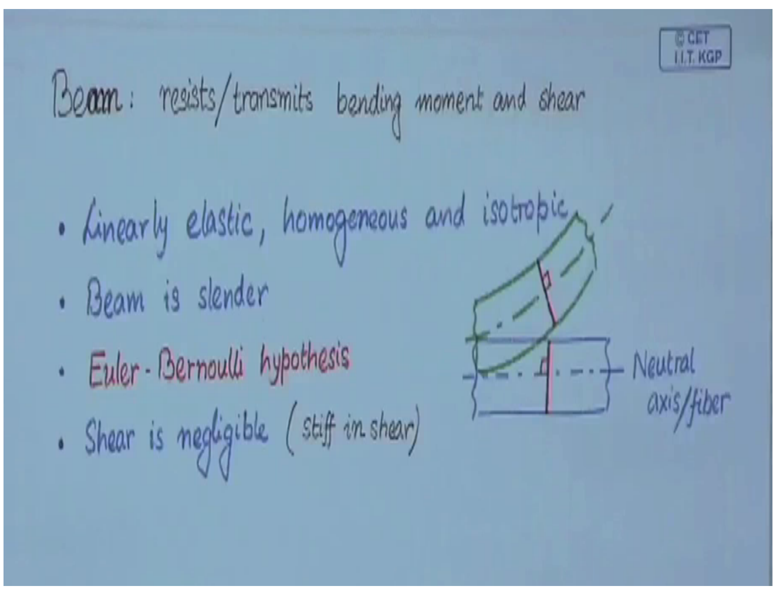

So a beam can resist or transmit bending moment and shear and it is a one dimensional elastic continuum, so this is a distinctly different from the definition of a - of a string which cannot resist bending so and such examples we have seen in - in our previous lectures. Now when we model mathematically model beams we must make certain assumptions.

Because first of all we are considering it to be a one-dimensional elastic continuum and we are also saying that it can resist bending moment so we must in order to take care of this various things we have some simplifying assumption in our model for our mathematical model, so let us see what these assumptions are.

The - the first assumption that we make is about the material which is we will restrict ourselves to Linear elastic material which is of course and also homogeneous and isotropic, the second assumption that we make is that the beam is slender, so in today's lecture this will be an important assumption under which there is this well-known Euler Bernoulli hypothesis, this Euler Bernoulli hypothesis holds.

So what this tells us is that suppose I have a beam so this is a section of beam and that is something called neutral axis or neutral fiber when this beam deflects so I takes up shape something like this. So if you take a section of this beam before deflection a plane section of this beam before deflection which is perpendicular to this neutral fiber and neutral axis then in the deformed configuration this section remains plane and remains perpendicular to the neutral deformed neutral axis so this is Euler Bernoulli hypothesis.

Then we make the final assumption in the model that we are going to discuss today that's shear is negligible or another way of saying this is that the beam is infinitely stiff in shear, so we will assume that the shear strain is in the - in the beam is negligible and another way of saying this is that it is infinitely stiff in shear, so now we with these assumptions let us get down to modeling of the beam.

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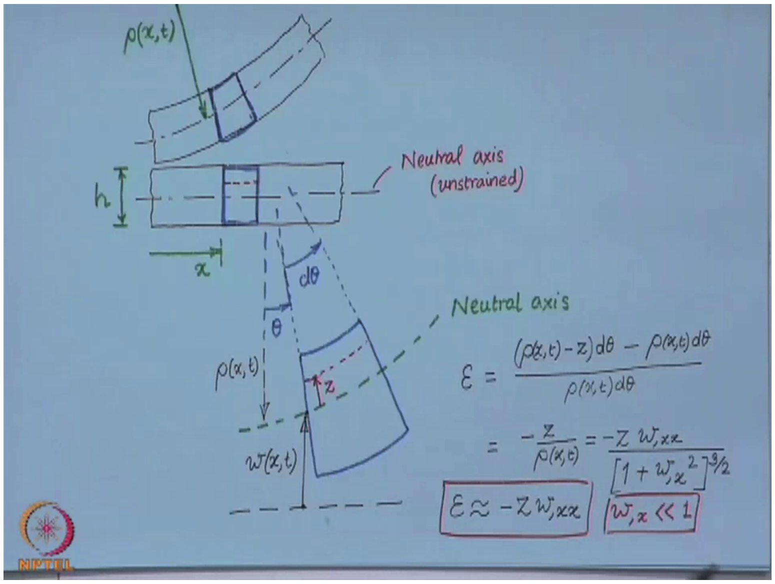

So once again let me draw this section of the beam, so this is our neutral axis or neutral fiber now this is neutral which means it is unstrained it reminds unstrained so this access in the undeform configuration and in the deform configuration it remains unstrained, so once again let us consider this element which upon deformation. So this is an element which undergoes deformation let the depth of this beam or height of the beam be denoted by h.

And once this beam deforms let us assume that the radius of the curvature at this point is given by at time t, show the radius of curvature at x at time t is rho of x and t. Now let us look at this element let us draw the free body diagram of this element so let this element be of angle d theta and this plane so this is an exit exaggerated figure so this angle is theta and angular length is d theta.

Now let this be the neutral axis then at a distance z is measured from the neutral axis or the neutral fiber let us look at another fiber so we can write so first we are going to find out the strain in these fibers of the beam. So as I mentioned that this neutral axis is unstrained so let us look at another fiber which is at a height z from the neutral axis or the neutral fiber, then I can write the strain as so the length of this fiber.

So first I have to write the length of this fiber now this radius of curvature is rho so length of this fiber can be written as so deform the length of this fiber is rho minus z d theta, so that is the length of this fiber. It is undeformed length before deformation as you can see this is same as rho times d theta, so since this length remains unchanged. So this in the undeformed configuration was of the same length as this so therefore its initial length was rho times d theta and the initial length.

So therefore this turns out to be now one over rho, one over the radius of curvature is known as the curvature and the expression of the curvature in terms of the equation of this neutral fiber. So if you represent that this equation of this neutral fiber in terms of the deflection of this neutral fiber from the undeformed neutral fiber so w x,t is the - is the displacement of this point from the - from the undeformed configuration so the equation of this curvature is given by w x,t at any time t.

So in terms of the equation of this curve the curvature as you know can be written as the double derivative with respect to x of w divided by 1 + del w del x whole square rise to power 3 over 2, now we have considered that we will consider a in this derivations that the slope of the beam is small. So under that assumption we can neglect we can drop by drop higher powers of the - the displacement variable w or the derivatives of the displacement variable or the field variable.

We can drop the higher order terms, and we can write the strain in this fiber as minus of z times the second derivative with respect to x of w, so w is our field variable so this is under the assumption that so we make this assumption that the slope of the beam at any location is much smaller than 1, so under this assumption our strain simplifies to this expression. Now so this is the strain in this fiber at the height z measure from the neutral axis.

And thus we have the strain in terms of the deflection of the neutral axis which is what we are going to track when we - when we represent the deflection of the beam so the deflection of the beam will be represented in the terms of the deflection of its neutral axis.

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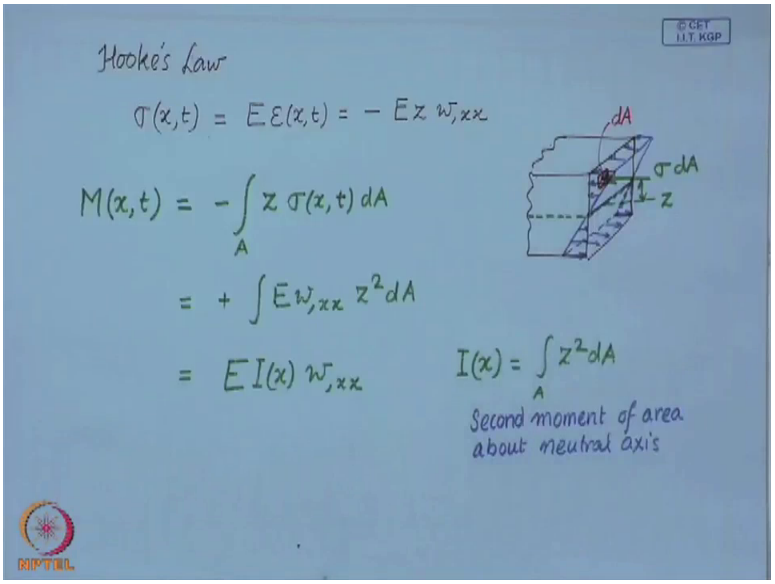

So once we have this expression of strain we can write we can bring in the constitutive relation using Hooke's law and we can write the stress in that fiber at a location x at time t, so this is the axial stress or stress in the actual fibers of the beam at a location z measured from the neutral axis. Now we have so this stress as you see here is linear in z so let us consider a cross section of this beam.

So this is the neutral fiber plane and if you represent the stress then so if z is positive we are measuring z from the neutral axis positive upwards, so now what we want to find out is the moment that comes because of this stress distribution on the cross section of the beam. So moment at any location x at any time t will be given by so if dA is a little area elemental area on this cross section then the force acting on this area is given by sigma times dA.

And the moment - moment about the neutral axis so we are going to find out moment about the neutral axis because of this stress distribution is given by so this arm so z cross this force. And if you do that calculation that turns out to be negative of and this has to be integrated over the total area A, so if you now substitute this expressions, so this negative and this negative turn out to be positive.

Now Young's modulus and the curvature the approximate curvature does not depend on this area integral so that can come outside and what we have here is where I this is known as the second moment of the area about the neutral axis, So this is the moment that is acting on this cross section.

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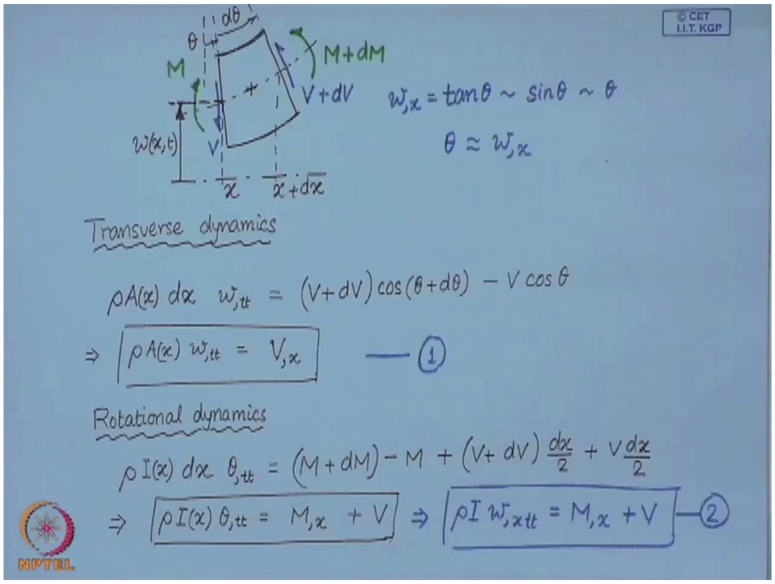

So now let us once again look at the free body diagram of this little element of the beam and introduce the interaction forces, so we have the shear forces at these faces and the bending moments so the sign convention what we follow so this is the positive shear force and bending moment and this so this is the undeformed neutral axis so this deflection is w x, t.

Now let me write down the Transverse dynamics the equations of Transverse dynamics of this little element whose free body diagram I have drawn, now so from Newton's second law if rho is the density of the material and if A is the cross-sectional area at this location x then rho times A is the mass per unit length so into dx which is the length of this element then this is the mass of this little element times its transverse acceleration must be equal to.

So I can write where what I have indicated before so this angle is d theta and this angle is theta, so I have taken projection of this forces in the transverse direction and you divide by dx and consider that cos theta is approximately equal to 1, then so this is the equation of transverse dynamics for the beam element, now along with the transfers dynamics now I will also write rotational dynamics.

So if rho is the density of the material and if I is the second moment of the area about the neutral axis, then rho times I give us the moment of inertia of this element about this - moment of inertia of this element about this access which is perpendicular to the plane of this paper and this is per unit length moment of inertia per unit length so times dx will give us the moment of inertia of this element times the angular acceleration.

So the angular acceleration can be written as the double time derivative of this angle theta of the element and this must be equal to the sum of all moments about the center of mass of the element we are writing the rotational dynamics about the center of mass so that turns out to be, so this implies upon dividing throughout by dx now this theta has to be somehow represented in terms of we know that this tan of theta for small slopes is almost equal to sin of theta which is almost equal to theta.

And tan theta is nothing but slope of the beam so therefore theta is approximately the slope the beam therefore from here I can rewrite this equation as, now you see in this two equation the bending moment M, we have represented in terms of the stress and which was calculated in terms of strain and which was calculated in terms of the deflection of the neutral axis our field variable w, now this V which is the shear force is determined from this equation of equilibrium.

Because this element is not sharing so this has to be determined because from the equations of equilibrium and that is what we have in this two equations so we will eliminate this shear force between these two equations.

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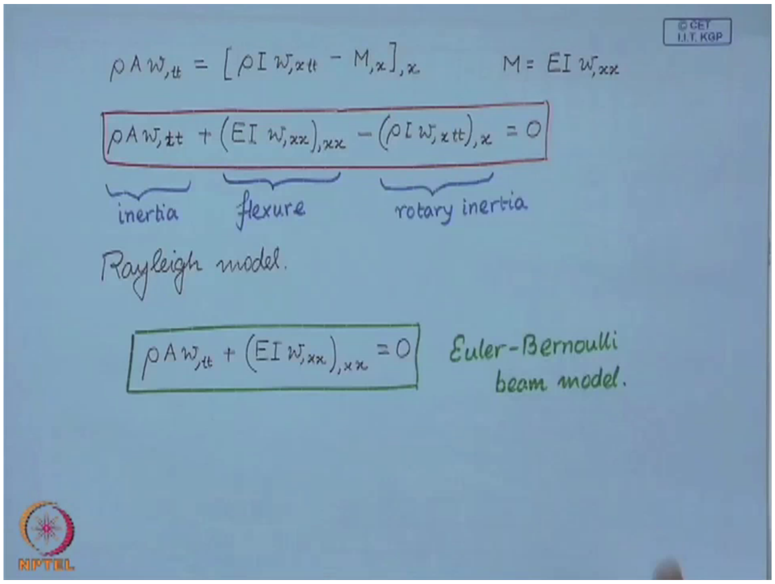

And if we do that so this was del v del x so this is and moment the bending moment we have determined this so therefore finally, so if I rearrange this equation so this is finally our equation of motion of the beam, now here this term is known as the flexure term, this term is the rotary inertia term and this is the normal transverse inertia term or translation inertia term, so this model of the beam is known as the Rayleigh model of the beam or Rayleigh beam model.

So in the Rayleigh beam model we have inertia term, the flexure term and the rotary inertia term of the element for very slender beams this term can be neglected and in that case this is known as the Euler Bernoulli beam model, so the Euler Bernoulli beam does not have this rotary inertia which is present in the Rayleigh beam model, now when you have forcing then instead of this 0 on the right hand side you have the force distribution.

So these are two very simple models for beams, next we are going to discuss the variational formulations of for beam dynamics so as we have seen before that this variational formulations gives us a very powerful alternative way of deriving the equation and not only that we also get the we have which we are not talked as yet which are the boundary condition so we will also get the boundary conditions the possible boundary conditions for the problem.

And this variational formulation also leads us or gives us some very powerful techniques for approximately solving discretizing the equations of motion as we have seen in our previous lectures.

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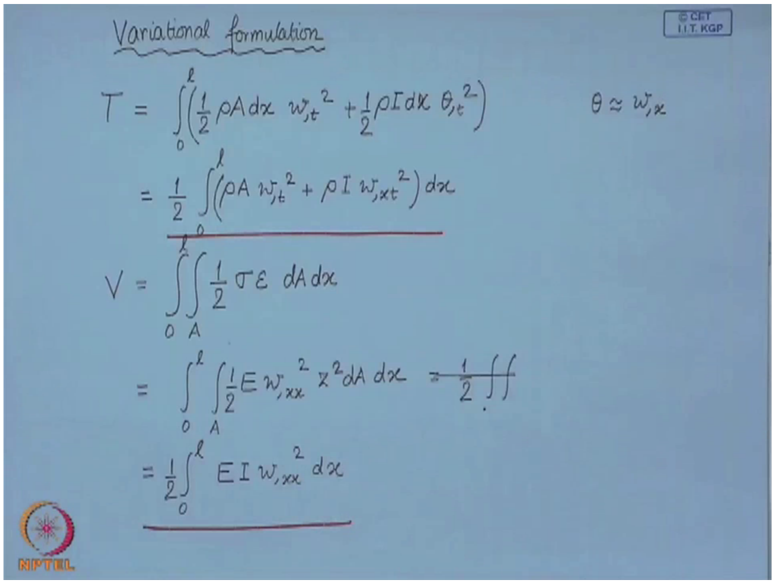

So let us discuss the variational formulations for the beam model, now so the first thing that we do in this variational formulation we write down the kinetic and potential energy expressions, so the kinetic energy of beam can be written as the translational kinetic energy so rho A is the mass per unit length so into dx will give us the mass of the little element times the velocity square.

So that is the translational kinetic energy plus so this gives us the second moment of - the moment of inertia of the little element times angular velocity square and a half. And this when integrated over the length of the beam will give us the total kinetic energy now using this approximation we can now rewrite.

So that is the kinetic energy expression of the beam element, now the potential energy of the beam can be we know that the potential energy per unit volume is for a linearly elastic material is given as half the strain so over the volume so dA is the small area of the cross section and dx is the small length of the element. So if we integrate this over the area and over the length then we should have the potential energy.

Now if you insert the expression of stress and strain in this expression, then this is what you are going to get and therefore there is of course this one half, so this Young's modulus and the curvature square this will have nothing to do with this area integral so they can come out. And what we have is z square dA integrated over the area and that we already know is the second moment of the area about the neutral axis of the beam. So these are the expressions of the kinetic and potential energies.

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$$\frac{1}{2}\delta\int_{t_1}^{t_2}\int_{0}^{l}\left[\rho A w_{,t}^{2} + \rho I w_{,xt}^{2} - EI w_{,xx}^{2}\right]dx\,dt = 0$$

$$\int_{t_1}^{t_2}\int_{0}^{l}\left[\rho A w_{,t}\,\delta w_{,t} + \rho I w_{,xt}\,\delta w_{,xt} - EI w_{,xx}\,\delta w_{,xx}\right]dx\,dt = 0$$

$$\Rightarrow \; + \int_{0}^{l}\rho A w_{,t}\cancelto{0}{\delta w}\Big|_{t_1}^{t_2}dx + \int_{0}^{l}\rho I w_{,xt}\cancelto{0}{\delta w_{,x}}\Big|_{t_1}^{t_2}dx - \int_{t_1}^{t_2}EI w_{,xx}\,\delta w_{,x}\Big|_{0}^{l}dt$$

$$+ \int_{t_1}^{t_2}\int_{0}^{l}\left[-\rho A w_{,tt}\,\delta w - \rho I w_{,xtt}\,\delta w_{,x} + (EI w_{,xx})_{,x}\,\delta w_{,x}\right]dx\,dt = 0$$

$$\Rightarrow \; - \int_{t_1}^{t_2}EI w_{,xx}\,\delta w_{,x}\Big|_{0}^{l}dt - \int_{t_1}^{t_2}\left[(EI w_{,xx})_{,x} - \rho I w_{,xtt}\right]\delta w\Big|_{0}^{l}dt$$

$$+ \int_{t_1}^{t_2}\int_{0}^{l}\left[-\rho A w_{,tt} + (\rho I w_{,xtt})_{,x} - (EI w_{,xx})_{,xx}\right]\delta w\,dx\,dt = 0$$

Now when we derive the equation of motion using the Hamilton's principle we write it like this mathematically so this turns out to be so this is the statement of the Hamilton's principle so when you take the variations now we will integrate by parts this term with respect to time this time once with respect to space and once with respect to time and this time term twice with respect to space.

So if you do that and rearrange then what you are going to obtain so when we integrate by parts with respect to time this term so and this term once with space and ones which time so let me first integrate with respect to time and this I have to integrate with respect to space so this will get a derivative with respect to time with the negative sign this will get similarly, so this is what we are going to get.

Now here as we know that the variation must vanish at these two time points so this terms will vanish so we have so we will be left with this as a boundary term, now here still we have this space derivative which we can once again integrate by parts with respect to the space and if you simplify, so these terms are going to vanish now if you integrate by parts once again and do the simplifications then you can check so this these are the boundary terms, and so this is what we are going to get.

Now we invoke our statement of the variational formulation which says that boundary variations and the variation over the domain if this whole thing has to vanish then these two must vanish individually so the integrant must vanish for arbitrary variations over the domain and that as you can see will give us the equation of motion so this integrant is going to give us the equation of motion what we had derived before.

Now look at the boundary terms so equation of motion we have already written let us concentrate on the boundary terms.

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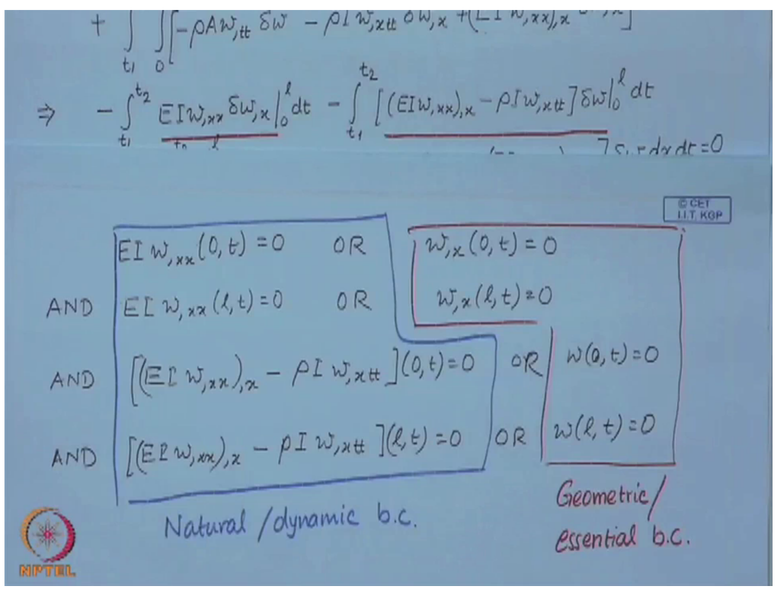

So - so here we can have so these are the boundary terms so we can have this at 0 to be 0 or we can have the slope at 0 to be 0 are fixed then at L again or and these two are connected by an and so you can have this and this or this and this or this and this is so combinations so this is from the first boundary term.

And from the second boundary term which must again be connected with an and or so here we have this displacement and so these are the possible boundary conditions so this as we can recognize the bending moment is 0 or the conjugate of the bending moment is the - the angle with the deflection that must be 0 or this is the shear force now this is an additional term because of rotary inertia so this must be 0 or the displacement which is conjugate of this by the conjugate.

I mean the product which gives us energy or work so either force or its conjugate this is displacement is 0 either moment or its conjugate angular displacement must be 0. Now this set of boundary conditions are the Geometric or also known as the essential boundary conditions and this set is known as the dynamic boundary conditions.

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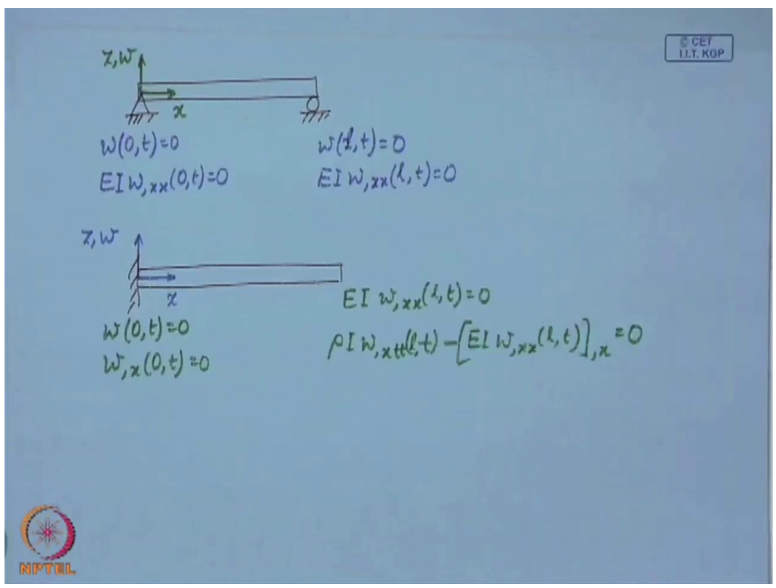

Now let us quickly at an example so we know that we have simply supported beam like this, so the boundary conditions here and the bending moment similarly here, you again have the deflection to be 0 for cantilever beam the boundary conditions here deflection 0 and you know that slope is also 0 here we have the shear force and the bending moment to be 0. So these are two examples that we have considered where we - we have written out the boundary conditions.

There can be other examples and we will discuss this in the subsequent lectures so what we have discussed today we have looked at some models of beam transverse dynamic of slender beams and we have derived the equation of motion for - for the Rayleigh beam and the Euler-Bernoulli beam and we are looked at the variational formations from where we can also derive the equations of motion so we will continue this discussion in the next lecture.