Vibrations of Structures

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Lecture – 14
Forced Vibration Analysis – III

So, we have been looking at Forced Vibration Analysis of one dimensional structures in the past few lectures. Today, we are going to look at, we are going to continue our discussions on forced vibration. In the previous lectures, we have seen harmonic forcing, systems with harmonic forcing which we have separated in space and time, there is a distribution function for the force and then there is a multiplying harmonic function of time.

Today we are going to look at general forcing which means forcing which need not necessarily be separable in space and time. So we are going to look at general forcing in today lecture.

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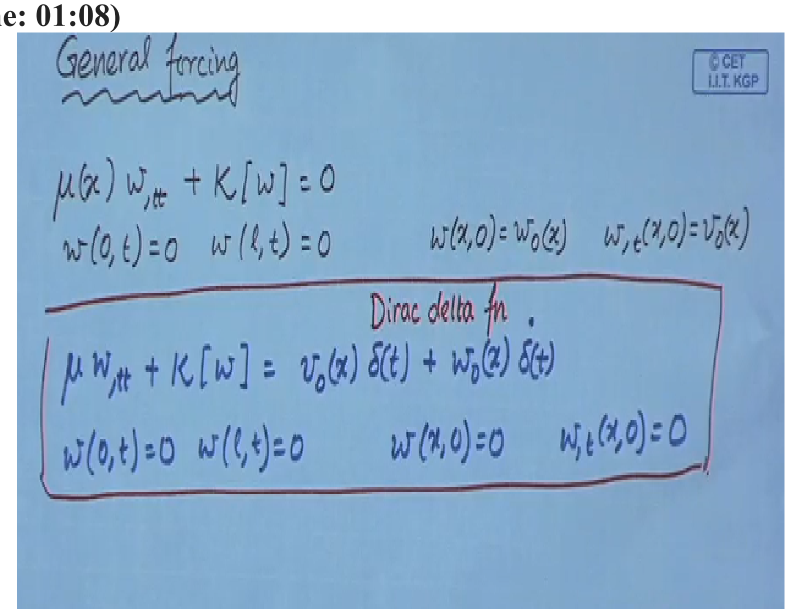

Now, let us consider some examples of this general forcing, in one of the previous lectures you have already seen that an initial value problem of with certain boundary conditions and initial conditions, so the initial value problem can be equivalently stated as. So, these two statements are equivalent so essentially, we have converted an initial value problem to a forced vibration problem, here is the Dirac delta function and this is the time derivative of the Dirac delta function T=0.

So this problem is a forced vibration problem equivalent to this initial value problem, so in this case as you can see this kind of forcing is special in a very special form and this is not harmonic forcing. So, forcing of this type or when a force is travelling on a string for example on a bridge you have a car travelling at a certain speed so you have a travelling load problem or a travelling mass problem.

So, such forces load the structure and they are not harmonic forcing in fact in such problems space and time are coupled they are not separable.as we will presently see in this lecture, so problems of this type can be considered of general forcing. These are problems with general forcing.

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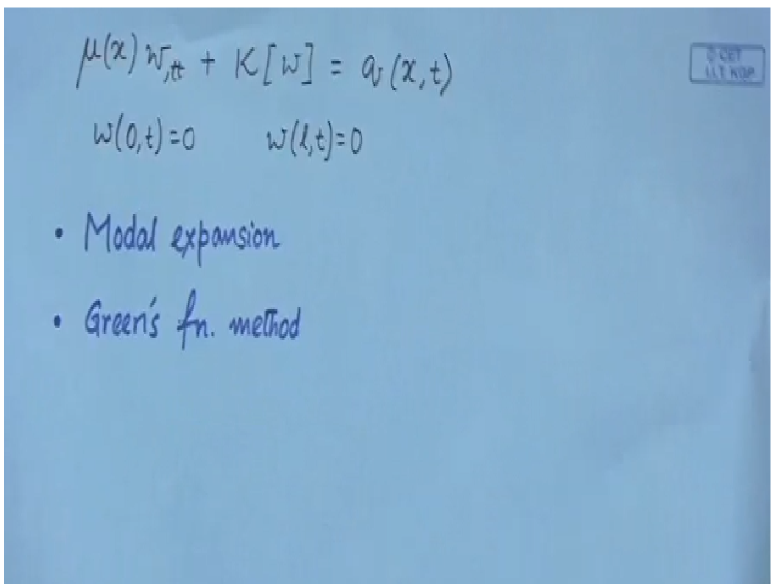

So, let us look at how do we solve a general forcing problem, we consider an example, let me state the problem, so we have the equation of the differential equation of motion with of course certain boundary conditions, so this q x, t is a general form of a force which we are now considering, to solve to prove such problems we have various methods, what we are going to consider here today are the modal expansion method, we are going to concentrate on this modal expansion method.

And towards the end we are also going to look at the Green's function method, so essentially today we are going to look at these two methods.

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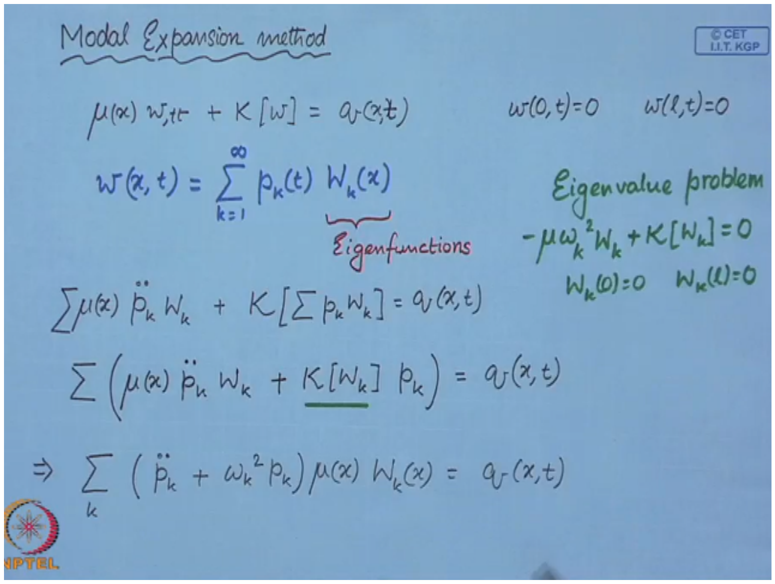

So, let us start with this modal expansion method. We have discussed in an initial in previous lecture, how and why the modal expansion technique works, the modal space and the eigen value problem of the systems form a complete basis which means any shape or any configuration of the system can be represented by these eigen functions, so this is the basis of this modal expansion technique, so we are going to look at this problem with certain boundary conditions.

And as we have discussed the solution we will represent in this form, so these are the eigen functions of the corresponding eigen value problem, now when we this in the equation of motion, so we have to represent so we need not worry about the boundary conditions, so here we now have to determine the coeffience, so that this expansion satisfies the equation of motion. So, we have substituted this expansion in equation of motion.

And now here this K linear differential operator and therefore what we can do is we can commute this summation and the operator and we can rewrite this equation in this form, now recall that the eigen value problem for the system, so this was the statement of the eigen value problem, so here differential equation of the eigen value problem, from here we can see this expression of the stiffness operator operating on the eigen function.

So, we can replace it from here and then what we obtain, we obtain this.

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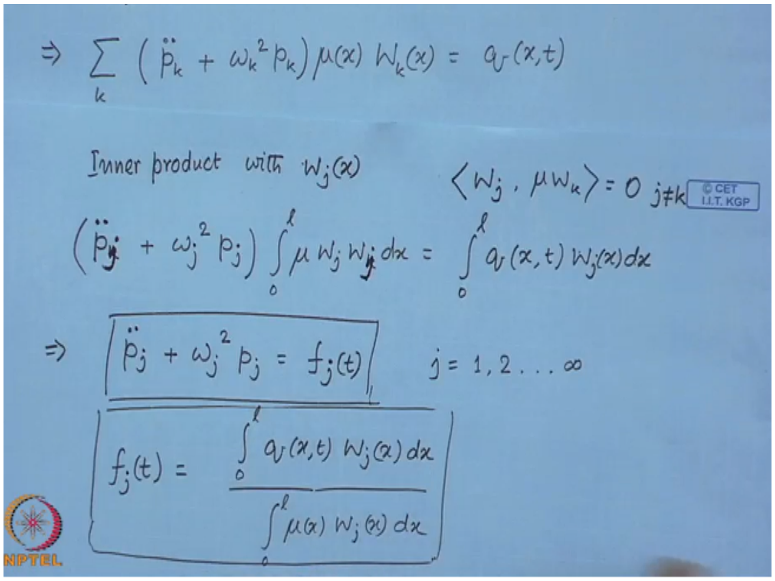

Now if I use the orthogonality conditions, so I take the inner product, so what I do is, I multiply both sides of this equation with wj and integrate over the domain of the problem, so then we know that from the orthogonality of these eigen functions we already know that so if j is not equal to k then this is 0 then what this is going to do is filter out the j th coefficient expansion, so we are going to get the j th equation.

And therefore, I can write this in this form, where this force term fj is given by and so and this I can take for a different wj generate all these equations corresponding to the coordinates of the j th of the individual modes, so I have these equations governing the coordinates of the system in the configuration space or the modal space and that is a forced dynamics problem where the forcing of the j th mode is expressed in this form.

Once we have applied the external force where we can easily compute by performing this integration, now this is a second ordinary differential equation, so we will need two initial conditions for this problem.to solve the problem completely.

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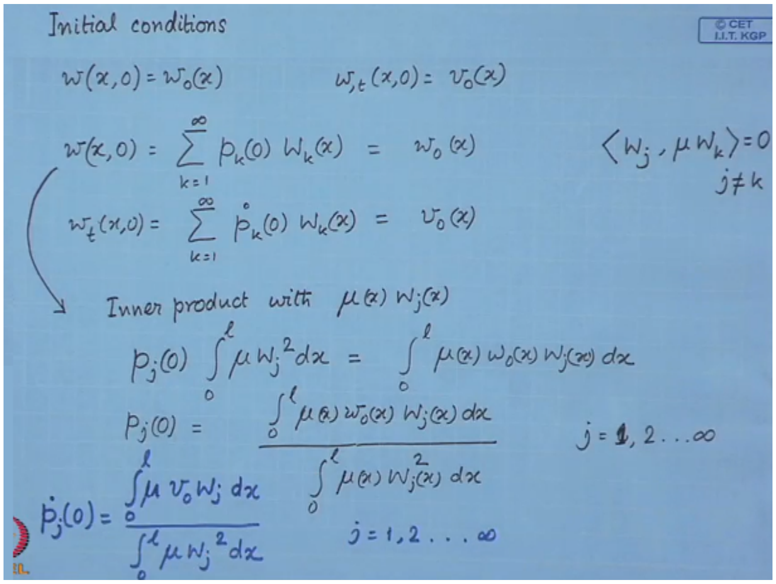

So, what we need are the initial conditions, now we are given these initial conditions on the field variable, so we once again have this expansion and similarly from the velocity initial conditions we get this equation.so we have to solve this equation. With initial conditions on p at 0 and p dot at 0 which we are now going to solve from these two equations. So once again if we use the inner product, now here remember that this is the orthogonality condition.

Therefore, what we must do to solve these coefficience in a straight forward manner, so what we do is, we multiply both sides with mu and wj and integrate over the problem, once we do that is going to filter out the j th term in this expansion. And therefore, so that is how we can solve pjo and we can taken we can obtain all these infinitely many initial conditions on the model coordinate, similarly we can also find out the initial velocity condition.

So, once we have all these initial conditions p and p dot then we can solve this problem

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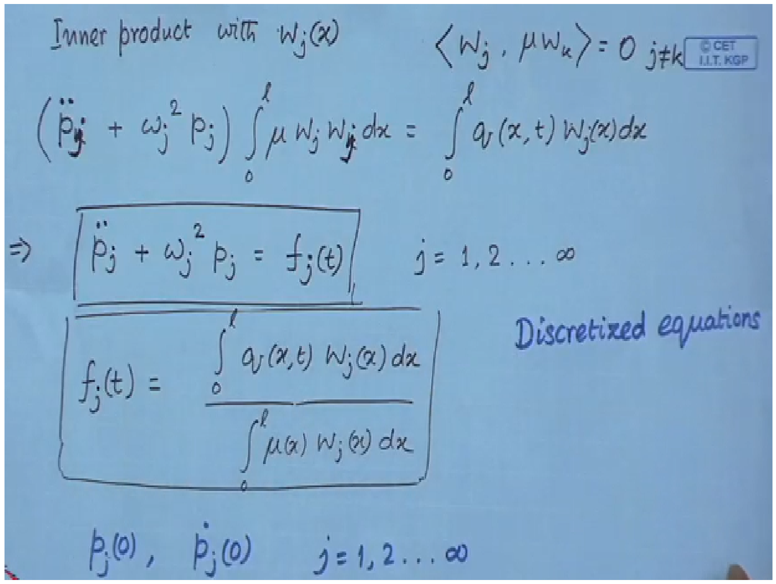

Once we have this initial condition, then we can very easily solve, we already know the solution of this kind of forced dynamics, what we have achieved in this process is the discretized equations of motion, so we have discretized the forced vibration problem.

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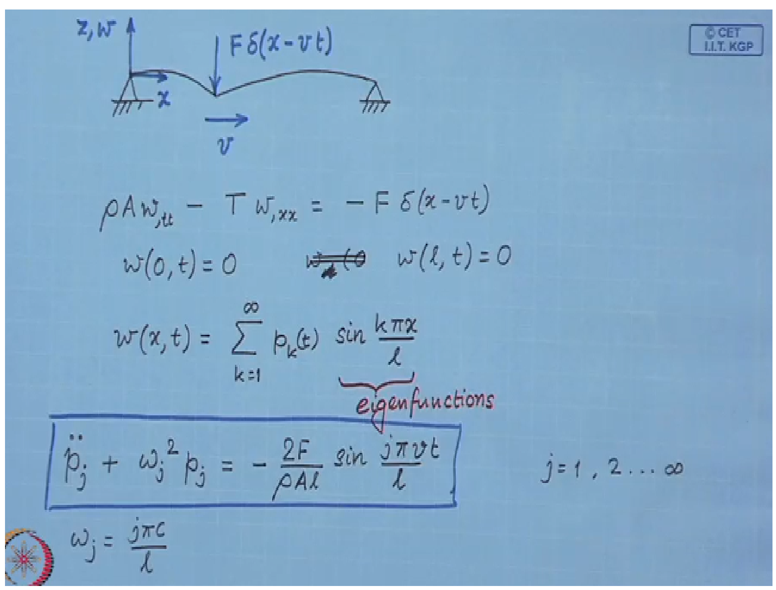

Now let us look at some examples, so the first example that we are going to consider is that of a force travelling on a string, so let us look at this problem, so we have a tot string with a considerate force travelling at a speed v, so this f is a constant, so let me write down the mathematical formulation of this problem, so we have the string equation of motion with this forcing and of course the boundary conditions.

So, these are the boundary conditions and there can be certain initial conditions, now here we will be looking at the forced problem, using this expansion in terms of the eigen function of the tot string, now if you substitute this in here and follow the procedure that we just now discussed then for the jth mode discretized equation of motion appears to be in this form, where this omega j so this is our equation of motion.

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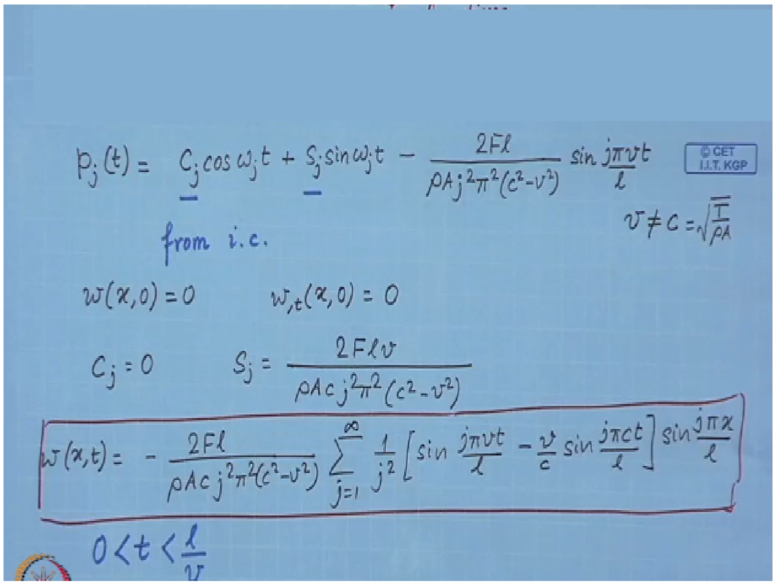

Now the solution of this equation can be easily written as, so this is the solution of this equation provided v is not equal to c, where c, is the speed of transverse waves in the string, so c is the speed of transverse waves in the string as long as the velocity or the speed of travel of this force is different from c, is less than c for example this is going to be the solution, so you see when v equals c, and if you look at this omega j.

Then you will find you will realize when v equals c this becomes a resonant forcing of the string in the j th mode and therefore in all modes all modes will be resonant if v equals to c, resonant forcing of c, as you know already in a different form, so this solution is valid as long as v is not equal to c, you can also find out the resonant solution, using the standard techniques that you have studied in discrete vibration problem.

In a course on discrete vibration, now so this is a solution form therefore here these are unknown coefficient which form the homogenous solution for the system which are to be found out from the initial conditions, so we will consider a stationery string before the force starts travelling on the string, so initial displacement is zero, initial velocity is zero and if you solve for these coefficients.

Then you will see that the coefficient turns out to be and therefore the final solution of our problem. So, this is our final solution and this solution is valid of course when v is not equal to c not only that but also the time interval for which the force is on the string as soon as the force crosses the string this solution is no longer valid, you have to solve for the string with certain initial conditions which are the final conditions at t equals to l over v of this.

So you will get the final displacement of the string and the final velocity at t equal to l over v from this expression that goes in as the initial conditions of the string in free vibrations. So that would be an initial value problem, so after this you have to solve an initial value problem to understand how the string behaves after the forces left the string, so that is an initial value problem which you have to solve using the final conditions from this solution.

And we have already seen how initial value problems are solved, let us look at certain configurations of the string.

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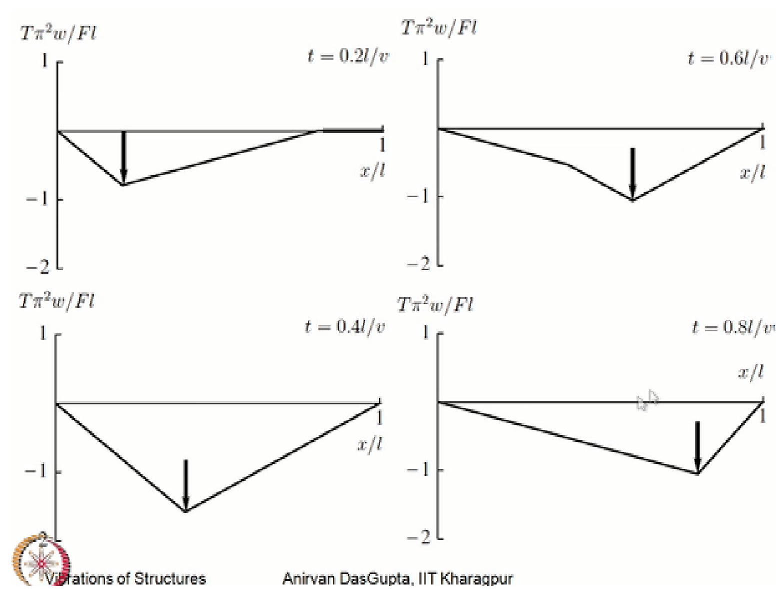

Let us consider, you will see the configuration of the string at certain time instance, for example here this is point 2l over v so this force has travelled one fifth, the total length of the string and this is how the string looks, like you see that this portion of the string is as yet undisturbed but when t reaches point four l over v, forty percent of the distance has been covered, so this information that a force is travelling on the string has reached the end here at point 6 l over v.

We can see cling forming here actually the information is now reflecting back and forth on the string, now to this shows that point l point 8 l over v. Now these are snap shots.

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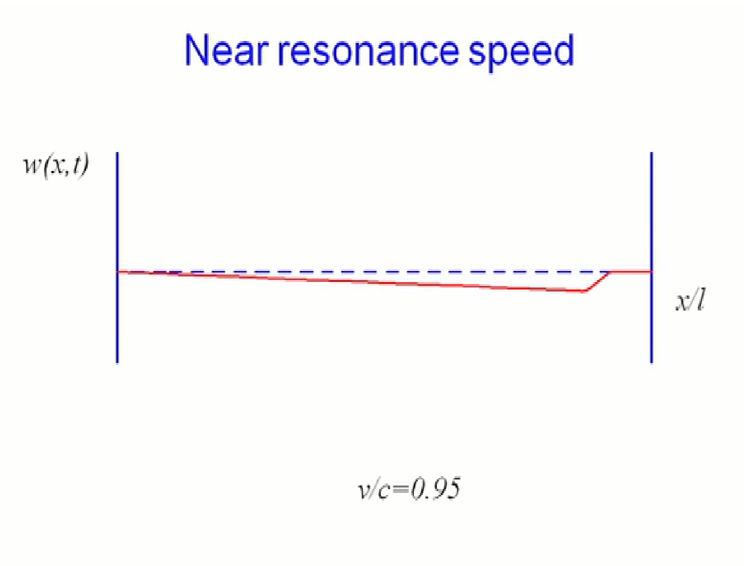

Here I have simulation or animation of the same thing, so I have taken number of terms in our expansion, I just now discussed and I have plotted the shape of our string in different time instance and prepared this animation. Now here you can see that the string appears to move in a very strange manner, it looks very strange that it looks static, certain portions of the string looks static for certain time instance.

So, what is happening is this disturbance which has created by this travelling force travels on the string as a wave and it reflects back and forth on the string from the boundaries, unless the disturbance reaches a portion of the string, the string does not know that the force is travelling, so it remains, it looks like static, remember two things in this animation, one thing is that this is a slow motion of the actual thing that is happening.

So, I mean it is very difficult to observe this kind of motion by naked eye, you have to record it and run it in slow motion, the second thing is this animation does not take care of the motional string after the forces left, this is just a repeat of the animation when the force is travelling on the string, so here you do not see that initial value problem solution, now this is at low speed where v over c is 0 point 1, so it is ten percent of the critical speed.

In this animation, this is a near resonance, so point 95, v over c 0 point 95, so it is 95 per cent the critical speed, here you will find those reflections of the clings etc. they are not so easily observable at least, and the amptitude is growing as the force moves from left to right which is a typical behavior of near resonance solution, so at resonance you know the amptitude goes linearly with time, this is near resonance slightly lower, so you can see the behavior of the string.

It is distinctively different from the previous behavior, these motion, this is an transion motion, this has to be understood in terms of the propagation of waves in the string, which we are going to take up later in this course.

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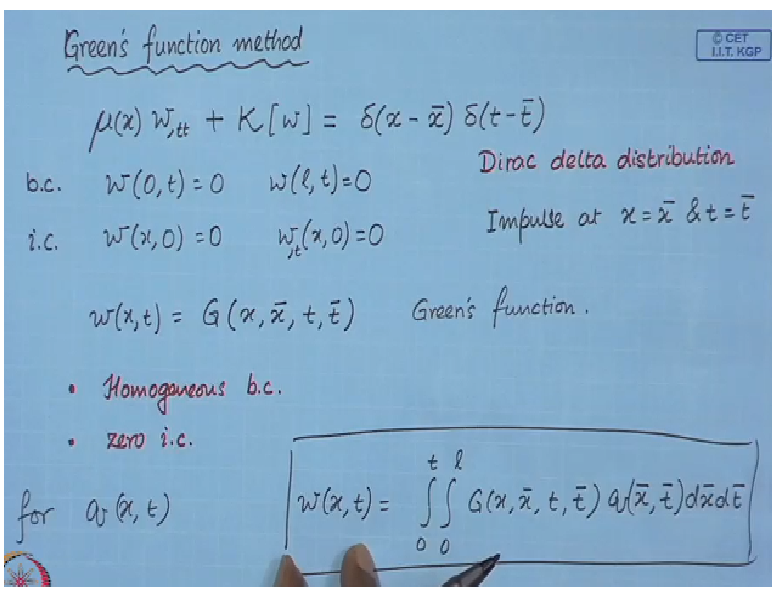

So we have looked at modal expansion solution, there is another method which we are now going to look at briefly, we have already come across this Green's function method in our previous lecture, so what we are going to solve is , so in the Green's function method we solve a problem of this type, where the forcing has a very special form, these are Dirac delta functions, or Dirac delta distributions.

So the forcing is of a very special form like this, so what in effect this represents, is it represents an impulsive loading at the location x equals to x bar and time t equal to t bar. So, we give an impulsive loading at the location x equals to x bar in the system and time t equal to t bar and we are going to look at the solution of the system a loading, here the boundary conditions are homogenous and initial conditions are all 0, the solution of this system is known as the Green's function.

We will represent this in this form, so these are, these points are important that we have homogenous boundary conditions and zero initial conditions, so always this Green's function is calculated with these conditions, so this will be classically assumed whenever homogenous function is calculated, so homogenous boundary conditions is zero value condition. So we know in previous lecture, how non-homogenous conditions can be converted to homogenous boundary conditions.

So if required we have to do that and then calculate the Green's function. So why let us first look why this or how this Green's function method works. So, we claim that for any arbitrary forcing, when we have forcing in the form of qx, t general forcing, then the solution actually can be obtained in this form, so once we have the Green's function, the solution to an arbitrary forcing can be determined like this. So let us see why this happens, or how this is happening.

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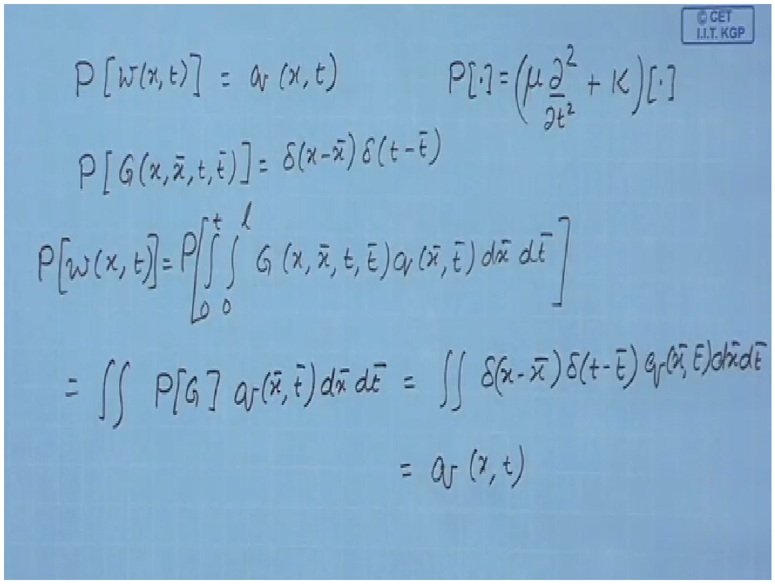

So, let us see why this happens, or how this is happening, so let me write this differential equation, so let me write this differential equation as, so this is our problem, where p, so I am representing this full operator using p, now we do is we are actually solving this problem first with this of course the homogenous boundary conditions and zero initial conditions and then what I said was the solution is given by this.

So, let me apply the operator t on this, not his integral over x bar and t bar so this operator can go inside and it will operate only on g and we know that p of g is and this is equal to therefore, so therefore this is definitely a solution of, so w given by this argument is definitely a solution of this system, so this shows that we have the solution w in terms of the Green's function.

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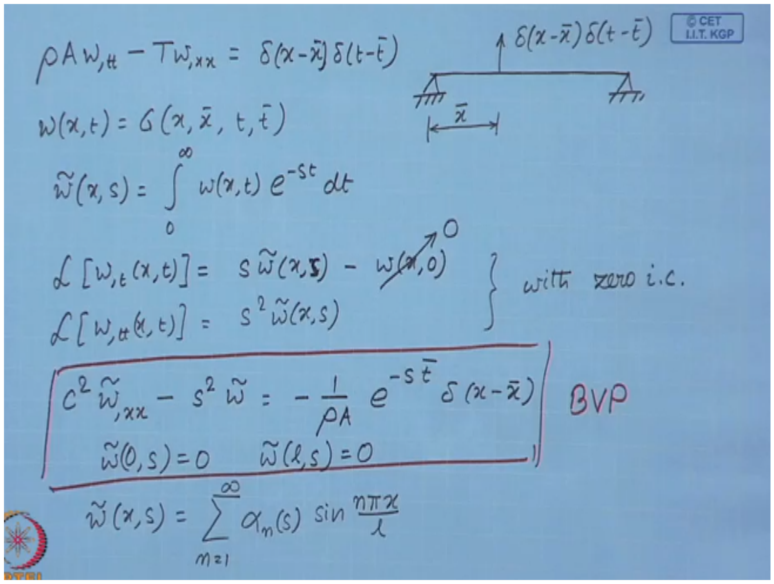

Now let us calculate the Green's function for string, so we are giving an impulsive loading t equal to t bar is equal to x equal to x bar and the solution is a Green's function and we are considering homogenous conditions and zero initial conditions, so , first what I will do to determine Green's functions, so we have to solve the Green's function now by solving this system, this problem, so what we are going to do to simplify things is we will take the Laplace transform of this equation.

So we define Laplace transform as follows, we know that the laparce transform of, here this is zero because the initial conditions are zero, similarly laparce transform is given by this width zero, initial conditions, now once you take the laparce transform of this equation then and rearrange, we obtain this equation along with the boundary conditions like this, so this is our boundary value problem that we need to solve now.

So this is, we have homogenous boundary conditions, so therefore we can look for solutions in this form.

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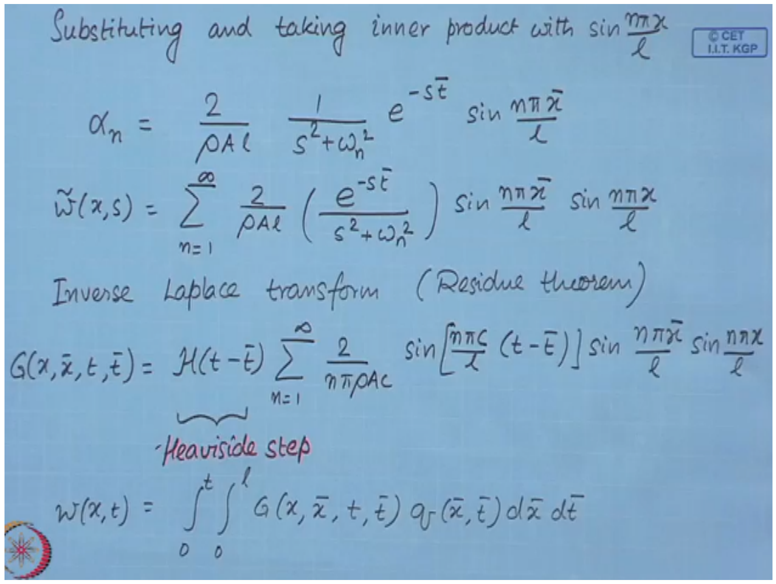

So, substituting in this boundary value problem and when we rearrange, take the inner product both sides sin n pi x over l, so we substitute and take inner product, so we substitute this in differential equation, take inner product with sin n pi xn over l on both sides and then it filters the n th coefficient, which I can write and this can be done very easily, so we will obtain this co efficient and finally the solution, therefore is obtained.

Now this is this solution in the laparce domain, now we have to take inverse laparce transform, which can be done by residue theorem and the final solution, these are simple steps, I will write the final solution, here this is the heavi side step which comes from the causality which means the solution does not exist before the impulse is provided, t less than t bar is zero, the solution is zero, so this solution is not zero only when t is greater than t bar.

So, this is your solution on the Green's function and as I have discussed for any arbitrary forcing you can determine the solution using this integral, so this is a very powerful method of approaching forced vibration problems with arbitrary forcing, now you can so we know how to solve the Green's function method here, here we have used the modal expansion technique to solve the boundary value problem.

In a previous lecture, we have seen how the Green's function was solved in a different method, so this approach gives us a very powerful method of solving forced vibration problem of continuous systems, now in this lecture we have considered the general forcing, of one dimensional continuous systems and we have looked at the modal expansion technique and the Green's function technique for solving the forced vibration problem of one dimensional systems, so with that we conclude this lecture.