Vibration of Structures
Prof. Anirvan DasGupta
Department of Mechanical Engineering
Indian Institute of Technology – Kharagpur

Lecture – 10
Modal Analysis: Approximate Methods - II

So, in the previous lecture, we have discussed about the approximate methods of modal analysis, one class of methods which is based on the energy the kinetic potential energy, the Lagrangian etcetera. So which are broadly classified as energy based methods for modal analysis. Now in today's lecture, we are going to look at yet another class of methods which are known as Projection methods.

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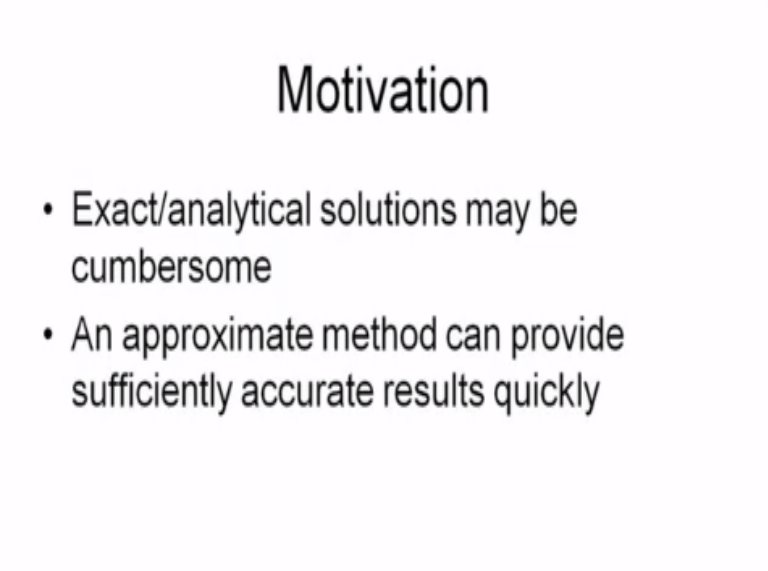

So, the motivation for studying approximate methods we have already discussed that analytical methods though they are more preferable but are quite cumbersome and we can have approximate methods which can give quick and sufficiently accurate results. So in today's lecture, we are going to look at these projection methods which work directly with the governing differential equation of the system.

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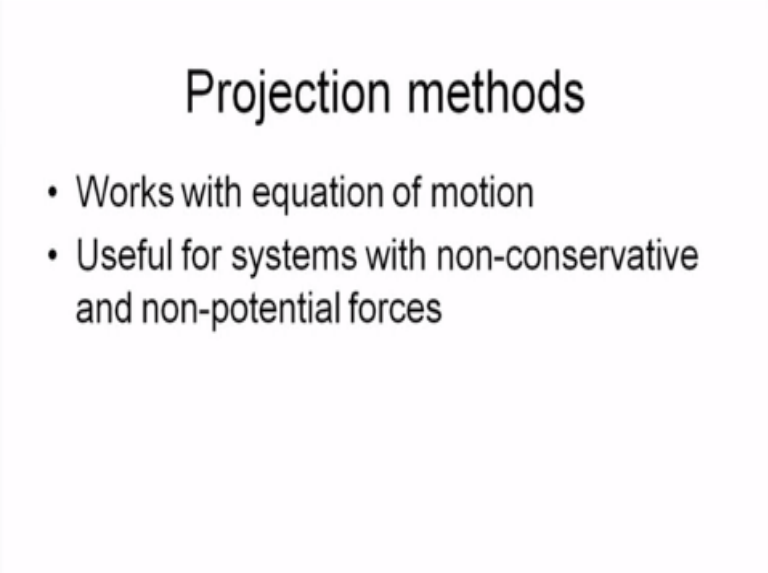

So these—since this method work with the differential equations directly they can be used very easily for dealing with non-conservative forces, non-potential forces et cetera. So since these terms they can directly written in the equation of motion and they are little tricky to introduce in the Lagrangian as such though they can be done in that way. But if method works directly with the equation of motion then these non-conservative terms and non-potential forces they can be handled quite effectively.

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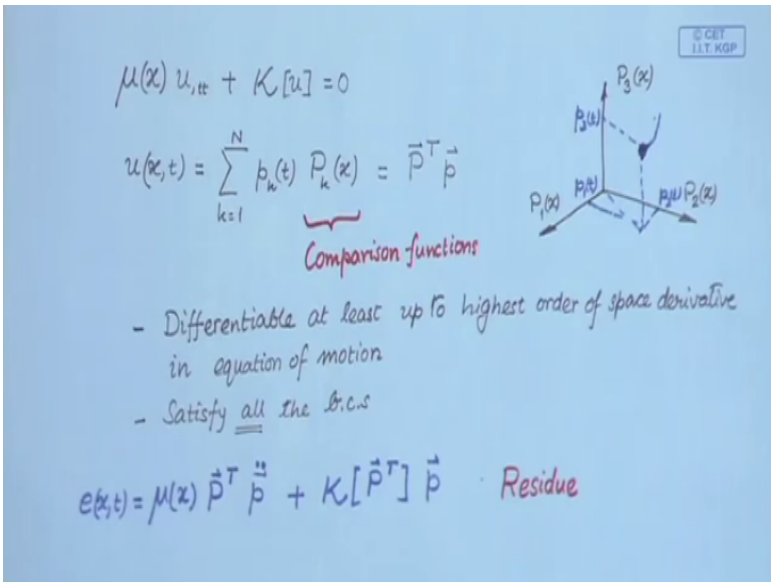

So what do we in this Project methods. So let us discuss in the context of this equation of motion. So – so let us consider a differential equation which has a structure like this so Mu (x) represents the inertia operator let us say and K is another linear differential operator. So

essentially what we are going to do is discretize, so this is the continuous system described by just partial differential equation. So what we aim to do is to discretise this equation of motion.

So we use the idea of expansion of the field variable in this form. One thing that maybe mentioned here that even though this looks like a separable solution but actually it is not because this is an expansion had it been only one term then it is solution that is separated in space and time. But once you take this expansion it is no longer in a separable solution. Now this maybe written in terms of vector multiplication like this.

Now here in this projection methods there is a restriction on the kinds of functions in which using which we do this expansion. These functions are known as comparison functions. Now what are comparison functions? These are functions that satisfy two important properties. They must be differentiable at least up to the highest order of space derivative in the equation of motion.

The second important property that they should satisfy is that—they satisfy all the boundary conditions of the problem. Now this is very important to note that these functions must satisfy all the boundary conditions of the problem. This actually makes this method little more difficult to apply compare to the energy based technique where we were using admissible functions.

So we expand our field variable in terms of this comparison functions and unknown coordinates of time. So if I substitute this expansion in the equation of motion then of course I do not expect that this solution will satisfy this—this equation of motion because in any case this an approximate solution. So what we generate is known as Residue. This is known as the Residue. So this we do not expect that this will be zero throughout the domain. So this function is known as the residue.

Now we project this residue in a certain space. So we are looking—I mean this can be thought of as expanding the solution. As a linear combination of certain functions as we had discussed previously as well and so on. So this point represents the configuration of the system and as this temporal functions they change this point moves in this space. Now we have generated this

residue because this solution is approximate. So what we can do is we can try to make this residue zero at certain points.

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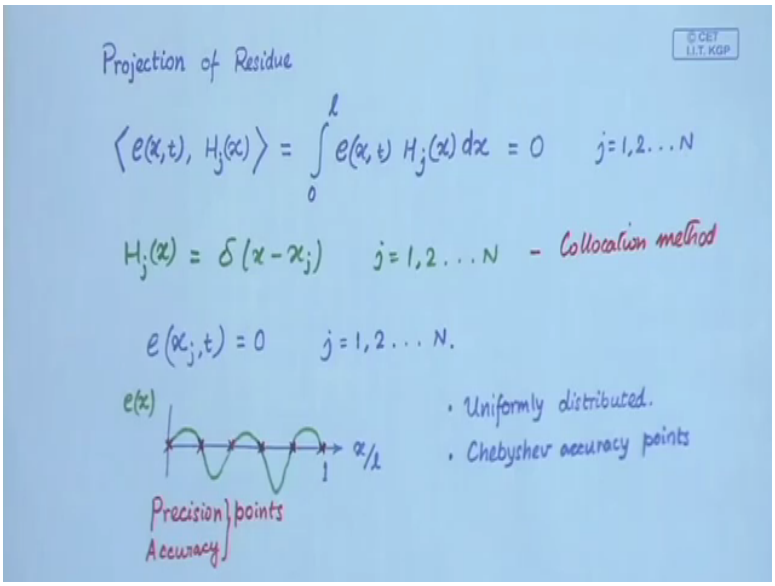

There are various ways of doing it, so here we first introduce the idea of Projection. So we have defined a suitably inner product of two functions, so one is the residue of residue that we have generated and another function H of x which we can simply defined in this case, and say that this projection is zero, rather than the residue vanishing identically we say that the projection of this residue—on certain functions that we will qualify shortly, so this projection is zero.

Since we are searching for approximate solutions so we this solution structure we have considered that is approximate since its finite expansion. Once we substitute that in the equation of motion we generate this residue since this this will not be identically zero. So what we say is instead of having this identically zero we have weaker condition which says that the projection of this residue along certain function direction this is zero.

So that is so we can take N such functions suitably chosen to generate N equations. And thereby we can attempt to solve for this N unknown pk that we have in the expansion the temporal function. Now this choice of this functions H that decides the method. So what are the different ways of choosing these function on which we project? The simplest choice is this Dirac delta function.

Once we choose this function as this Dirac delta functions this method is known as the Collocation method. Now what does it mean to choose Hj as Dirac delta function. So if you substitute these functions here then what you will obtain is—what this means is that this residue is zero at certain points not at all points but at certain points over the domain. Say for example for the bar if I choose this xj these points something like this so at these points the residue must vanish.

So now there can be various ways of choosing so these points are known as the precision points or also sometime known as a Accuracy points. Now there can be various ways of choosing this precision points or accuracy points, they can be uniformly distributed or there can be other methods of choosing precision points good way of choosing precision points is given by Chebyshev method and they are known as Chebyshev accuracy points.

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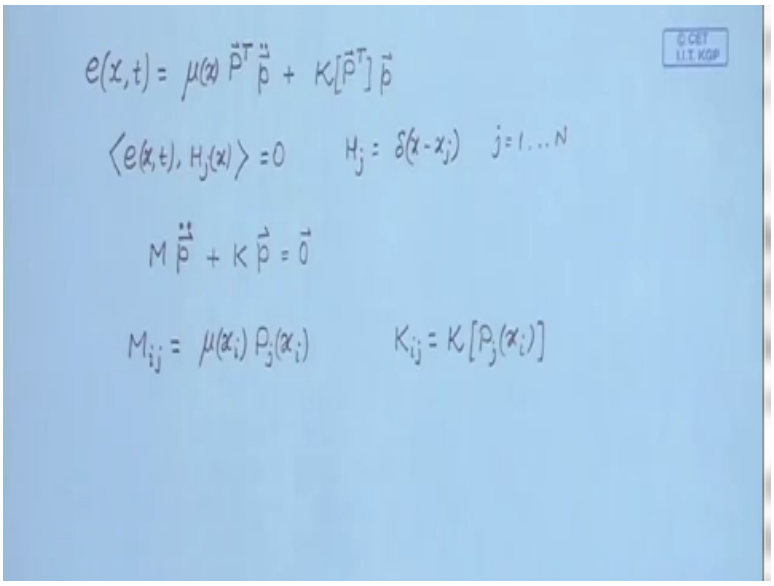

Now let us first look at what happens when we substitute this residue in the projection equation. So I rewrite the residue and if I consider the projection the functions in this form using Dirac delta functions. Then we arrive at this equation of motion, this is the discretise equation of motion where this metrics element are obtained like this. So when we use collocation method we obtained this discretise system where the metrics elements are given like this.

So let us once again look at the example of the tapered bar.

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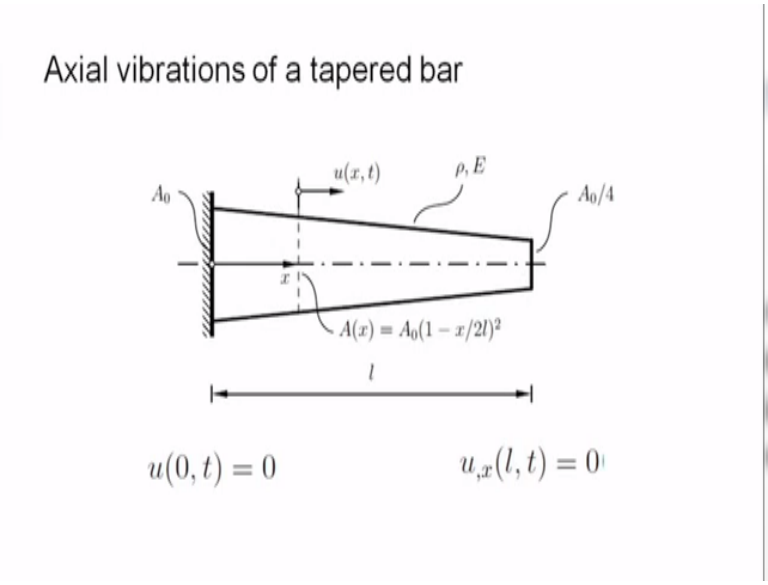

So here we have the fixed free bar once again and we have the geometric boundary condition at the left hand and the natural or dynamic boundary condition at the right hand. Now we have to choose functions which we have to choose the comparison functions which satisfy – so comparison functions must satisfy the boundary condition all the boundary conditions of the problem which means they must satisfy the geometric as well as the natural boundary conditions.

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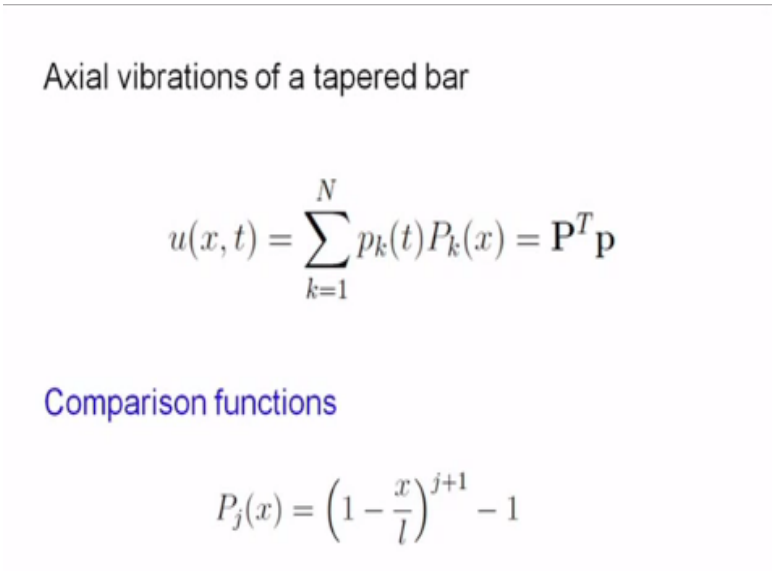

So let us look at particular choice as shown here. So we have this functions Pj you can check very easily that.

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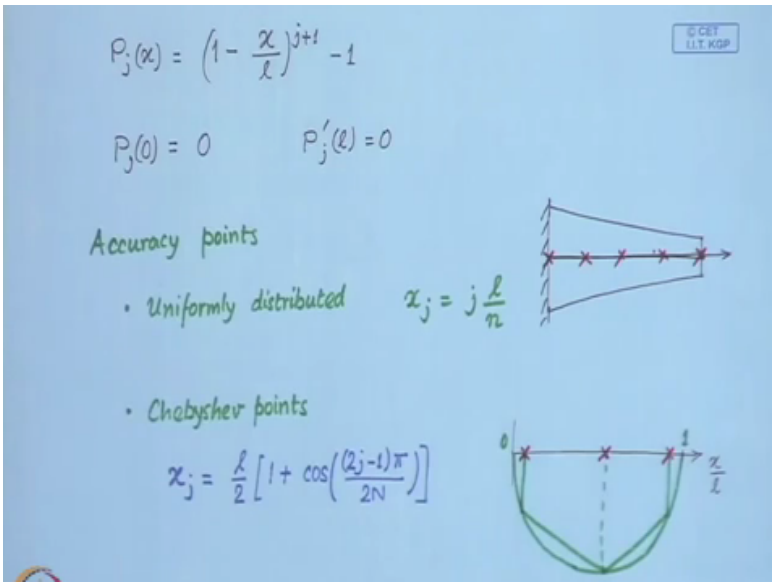

So Pj of x, so you can check. So these functions satisfy the geometric boundary condition as well as the natural boundary condition at the right end. Now we can choose these accuracy points as we have discussed. We can choose them as uniformly distributed. So if you divide the domain of the bar in N parts then you can take these uniformly distributed or you can also have the Chebyshev accuracy points which are determined by this expression.

Now this has a nice geometric visualization so this is the domain of the bar. So if you draw a semicircle with this domain as the diameter and you put in a regular polygon. For example, if you want to take three accuracy points you have to inscribe half hexagon and the projection of this corner points on the domain will give you the accuracy point. So this is the Chebyshev, these are the Chebyshev accuracy points.

So as you can see this Chebyshev accuracy points will never fall on the ends of the bar. So once you use either this uniform spacing or the Chebyshev spacing and calculate the Eigen frequencies.

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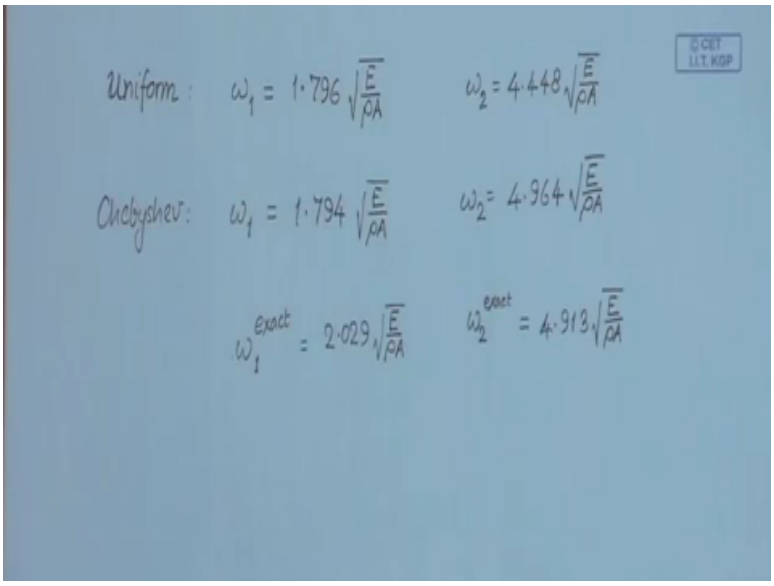

So for uniform spacing what you obtain. So these are the first two circular natural frequencies when you consider uniform spacing. When we consider Chebyshev spacing then the first two natural frequencies are obtained like this. However, if you look at the exact circular natural frequencies there obtained like this. So here what we find is the fundamental circular natural frequency is in some error from the exact circular natural frequency while the second one is more close.

Now in this collocation method has this disadvantage that this does not have that Upper-bound property as we saw in the energy based methods. So this approaches the exact natural frequency from below. But I mean it can also – for example if you see this comparison this is approaching it from above. So this is a disadvantage of the collocation method. Now, if you calculate the Eigen functions corresponding to these Eigen circular natural frequencies. then what we obtain is shown in the screen.

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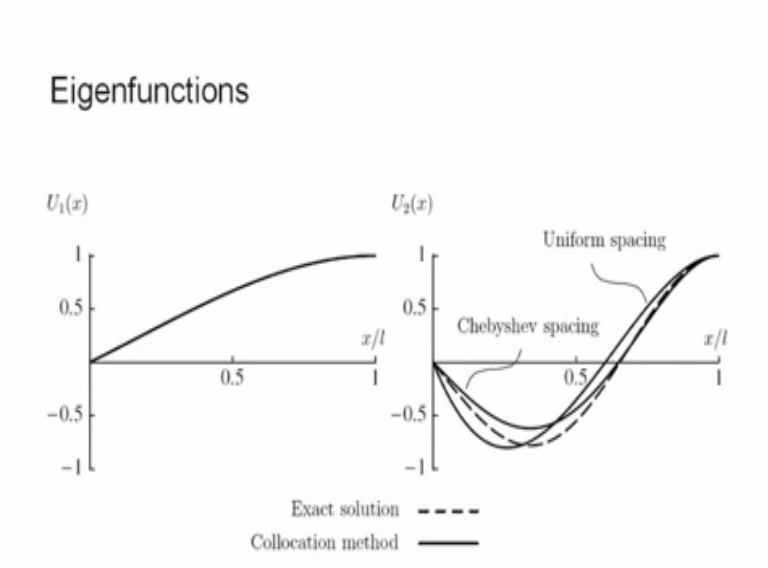

So if you look at the fundamental Eigen functions the Eigen function corresponding to the fundamental mode. Then within the accuracy of this plot you see you cannot distinguish the exact method with the collocation method. Though the natural frequency is in some error. But this can – of course we rectified by considering more and more turns in the expansion and making it more accurate.

On the other hand, if you look at the Eigen function of the second mode you see with the Chebyshev spacing it appears to be more accurate because the location of the node matches quite well with the with that obtained from the exact solution. While that of the uniform spacing is in some error. So we have looked at-- so this choice of—we have looked at one choice of the projection functions H of x.

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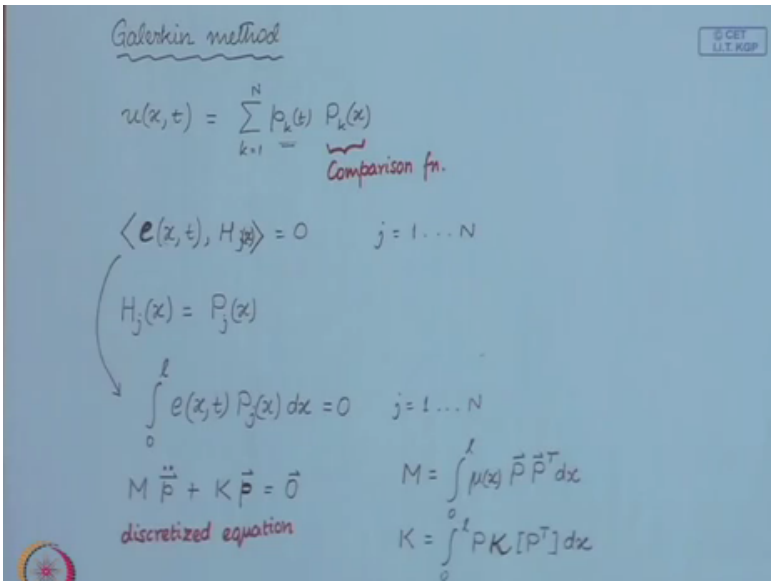

Now we will look at another choice which gives us what is known as the Galerkin method which is another powerful method which uses the projection technique. Now in the Galerkin method—so remember that we have expanded where these capital these are comparison functions. Now – then we generated the residue and we projected the residue on certain functions and put them to zero.

Now in the Galerkin method we take this projection functions same as the comparison functions we use for this expansion of the field variable. So this—so this projection gives us is N equation from which we are going to solve for this N temporal function p of T. Now this – so when we choose this projection functions same as the comparison functions in expansion then we have the Galerkin method.

And when we do this projection we obtained this discretize equation of motion where this metrics M and K are obtained in this form. So this structure so this is again, so we obtained the discretize equation of motion of the system.

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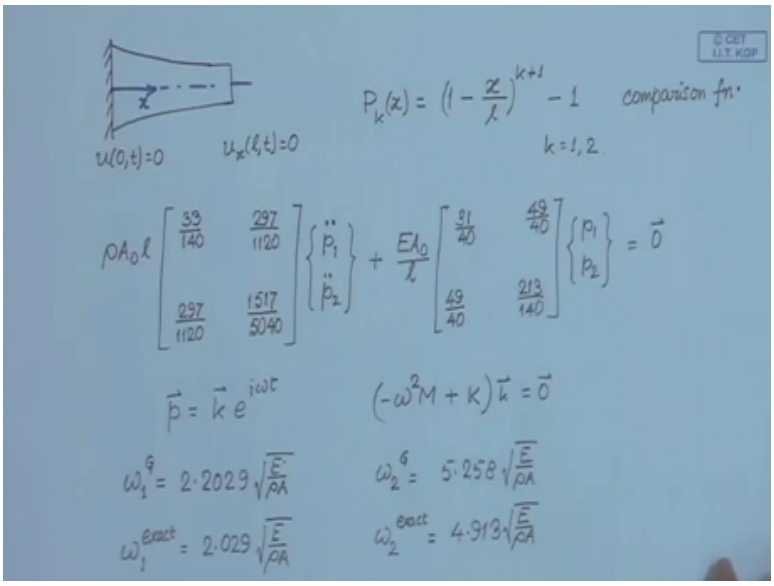

Now let us look at this example once again. So these are the boundary conditions at the fixed and the free end. So once again we choose the comparison functions. We will take two comparison functions and discretize the equation of motion. So if you do that you will obtain the discretize equation in this form. Then as usual we do the modal analysis assuming this structure of solution. And we obtained the Eigenvalue problem, Discretize Eigenvalue problem and from where we obtained the – the first two circular natural frequencies.

Now this superscript g indicates obtained from Galerkin. Now if you compare—so as you can see that the fundamental frequency compares very well with the exact while the second modal frequency is in some error. Now here again if you want to have accurate modal solutions for the first N modes you use an expansion with 2N turns. So here we can see that the error in the fundamental frequency is very small as compare to the second.

Now when you solve this Eigenvalue problem you also obtain the Eigen vectors K.

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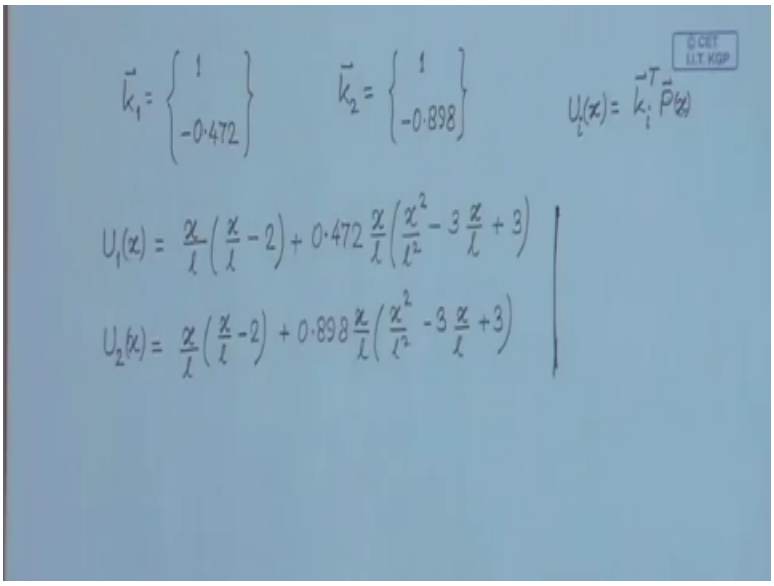

Now these are obtained, so once can easily obtain this Eigen vector corresponding to the Eigen frequencies. And the corresponding Eigen functions—so the Eigen functions are—so this Eigen functions is obtained as using this Eigen vectors. So for the first Eigen vector for the first Eigen function you use the first Eigen vector in this vector product. Similarly, for the second Eigen function use the second Eigen vector in this vector product.

So these are the two Eigen functions that are obtained from the Galerkin method.

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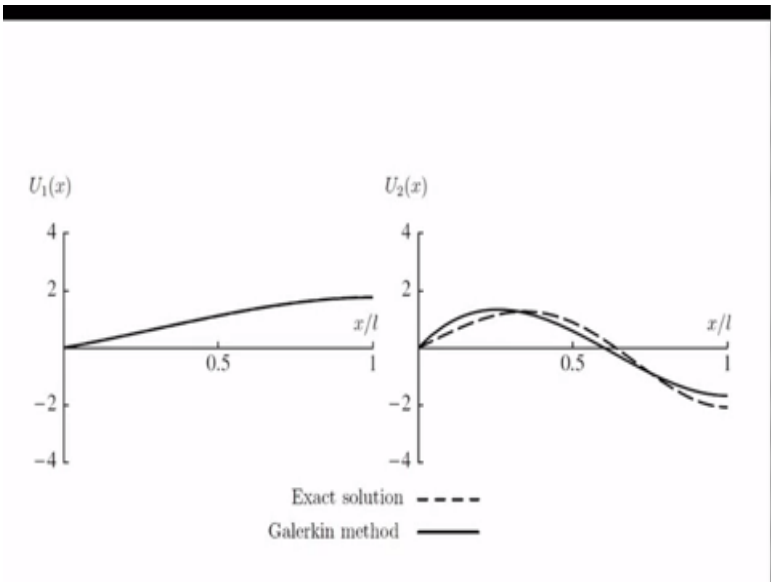

Now here we can see a comparison of these Eigen functions with the exact solution that we have obtained previously. So as you can see the within the accuracy of the plot the fundamental Eigen

function is indistinguishable from the exact while in fundamental Eigen functions solved from the Galerkin method is indistinguishable from the exact solution. For the Eigen function of the second mode this is fairly close and you can see now that the boundary condition at the right end the free end is also matching though the location of the node is in slight error.

So in these plots we have compared these two Eigen functions obtained from the Galerkin method with the exact solution. Now this Galerkin method since as we discussed that this is a method for discretising the equation of motion of the continuous system. So we can use this also when we have external forcing and which we will discuss later on in this course.

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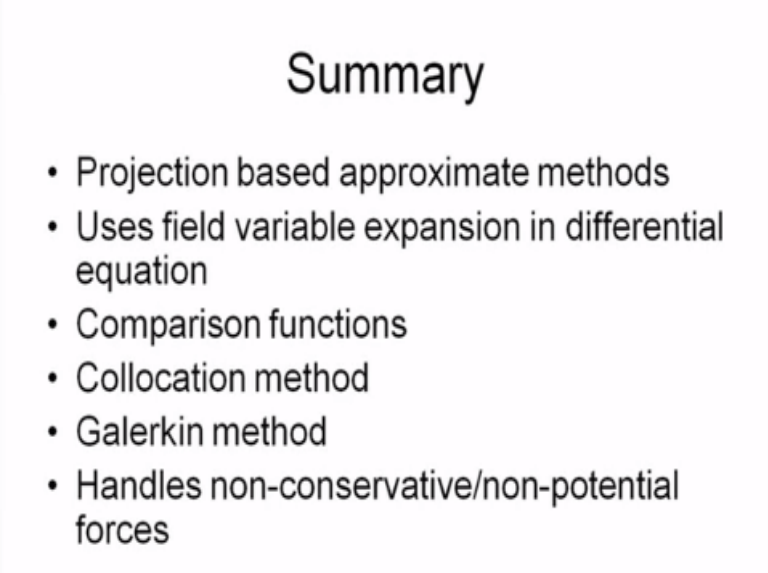

So to summarize we have studied in this lecture projection based approximate methods which in which the field variable is expanded in terms of comparison functions and substituted in the partial differential equation of motion of the system. Then what we generate is the residue and this residue will not be zero uniformly over the domain so we use a weaker condition we project this residue onto certain functions and the choice of these functions decide the method.

So when we use the Dirac delta function we have the collocation method while if you choose the comparison function used in the expansion of the field variable themselves as the projection function then you have the Galerkin method. And these methods of the projection methods they can handle non-conservative and non-potential forces. So with this we conclude this lecture.