Vibrations of Structures
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Lecture – 12
Forced Vibration Analysis - I

We have been looking at the response of one dimensional continuous systems to initial conditions and we have also looked at how this initial value problem can be recast as a forced vibration problem. Now in this lecture and in the next few lectures, we are going to concentrate on the forced vibration analysis of continuous systems. Now the question actually arises what are the sources of forcing, how or why we should study forcing.

So there are various reasons, for example, you can have a system with an actuation, say for example, for vibration control or for some other control. So these actuators they will excite the system, the modes of the system. Secondly, you can have a fluid forcing for example, a bridge or high rise building that will be excited by the flowing wind. So that provides forcing to structures.

Then there are earthquakes and such natural sources of forcing and finally and very interestingly forcing is also used for evaluation and testing of materials. For example, to detect flaws or phases or faults in material or in a structure. So from these considerations, it becomes important to analyse forced vibrations of systems. So let us briefly look what the ways of forcing a structure.

So you can have an actuator, you can just attach an actuator to on the structure and you can force it. You can force a structure like a string by bowing, so in a violin for example, you use a bow to excide the string by bowing or you can put, you can hit the structure with an impact hammer and that gives us impact or impulsive forcing to the structures. So today, we are going to look at forcing, we are going to start our discussions on forced vibrations of continuous systems, one dimensional continuous system.

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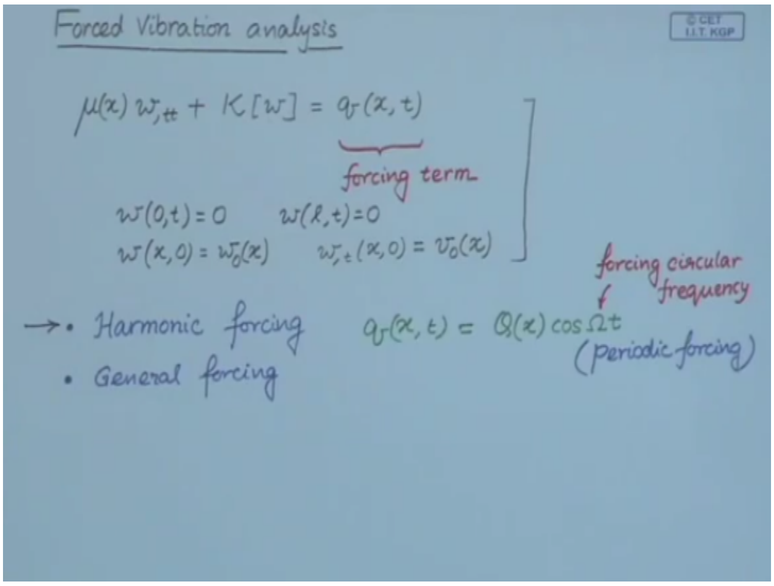

So let us consider a system, let me first formulate the problem mathematically. So for a system which can be put in this form, the kinds of system we have been discussing can be put in this form. So here, this represents a general forcing. Now along with this of course you have the boundary conditions, let us say zero, just to take an example and you have initial conditions.

So this is the complete formulation of the forced dynamics of a system that can be represented by this differential equation. Now this forcing terms as you can see makes the equation of motion in homogeneous. So we no longer have w equal to zero which is the trivial solution, as a solution of this system.

Now there can be various kinds of forcing, you can have harmonic forcing, which is the most common kind of forcing specially when we are evaluating or testing a structure we provide harmonic forcing and we try to see the response of the structure whether it matches with our expected response or not. So this harmonic forcing is one of the very important types of forcing which we are going to look at.

The second is the general forcing which can be, so in the harmonic forcing for example, your q of x t can be for example, q of x Q of x times cos sign of omega t, where omega is the forcing frequency. So is the forcing circular frequency. So here as you can see, this kind of forcing is separable in space and time, possibly separable. So for example, one term, one frequency forcing like this, so you have this as a separated in space and time.

Now this is the amplitude function or the distribution of the force and this is the temporal variation of the force and any periodic forcing as you know can be represented as a series of harmonic forcing. So if you know the solution for the harmonic forcing, then you can also find out the response to any periodic forcing. So we can also deal with periodic forcing if we know how to find out the response to harmonic forcing.

Now when you have this non-separable, the space and the time part non-separable and we are going to look at some examples of this forcing and we have actually looked at one of the examples of general forcing when we write the initial value problem as a forced vibration problem. And we are going to discuss this shortly in the later lectures. So today we are going to focus on the harmonic forcing.

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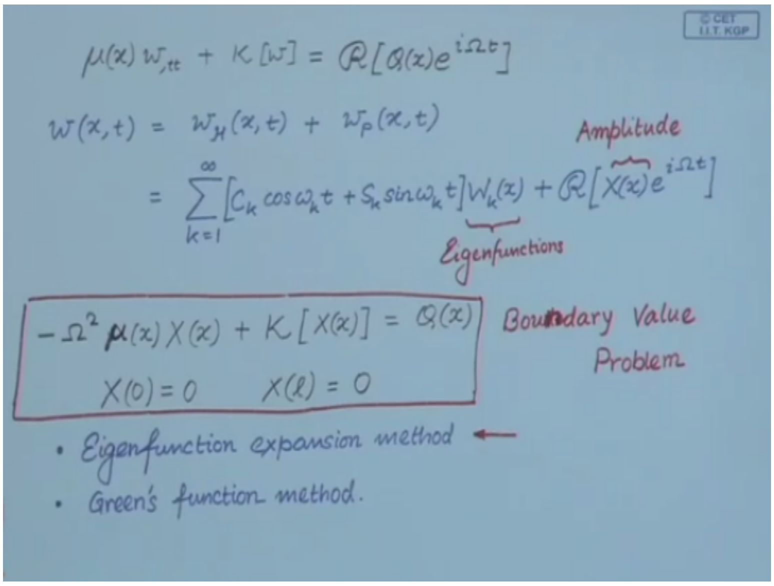

So let me write the differential equation of motion as, so the forcing that I discussed just now can be represented as a real part of this complex forcing term whereas I mentioned, this is the force distribution function, this is the circular frequency of forcing and this are R represents the real part of this argument.

Now the solution, so this of course along with the boundary and the initial conditions will completely specify the forced vibration problem. Now first we write down, must write down the general solution of this differential equation. So as you know, the general solution of such a differential equation can be written as the homogeneous solution, which means the solution with zero forcing plus the particular solution which is due to this forcing.

So this kind of a solution satisfies this differential equation since the homogeneous solution will actually go to zero once you substitute here, the particular solution will satisfy or equate the right hand side. So the solution, the homogeneous solution we have been looking at this homogeneous solution in the last few lectures, it can be represented as, so this is the homogeneous solution which is expanded in terms of the eigen functions of the corresponding eigen value problem.

So the eigen value problem was obtained by considering the homogeneous problem and searching for special solutions which are separated in space and time. So these, so from there we obtained these eigen functions and we have been representing the solution for solving various kinds of problems for example, the initial value problem. So here again we come across the solution, so this the homogeneous solution and this is the particular solution.

So which satisfies or which meets this non-homogeneous term on the right hand side of the differential equation. Here this is the amplitude function. So the amplitude functions of the response. So as you know that in a undammed system the response is proportional to this harmonic time function. So we have written this out as the real part of this amplitude times exponential i omega into t.

Now if we substitute this solution from in the equation of motion, the differential equation, then what we obtain, so this term is going to go to zero. So what remains comes from this term and if you substitute this and make a little bit of simplification of the equation then what you will obtain is, so this is the differential equation in x. So this amplitude function, so X of the space coordinate x.

So this is the differential equation that you obtain by substituting this solution in this equation of motion. Along with this, you also have the boundary condition which this amplitude function must satisfy this comes from the boundary conditions which we wrote out when we formulated the problem. So what we obtain is the differential equation in this amplitude function with, along with these boundary conditions.

This specifies what is known as the boundary value problem. So this is the boundary value problem corresponding to the amplitude function of the particular solution. So we must solve the boundary value problem in order to solve this amplitude. Now there are, there can be various ways of solving this boundary value problem. One is the eigen function expansion which is what we are, we have been looking at in the past few lectures.

This words on the premise or the fact that for self-adjoint problems, you have the eigen functions which are all real and which form a complete basis for the system. So by complete basis, I mean that, any configuration or shape of the system can be represented in terms of these eigen functions and we have been looking at this method for in the past few lectures.

So you can represent any shape using these eigen functions as we have done even for the homogeneous solution. So this eigen function method is one of the methods that can be used to solve this boundary value problem, the other is the green's function method. So this is another method which we are going to look in, in the next lecture. So today, we are going to focus on this eigen function expansion method for solving the boundary value problem.

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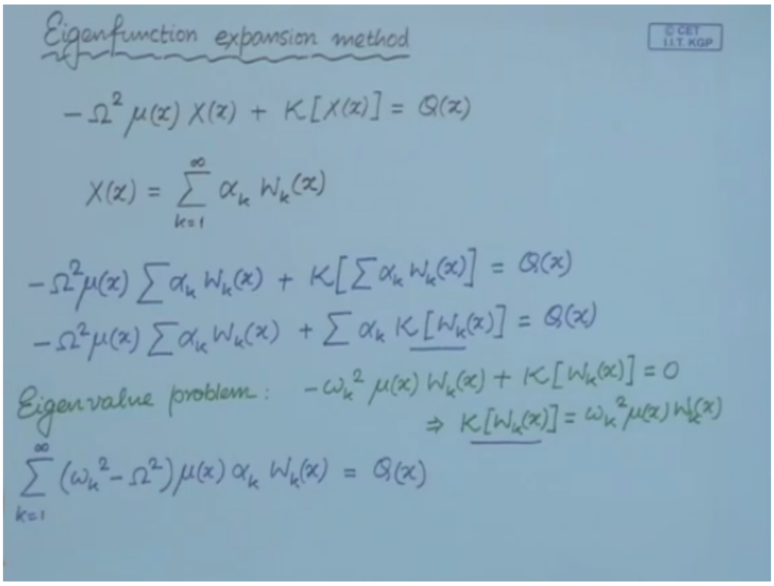

So what we are going to, so in the eigen function expansion method, so we have our differential equation of the boundary value problem in this form, we are going to expand this solution, the general solution of this differential equation in terms of the eigen function of the problem. So these are the eigen functions of the problem which we have obtained previously by solving the homogeneous problem.

So if you substitute this expansion in the differential equation then, so here of course these alpha k's are constants which are to be solved. So this is what we obtained. Now this k is linear differential operator. So I can exchange the summation and the operator and right it like this. Now recall that the eigen value problem for this operator for the system read. So therefore I can write the operator acting on the K eigen function in this form.

Now this is what I am going to replace here. So if I do that and simplify, so this is the equation that I obtained which has these unknown coefficients alpha k which I now need to solve. So for this, we can use the orthogonality condition for the eigen functions. So let us see how we can solve using the orthogonality condition.

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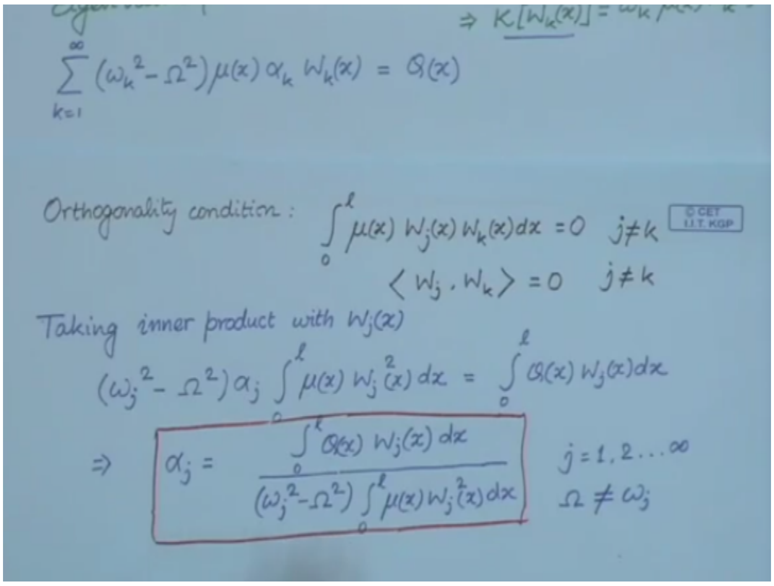

So the orthogonality condition of the eigen function for the system that we are considering that can be represented as, so for j not equal to k, we have this orthogonality condition. So we sometimes denote this as the inner product like this. So to solve these coefficients what we can do is we can multiply both sides with the jth eigen function and integrate over the domain of the problem.

So will say that we take inner product the jth eigen function and when we do that what this does in effect because of this orthogonality only the jth term is filtered out. So which means if we do this inner product, we take the inner product then what we are going to obtain this condition and therefore, so that is the solution for alpha j, now I can take j from one to infinity and solve for all these alpha j's.

But then this is contingent on the condition that the forcing frequency is not equal to any of the natural frequencies of the system. So the circular forcing is not equal to the circular natural frequency, any of the circular natural frequency of the system. So otherwise, this alpha, the corresponding alpha j will go to infinity. So you do not have a finite solution in that case. Now let us look at the situation when, so this completes our solution for the non-resonant case we may say.

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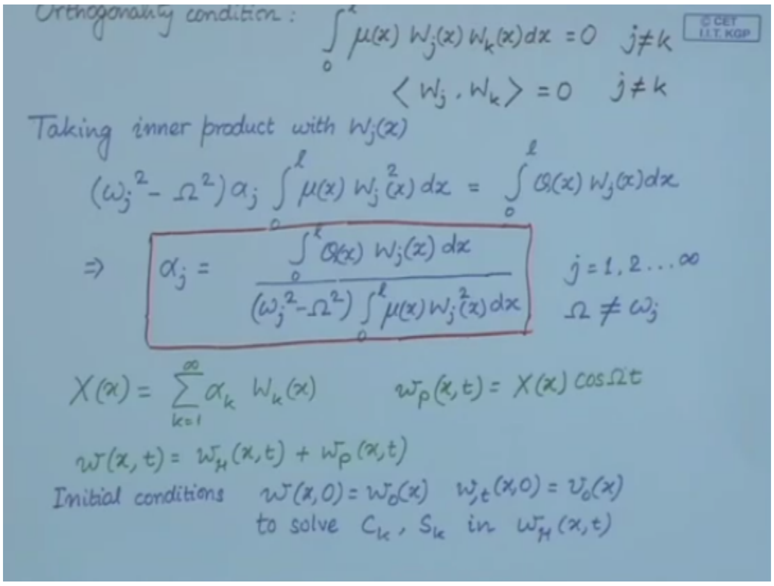

So then finally you can as I had wrote, I will write this again, so we have the solution and the particular solution will be obtained as in this form. Now this now has to be substituted in the complete solution and remember this homogeneous solution has these unknown constant c k and s k which I had written just a few moments ago. Those constants are to be determined from the initial conditions.

So we have to apply the initial conditions to solve for the c k and s k in the homogeneous solution. So that will complete the solution of the forced vibration problem. So this part we have already done in a previous lecture, how to solve for these constants using initial conditions so will not repeat that here again. Now we are going to look into this condition, what happens if we have a resonant forcing.

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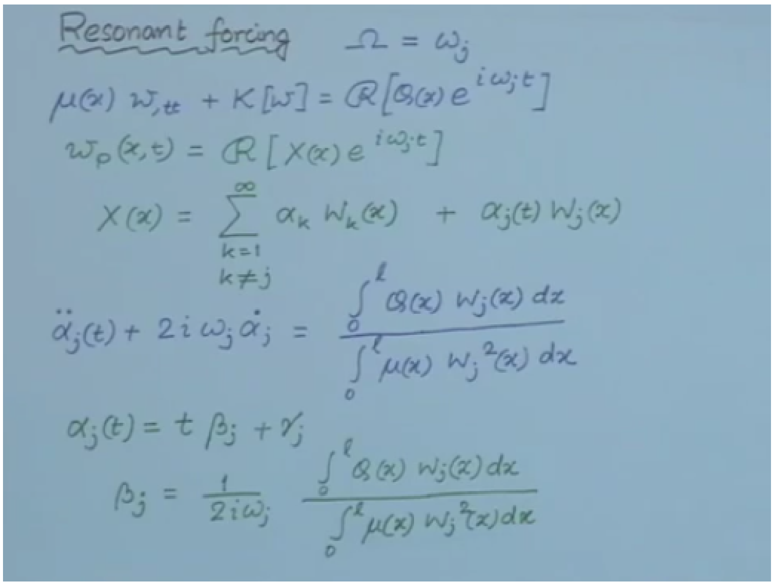

So let us consider the case of resonant forcing. Let us assume that the forcing frequency is equal to one of the circular natural frequencies of the system, let us say that the jth natural circular natural frequency. So the forcing frequency is equal to the jth natural circular natural frequency of the system. In that case, our solution, so let me write this equation once again. So we have is, we have, so this is what we have.

So therefore our particular solution we must now, we cannot consider this, so let me write down this particular solution as. Now we have been expanding this x, the amplitude function of the particular solution in terms of the eigen functions will do that but now because of this resonant forcing and if you look at the solution that we just now derived, when omega, this capital omega equals omega j, so this is going to be undefined.

So to prevent that we are going to expand this as this for all k except k equal to j. So the same expansion works for all the non-resonant mode, for the resonant mode we are going to consider, or assume that this coefficient is now a function of time. So this is going to be our expansion. Now here, so this we have considered to be a function of time as we do in for example, for (()) (35:32) method, so variation of parameters.

So we assume that this is a function of time and we substitute this expansion in here and the particular solution into our differential equation, then for all the non-resonant modes we have a way of solving just as we discussed just now. For the resonant mode, we are going to obtain a differential equation corresponding to alpha j which is obtained as, if you substitute this and take inner product with w j, this is what you are going to obtain on account of orthogonality.

Now, this differential equation we know from our previous studies that this differential equation admits a solution of this type where beta j is now a constant, it could be this but then if when you substitute and evaluate this vanishes. So we have this only beta j and beta j, if you substitute this solution from in here, then, so this is the solution for beta j and once you have the solution for alpha j then you can substitute this in this expansion.

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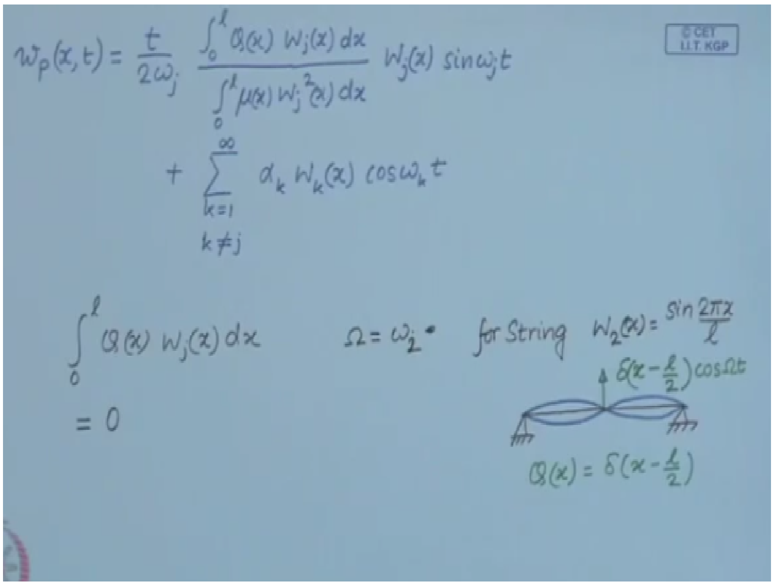

And what you will obtain is, so there as you can see that this beta j is, there is one 2 i in the denominator, so this complex imaginary and when you substitute this and take the real part as here. So when we substitute this whole expansion and take the real part what we are going to obtain finally upon simplification. We obtain the sin omega j t when we take the real part because of that i, sitting in the denominator of beta j.

And along with this we have the other terms. So this competes the solution of the particular, particular solution, now once again you have to add it with the homogeneous solution and use the initial condition to solve the constants in the homogeneous solution. Now here we see something interesting in this solution form.

In the numerator of this term, so as usual the resonant mode, the amplitude of the resonant mode has an envelope, which is linearly increasing with time. This is what we all know that this is what happens at resonant for the resonant mode, this is what happens. Now in a continuous system like this we have this integral in the sitting in the numerator of this resonant solution.

Now this integral in general can be, it will be non zero but then there are special instances where this will actually vanish. So let us look at some situation. So if you consider omega j to be the second natural frequency. So omega, the forcing frequency, the circular forcing frequency is equal to the second circular natural frequency of the system. In that case, as you know for a string let us say, for a string, for a taut string we have the eigen function sin of 2 pi x over l. So this is the eigen function of the string, the second mode.

And if we have a string which is being forced at the middle so this let us say is the string and suppose the forcing is of this form is being applied at the middle. So if you then substitute this so Q of x in this case is and if you substitute these two expressions here, then you can easily see that this integral vanishes. So if this integral vanishes, even though you are exciting at the second natural frequency of the string.

So this is omega 2, capital omega is omega 2. The second mode will not show the resonant behaviour which means that because of this integral vanishing so this term will drop out from the solution. So the response of the system will still be finite. So force like this cannot excite this mode, it cannot excite this mode because this forcing is at the node of that mode. So this is one situation where there will not be any forcing.

There can be other situations, and one such example we are going to look into very shortly. So this, what we see here is that in a continuous system just forcing the system at a resonant frequency does not mean that you will observe a resonant solution. So the location of the force is also important in these situations.

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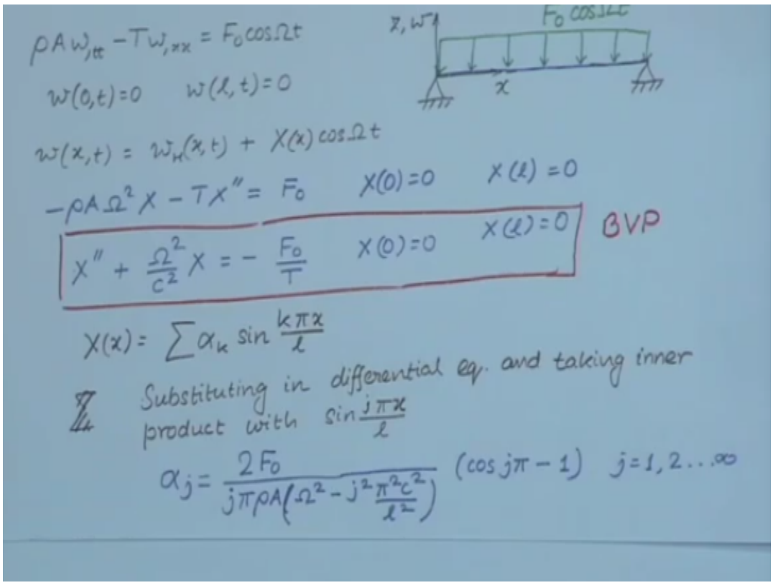

Now to look at what we have been doing, let us take an example, so this is an example of a taut string with uniform harmonic forcing. So let me represent the situation that we have, so this is the string and the forcing is a uniform, so the distribution is uniform. So the mathematical problem, the differential equation, the boundary conditions on the two ends, so it is a fix-fix string.

Now we are going to look at solutions in this form, if you substitute this in the equation of motion and removing the cos sin omega t term for throughout then we get this along with the boundary conditions. Now we can, we will simplify this little bit by dividing the whole thing by the tension and T over row A is c square.

So I will write it like this. Now this is the boundary value problem of our system which we are now going to solve. So as we have done, we know that the eigen functions of the taut string are of this form, so we are expanding in terms of this eigen function and when we substitute in here and let me so these steps are quite simple, let me write out the solution.

So I substitute this in the differential equation and take inner product with the jth eigen function. So when I do that I can obtain the solution of alpha j, these steps are straight forward. So you obtain the solution for alpha j and you can put in any value of j and you can get this alpha j and now you can substitute in the expansion to obtain the amplitude function.

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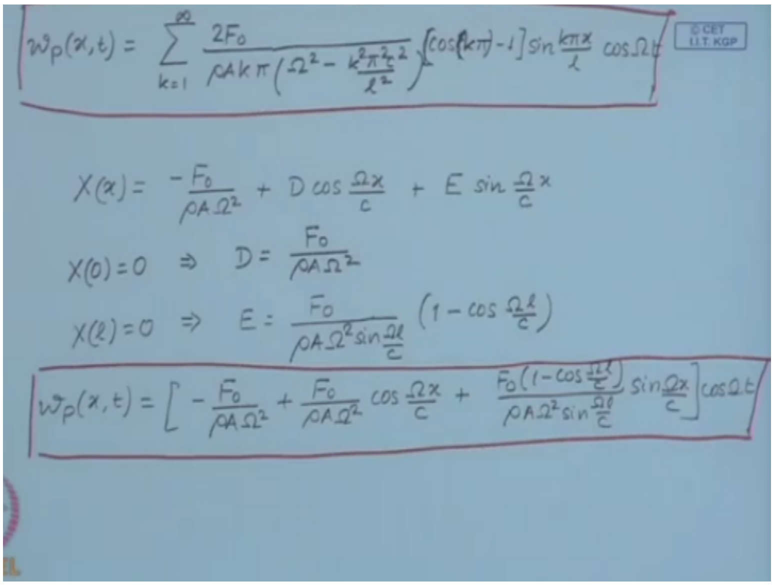

So let me write out particular solution then, so cos of k pi minus 1. So that completes the particular solution. So this is the particular solution of the problem. Now this boundary value problem can also be solved exactly as like this differential equation is straight forward and the solution of this differential equation can be easily written as. So this is a general solution.

Now see here, in the previous solution, we did not have to worry about the boundary conditions because we have expanded in terms of the eigen function which already satisfy the boundary conditions. But now we have solved this exactly now we have to satisfy the boundary condition, so if you solve for this constant D and E. You can easily obtain these constants which can be substituted here and you can once again get the particular solution.

But now the particular solution is close form expression, so this is the solution which is now in the closed form. So we have kind of summed over all these terms to obtain this. Now here, one thing to note is you have this cos k pi minus 1, so for even values of k this is going to go to zero, this bracketed term is going to go to zero. So therefore you will have only odd, so only for k odd, you will have non zero coefficients.

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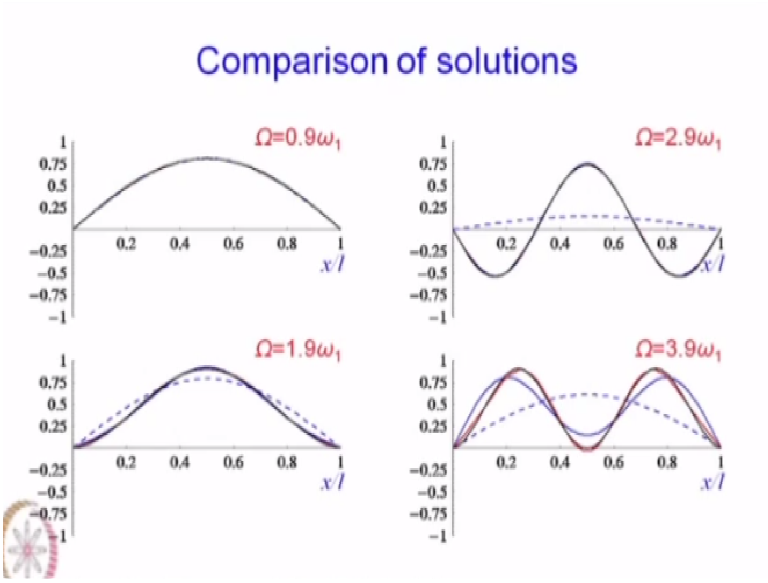

Now in this figure, in this slide I have compared these solutions, the exact solution, the closed form solutions with a series solution taking up to three terms. So if you look at, so I have taken the forcing frequency very close to the first natural frequency of the string is point nine times the first natural frequency and you can see that the exact solution and the series solution, they match.

Actually I have plotted this series solution taking one term, another solution with two terms and another solution with three terms. Now in the first plot, they are indistinguishable with the exact solution. This is when you force it close to the second natural frequency. So two times omega 1 is actually omega 2 for a string as you know that they are integral multiples of the fundamental frequency, so this is close to the second natural frequency.

The forcing is close to the second natural frequency this dash curve is the series solution with only one term. So you can see that this deviates considerably from the actual solution. While, when you take two terms or three terms in the series, then they are matching quite nicely. This is when the forcing is close to the third natural frequency, you see the one term series solution is quite off while when you consider two terms or three term.

Then they are matching quite nicely with the exact solution. This is when you are forcing at the close to the fourth natural frequency, again the one term solution is off. The two term solution is this blue solid line, the red line is the three term expansion series solution and the exact solution is given by this black solid line. So you can see that slowly as you increase the forcing frequency and consider higher modes, or then higher natural frequencies.

Then the two term solution is now error, the three term is still quite close but as you will go to higher and a higher, you will have to take more and more terms in the series to get close to the exact solution. So we see that the series solution actually converges on to the exact solution. So let us look at what we have studied today, we have discussed the force vibration analysis of one dimensional continuous system.

We have looked at harmonic forcing and we have solved this problem using the eigen function expansion method. We are going to continue this discussion in the next lecture. We conclude this lecture.