Vibrations of Structures
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Lecture - 11
The Initial Value Problem

Today, we are going to look at what is known as the initial value problem in dynamic or vibrations. So, we are going to look at the initial value problem for continuous systems.

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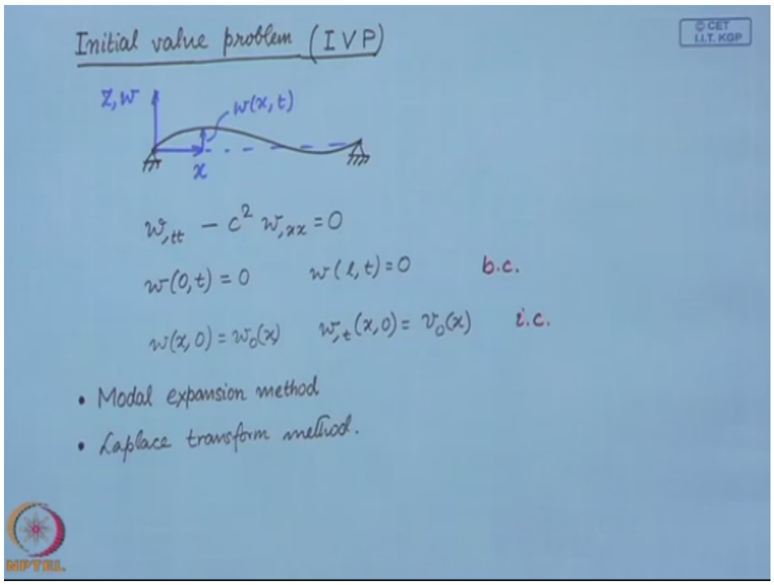

So, what is this Initial value problem? Suppose you have let us say a string and it is given some initial shape or initial velocity distribution over the string. How will the system evolve? So, we want to determine the evolution of this system. So, given this system described by this equation of motion. This boundary conditions and these two initial conditions how will the system evolve as time progresses?

So, this is the central problem in the initial value problem. Now, to solve this problem there can be various approaches today we are going to look at the Modal expansion method. The initial value problem can also be solved by Laplace transform method. We are going to look at this Laplace transform method slightly later. So, we are going to concentrate today on the mostly modal expansion method.

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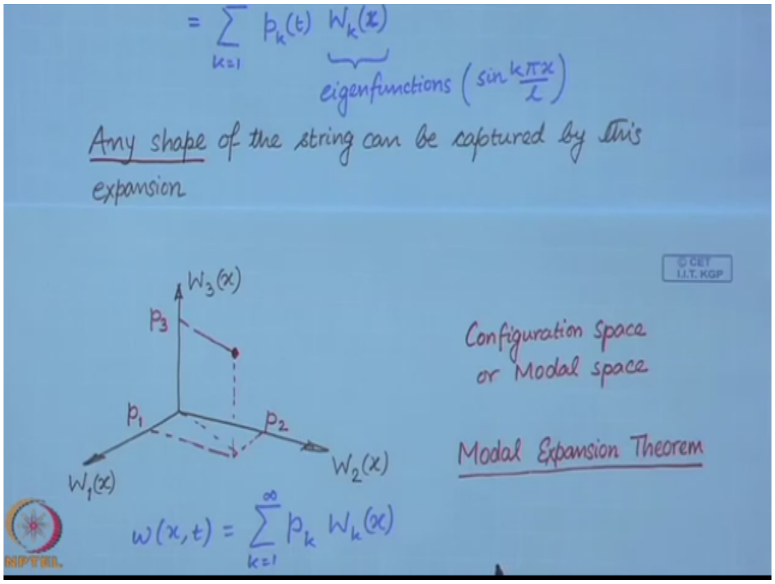

Now, we have performed modal analysis of this kind of systems and we have observed that when we have a self-adjoint system which is to say that the stiffness operator is self-adjoint then the eigen value problem has real eigen values and real eigen functions. Now, these eigen functions have this addition property that they form a complete basis. This is the underlying thing that is used in modal expansion method.

So, this property is of fundamental importance in the model expansion method. So, what is this complete basis we have discussed that suppose let us consider the string again. So, the solution was obtained was first consider –so the general solution we have expressed in this form which I can write like this. Now, what this says is that, say for example at a certain time instant t. The solution or this field variable is an infinite expansion in terms of these eigen functions which for the strength it happens to be –so for a fix string it happens to be sin k pi x over l.

Now, when we say that this is a general solution that means any shape of the string can be represented by these expansions. So, the key word here is any shape so any arbitrary shape of the string can be represented by an expansion of this type. And we have also discussed eigen functions are orthogonal.

So, if I make, if I want to have a visualization of this that these eigen functions are orthogonal and this forms a function space with this as we call as the basis. Now, of course there are infinitely many basis functions here I'm drawing only three and with the slight stretch of imagination you can very well imagine that there are infinitely many such access which are all orthogonal to one another to represent this orthogonality property with respect to a certain inner product that we have discussed.

So, in such a space which is known as the configuration space of the modal space in such a space a point with coordinates p1, p2, p3, p4 etcetera. So, this point represents the configuration of the string at the time instant when the coordinates are p1, p2, p3 etcetera. So, at a particular time instant this expansion therefore represents a shape of string at that instant. So, this point represents a shape a configuration of the system.

And any shape of the string is a point in this space. There is no shape that lies outside the space so this is the important thing. So, any shape of the string can be captured by this expansion. This is known as the expansion theorem. The modal expansion theorem. So, the Modal expansion theorem says that any shape of the string can be represented in terms of these eigen functions. So, these eigen functions they form a complete basis which means any shape can be represented in this basis.

So what I have shown here in three dimensions. Now, this is the key to the Modal expansion method for solving the initial value problem. So, for the problem that we have here. This and similar problems we will now try to solve using the model expansion method. So, let us see this.

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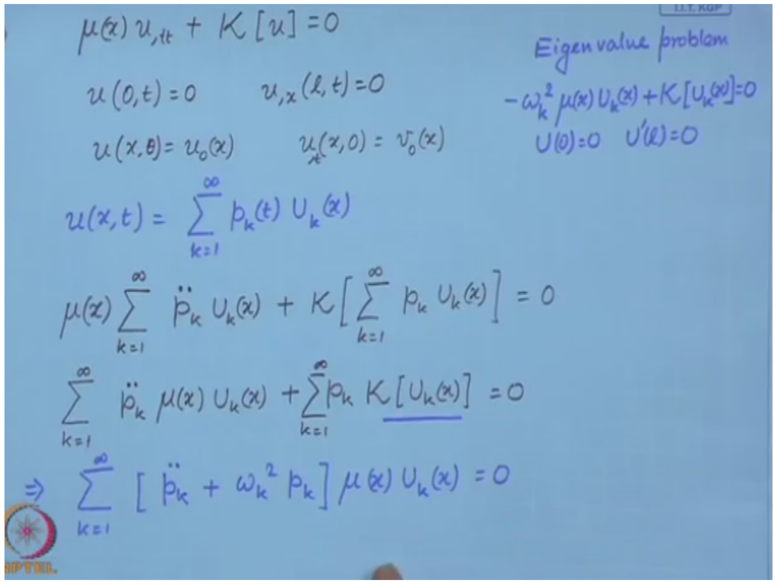

This general system with certain boundary conditions let us say this form and some initial conditions. So, we intend to solve this problem using the expansion this. So, if you substitute this expression here. So, this is what you will have. Now, K is linear differential operator and the kinds of system we are consider this k has only special derivatives and this is linear so therefor I can interchange the summation and the operator.

So, finally I can simplify this and write. So, this is what we obtained. Now, recall that this eigen value problem for this system. So, this was the statement of eigen value problem for this system. So, therefor in this summation I can replace this operator acting on the eigen function k at the eigen function with this. And therefore I can simplify the whole thing and write like this. Now, this is a summation again in terms of these eigen functions.

So, I can use now the orthogonality property. So, I will multiply both sides by the jth eigen function and integrate over the domain of the problem. So, this I will say you take the inner product with let us say Uj. If I take inner product with Uj then that filters out the jth term in this expansion. So, I will have and this I can do for all j's all values of j. And the solution, the general solution of this system can be easily written as –and therefor I will substitute this in the original expansion and write.

So, this is our general solution. Now, once I have this general solution now I have to use the initial conditions. Let us say these are the initial conditions that are specified for the system. Now, using these initial conditions we have to actually solve for these coefficients Cj and Sj. Now, let us see how we can do that. So, substituting this expression in the initial conditions//

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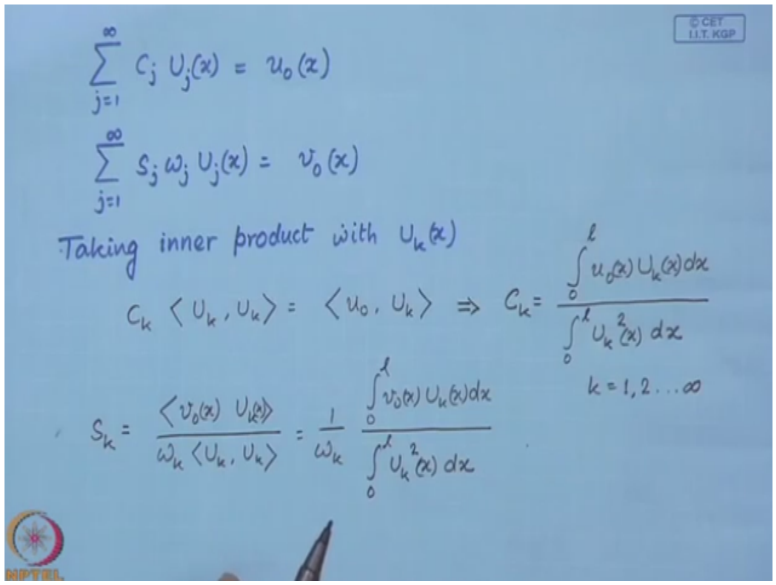

So, this is from the first initial condition and from the second initial condition this. So, we have to solve for the coefficients. So, there are infinitely many coefficients. So, for Cj and similarly infinite coefficient for Sj and we have these two equations. But then remember that these eigen functions they are orthogonal. So, we can use once again the inner product so suppose I take the inner product with let us say Uk then I can write –so once I take inner product with Uk this filters the kth term and that gives us.

So, if I write in the integral form. So, that solves for all the Ck similarly so that’s also all the values of Sk. So, once that I done I have all these expression, all these values of Ck and Sk so which I can substitute here and I have the final solution so we have solved the initial value problem using the modal expansion technique. Now let us look at some examples of actual systems. So the first example that we are going to see is shown in this slide.

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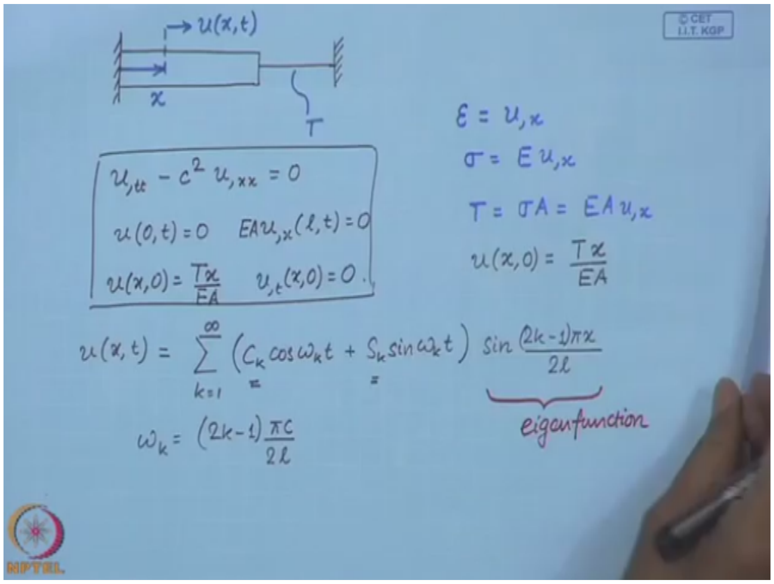

So, is the collapse of stretched bar. So, what we have is –so we have a fixed free bar which is under tension because of this string so here there is string which is attached to this free end of the bar and it is under attention T. So, naturally this bar is under extension. Now, imagine that you cut this string so this string snaps. So, when the string snaps this bar is going to collapse back so we are going to study the collapse of this bar.

So, let us mathematically formulate the problem so we have this bar this is a uniform bar. So, we are going to write formulate the problem at the moment the string snaps so in that case the right hand of the bar is force free. So, this boundary condition zero. Now, the initial conditions. So, you can very well you can find out the initial conditions so here what was the condition that this was under the tension T.

So, you can write the strain in the bar so that is delta u, delta x. The stress in the bar is the Young's modulus times the strain. And when this was under the tension T so the force was T which is sigma times the area of cross section of the bar. So when the bar was under this tension so this was the condition. This is the equation of statics of bar which can be very easily integrated out to determine since at x equal to zero u is zero so the constant of integration is zero.

So, this is the initial condition of the bar. And the initial velocity as soon as the string snaps the initial velocity is zero. So, this is our problem. This is the initial value problem for this collapsing bar. So, as usual we are going to as we have just now discussed we are going to express the solution of this collapsing bar in terms of its eigen functions. So the eigen functions of fixed free bar they are given by –So, these are the eigen functions of the fixed free bar that we already know.

Here Omega k these are given by these values so here this is the general solution and these are the coefficients that we must find out from these initial conditions. So, let us see. So when you substitute this in the first initial condition

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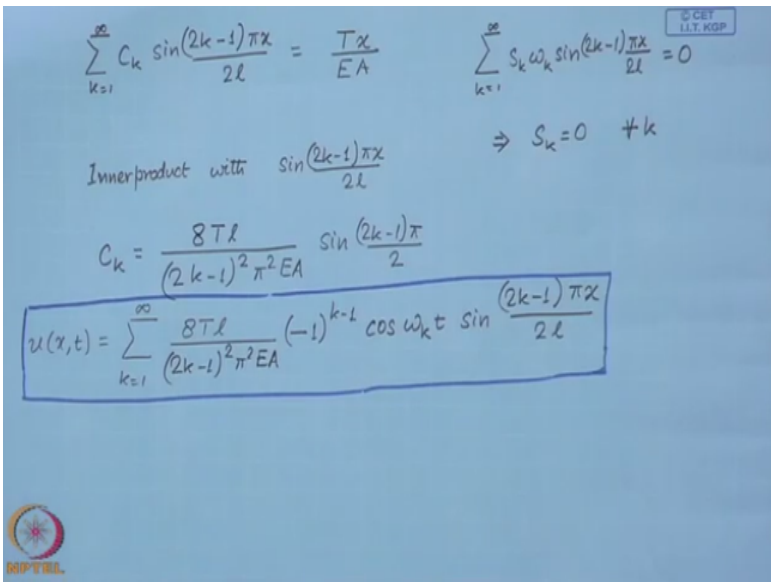

And similarly for the velocity condition result was zero. So, this immediately tells us for all k this coefficient as, k must vanish. So, we are left with this so we take inner product with this k th eigen function and if you perform this integral. So you multiply this and integrate over the length of the bar. So you can check. So these are the coefficients and finally therefor the complete solution is –now this solution I have plotted

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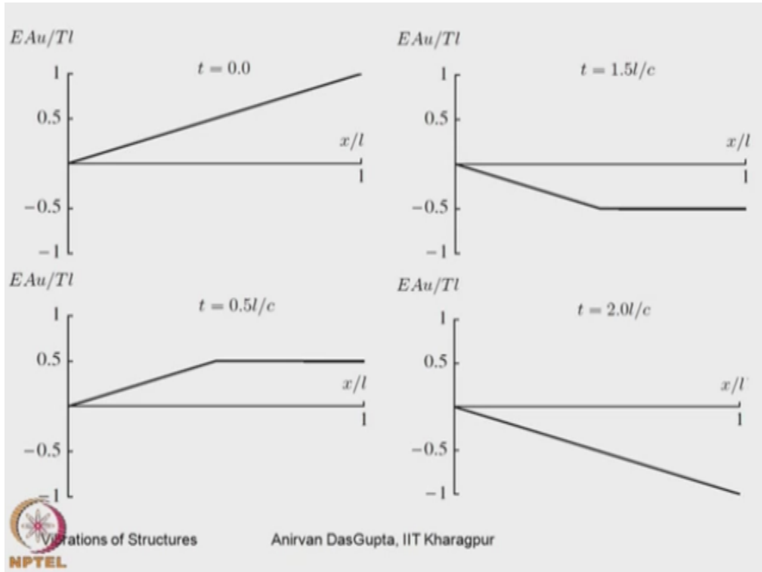

At certain time, instance in these sets of figure. So, you see at time t equal to zero. This is the actual displacement. You see this is the actual displacement of the bar. So, this line is nothing but tx over EA. This is nothing but TX over EA. So this is a straight line with x. So, here of course I am plotting it with non-dimensional free variable x over l. So, initially the actual displacement is like this.

So, as time progresses so at 0.5 l over c this is the actual displacement profile of the bar. So, here it is still linear the same as it was here so you can imagine that this portion of the bar does not know as yet that this end has been released. So, this information has progressed in 0.5 l over c time to approximately half so it is half, half of the bar knows that this end has been released. This half still does not know.

So there is a propagation of information from the end which was released at time t equal to zero into the bar. So, at l over c the bar comes to complete the full or whole of the bar is that the equilibrium configuration which is not shown in these figures and then at .5 l over c there is a compression taking place in the bars. Here these of all the bar is under tension. When it comes to the equilibrium configuration at l over c the bar is in equilibrium.

But then it has acquired velocity. So, it still goes and hits against the or compresses against the wall and then there is a compression generating in the bar and that compression is complete at 2 l over c. So, as you can imagine there is a propagation of information of this compression wave as it is usually know as compression and tension wave that progresses in the bar and it reflects back and forth and that way the bar is vibrating.

Now, this propagation of these waves we are going to discuss in this course. So, we are going to discuss that in detail and latter in this course.

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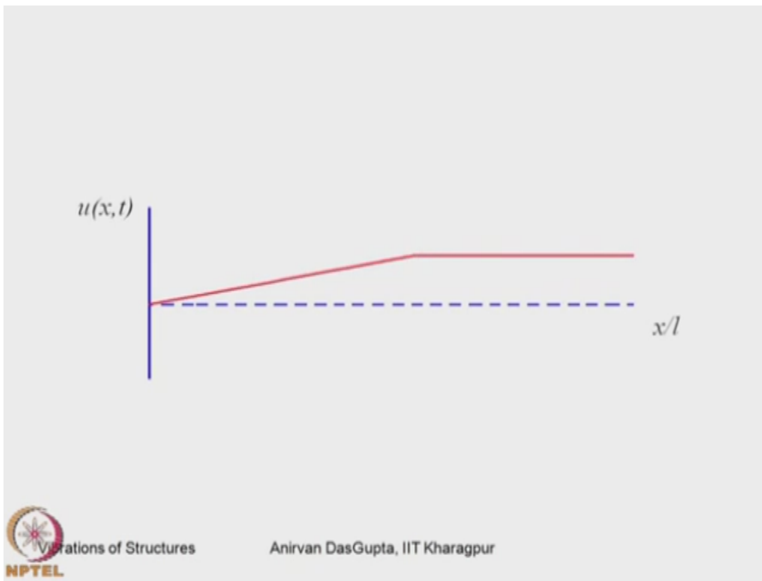

Now, here I have an animation which shows the same thing. So, as you can see at this instance now the propagation the compression wave is propagating. So, now it is in tension on this side it is compression and this wave propagates back and forth in bar which is shown in this animation. Next, let us look at another initial value problem where we are going to have velocity initial condition.

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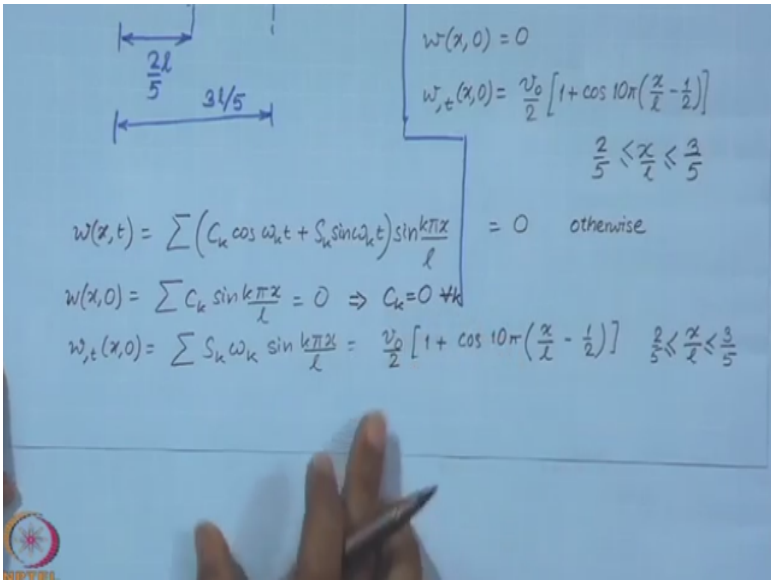

So, let us consider a string on which I specify a velocity profile. So, the velocity profile we will consider that the initial displacement of the string is zero but the velocity has this profile. So let me mathematically formulate the problem. So, the initial displacement of the string is zero while the velocity is given. So, we have velocity profile like this over this region. So, velocity initial condition is provided only over this small region which is of length one fifth the length of the string so central portion we have this kind of a profile.

So, for this problem once again so we consider a solution of this form. Now, when you substitute this solution in the initial conditions you will obtain these two equations. So, you have to solve these two equations in order to obtain the coefficients, from here it is immediate that all the Ck will be zero while if you once again take the inner product and solve this problem, I mean these conditions solve for the coefficients Sk then.

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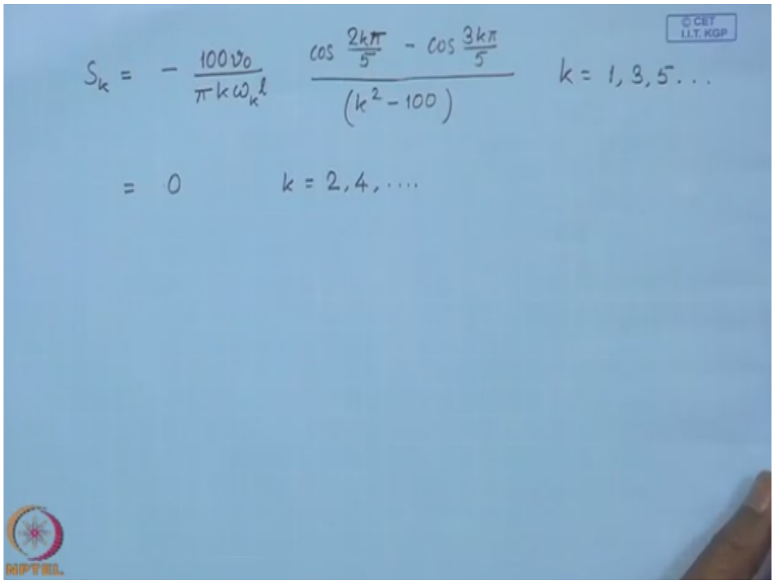

Then Sk turns out to be for K 1, 3, 5 so for all odd values of k we have this and zero for all the even values. So for all the even values Sk is zero for all the odd values we have this expression of Sk. So, now with these expressions in here you can write down the solution motion of the string. So, here I have the snap shots of the string at certain time instance. So, at time equal to zero as you can realize that that string is in equilibrium position.

So, I have not shown that at point 0.5 t equal to 0.05 there is a hump that is developed in the string and as time progresses 0.25 l over c this hump it spreads in the string. But see this portion of the string yet does not know that a disturbance has been created in the string. At this time instance you see the full string is displaced beyond this the disturbance reflect backs from these fixed ends and the hump shrinks and then it comes to the other side of the equilibrium position of the string.

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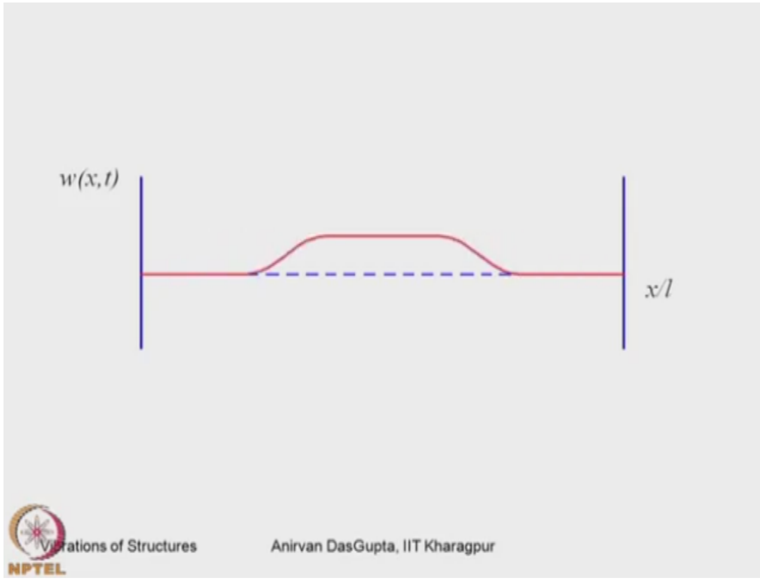

So, this shows this animation. So as you can see that the hump develops, spreads, reflects, collapses and comes on the other side and does the same thing. Remember that this is a slow motion of what actually happens I mean this progression of waves possibly cannot be observed by the human eye as such. You have to see it in slow motion to observe this kind of propagation of disturbance.

So, we have looked at the initial value of problem for continuous systems and we have considered two examples and using the model expansion technique we have solved this problem. Now let us briefly finally look at this initial value problem how an initial value problem can be actually converted to a forced vibration problem or a forced dynamics of the continuous system.

Now, this is very interesting because then once we discussed the force vibration analysis the method is that we discussed there will be applicable for solving the initial value problem as well. So, which mean that we can solve the force dynamics or the initial value problem as a force vibration problem. We then have a unified way of treating initial value problem, force dynamics etcetera.

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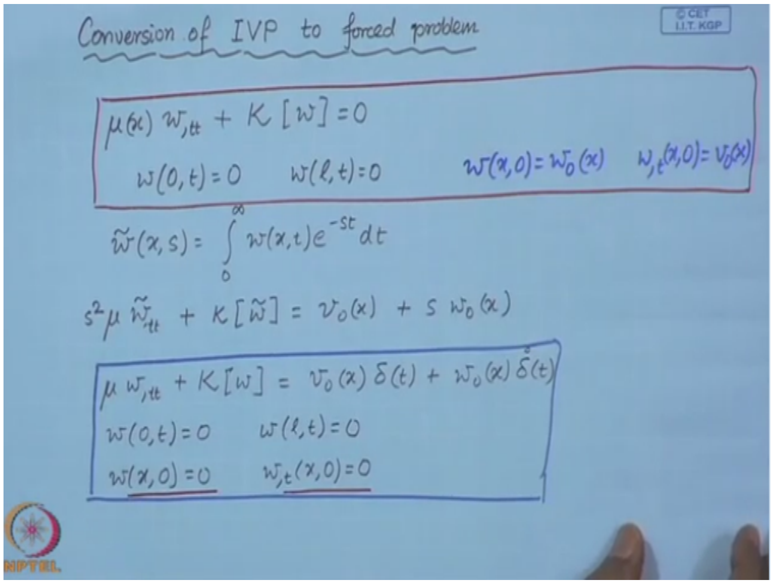

So let us look at conversion of an initial value problem to forced problem. Now, you see this we have been considering let us say this kind of a system and with certain let us say boundary conditions and initial conditions. Now if you take the Laplace transform of this equation. So, we define this Laplace transform. So, if you define the Laplace transform in this manner then in the Laplace domain so here S is Laplace variable.

So, in the Laplace domain you can write this you can very easily check. So, this is what you are going to get so this is the equation in the Laplace transform domain. So, is for this problem with this initial condition now the same equation of Laplace domain can be obtained for this system. So, in the Laplace domain the equation of motion for this system is same as the Laplace transform of this equation. But now you see this system has zero initial conditions.

So, here in instead of these none zero initial conditions we have this forcing. This inhomogeneity in the equation of motions. So, we have been able to convert a system with initial conditions to a system with zero initial conditions but with forcing. So, this is a system which is now a forced system with zero initial conditions and therefor the solution of this will be the solution of the original system with these none zero initial conditions.

So, this way we can convert or bring an initial condition problem or convert an initial condition problem to a forced vibration or the forced dynamics probably. So, in this lecture we have looked at the initial value problem we have solved using the modal expansion technique. We have looked at some examples of how the solution behaves and we have made some interesting observations in this solution how the motion actually takes places as a propagation of disturbance in the continuous system.

And finally we have looked at converting an initial value problem to a forced vibration problem. So with this we can solve the initial value problem and the force dynamic problems in a unified manner in a later lecture. So with that we conclude this lecture.