Conduction and Convection Heat Transfer

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Lecture - 08
1D Steady State Heat Conduction in Cylindrical Geometry

Good afternoon to all of you and I welcome you all to the session on Conduction and Convection heat transfer. In last classes, we discussed the one-dimensional steady state heat conduction through plane surfaces. We also discussed the plane surface where the area is cross sectional area, that is area normal to the heat flow is varying, how to deal with the problems and we have solved two interesting problems in the last class.

Now, today we will discuss one-dimensional steady heat conduction in cylindrical geometry. Now, in a plane wall, as we have seen the application where we can use the sufficient coordinate system, but in cylindrical geometry, we have to use cylindrical polar coordinate system. Now the cylindrical geometry comes mostly in case of pipe. When a hot fluid flows through a pipe.

then the heat is being transferred from the fluid by convection to the inner wall of the pipe and then by conduction from inner wall to outer wall of the pipe and in many occasions to reduce the heat loss, we have to provide insulation, insulating material and provide another thickness to the cylindrical pipe, so those are the application of steady one-dimensional heat conduction in cylindrical coordinate system. Let us see that.

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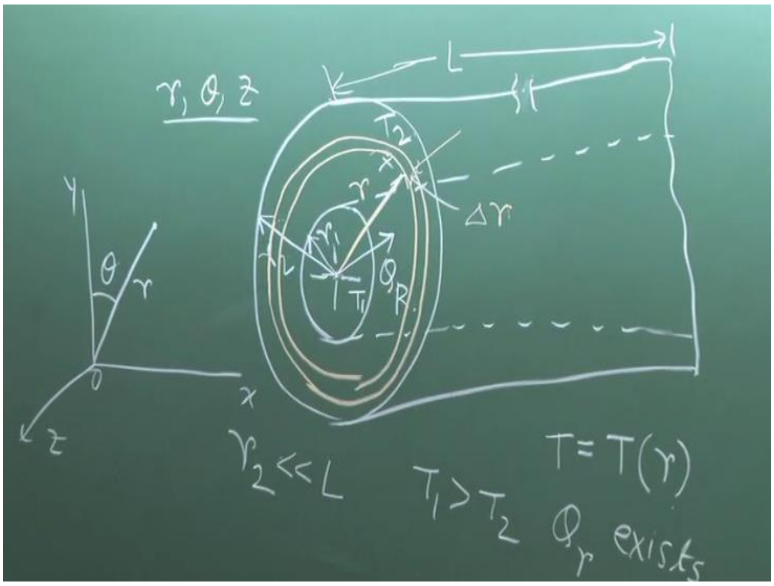

Let us consider a cylinder like this and the problem is specified like this, which we will be discussing. Let this be the center. The cylinder with inside radius r1 is cylindrical tube, outside radius r2, that means this portion is the solid cylindrical wall type of thing and the length of this cylinder is L and the problem is like this, r2 the outside radius of the tube, outer radius is very less than L, that means length is much more than the radius.

And the inner surface is maintained at a temperature T1 constant temperature throughout the surface. So, the inner cylindrical surface is maintained at T1 while the outer surface is maintained at a temperature T2 and T1 is greater than T2. Now, in this case since it is maintained at a constant temperature, there is no variation in the azimuthal direction, theta and also the temperature is same along the length of the cylinder.

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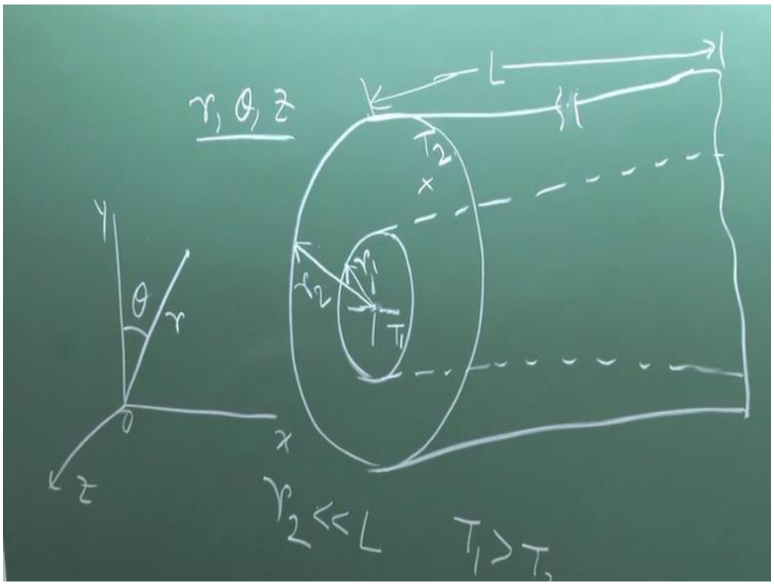

This can be expressed in terms of cylindrical coordinate system if it is sufficient coordinate the r, this is the polar azimuthal angle theta and z, r, theta and z. The coordinate system, in which any point here in the cylindrical wall, r1 to r2 can be specified, where z is the coordinate along the length and r and theta the radial and azimuthal coordinate.

Now, for this type of boundary condition, constant temperature both azimuthally and axially along the length, that is along z direction and r2 being less than L, this assumed one-dimensional heat conduction and we will consider a steady state, that means heat is going in such a way that the temperature is invariant with time, that is the situation when we are going to analyse the problem.

This problem in fact is assumed as state when the boundary condition are steady that we have to understand. The problem becomes generally unsteady continuously when the boundary conditions are unsteady, but the boundary conditions are steady after a transient, the system always attain a steady state, so we consider a steady state.

In this case, T is function of r only and only the heat flux in r direction exists, only Qr exists that means there is heat flux in the r direction, Qr, now our job is to find out the temperature distribution, heat flux distribution, what is the amount of heat flux across any section, all these, which we did for plane surface. The analysis is exactly the same as we did for plane surface, only difference is the mathematical state, how?

We proceed with that the x and element, annular element at an orbit value radius r. We take an annular element like this, a cylindrical ring at an orbit value radius r with the thickness delta r.

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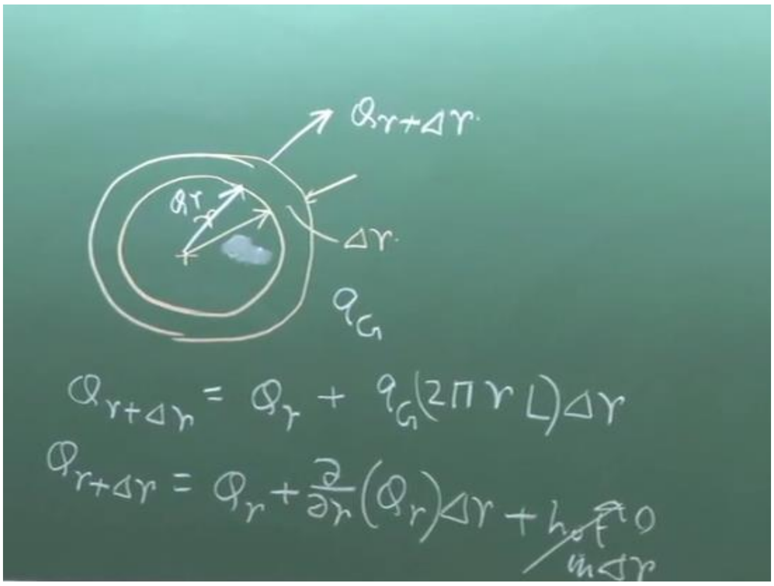

For clarity, I write here the annular ring. This is at an orbit value radius r, with this one delta r, then our same analysis what we do is that if Qr is the heat flux incident on this inner area, then this is the dimension, I am sorry. This Qr is the heat and heat going out is Qr plus delta r. The equation is very simple. If we consider a heat generation per unit volume qG within the cylindrical wall with extent from r1 to r2.

Then for this elemental ring, I can write Qr plus delta r at steady state is Qr plus qG times volume of this element. The volume of the element is the cross-sectional area is twice pi r into length where L is the length, that means it is a cylindrical ring, length is perpendicular to the direction of the board. This is the area times delta r, that means this is the total thermal energy generation, so Qr plus delta r going out.

At steady state, there is no other alternative, this is same equation, which we wrote for plane area and I have always told that, my suggestion that for steady one-dimensional heat conduction is greater to derive the equation from fundamental. This is okay. Now, if you write series expansion Qr plus delta r is Qr plus del/del r of Qr delta r plus higher order therm, which mean delta r, which we neglect, higher order therm in delta r, because delta r is very small.

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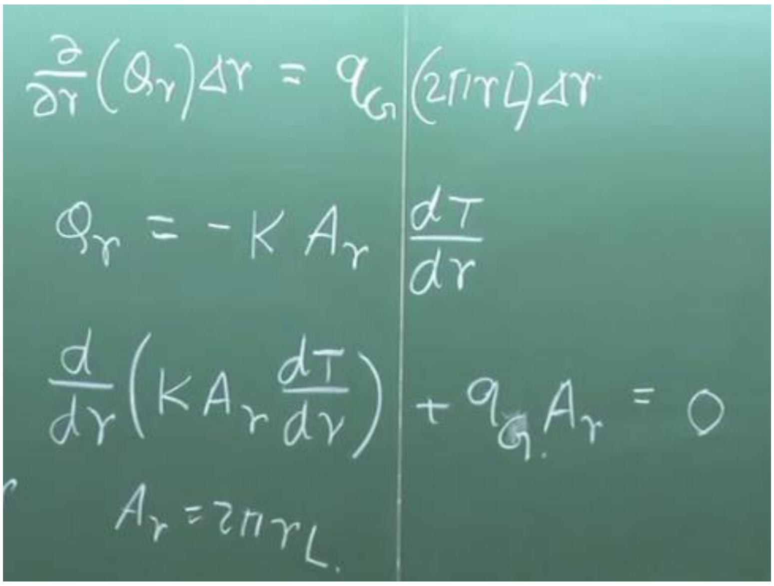

So, if we write this, Qr plus then, we can write del/del r of Qr delta r is equal to qG into twice pi r L delta r. So, this equation is the same as we did for plane wall. Now, we have to replace Qr by food-air conduction equation. What is Qr? At any radius, at an orbit steady location r, Qr is minus K into the cross-sectional area at that r into dT/dr, so therefore if we put that here, now here also twice pi r L, actually it is the cross-sectional area.

I can write in terms of the cross-sectional area, Ar. This is nothing but the cross-sectional area, twice pi r L, so that we can write. Now this del/del r, I can write as d/dr because it is a one-dimensional heat conduction Qr exists and it is a function of r only, d/dr of, take this here A this side, Ar dT/dr plus qGAr is equal to 0. Delta r gets cancelled. My sole intention to write in this fashion, again this right to Ar.

So, I wrote earlier twice pi r is very simple, that this is the same equation, which we have used in the plane surface with varying area, that d/dx in x and y coordinate system when T is a function of x in sufficient coordinate system KAx dT/dx plus qGAx 0, but the difference is that in plane area, this A may be constant under substance equation. It will come out and equation becomes d/dx K dT/dx plus qG 0.

In some cases, it is varying and the integration depends upon the type of variation of A with x even if K remains constant, but here Ar is fixed. Ar is two pi r L.

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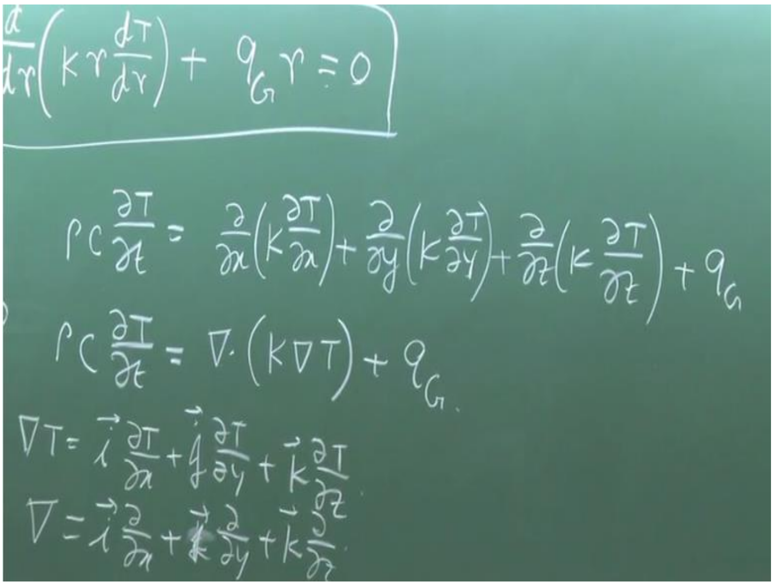

So, therefore, if you just substitute this, then you will get d/dr and if you take two pi L to be constant and L cannot be 0, so you can cancel that, d/dr of Kr dT/dr plus qG into r equal to 0 and this is our basic equation for temperature distribution in differential form. If K is constant, then K will come out. It may be going here by K. Now this same equation, I again tell you can be derived from the general conduction equation.

The way I told the plane surfaces also, that we can generate the same thing by it is greater always to maintain energy balance for a steady one dimensional and derive your own equation, but from the general equation also, we can come to this. In a cylindrical coordinate system, the same thing will appear, just for your interest I tell you, if you recollect the general energy equation, which was derived in the class in terms of Cartesian A.

Cartesian frame of reference rho C del T/del t is equal to del/del x of K del T/del x plus if you remember del/del y of K del T/del y plus del/del z of K del T/del z plus qG. So, this was the general energy equation where T was defined as a specific heat, which if you find in such a way that mass times a specific heat times the rate of change of temperature equals to the rate of change of internal energy of the material

And in a solid, it is the change of the internal energy because there is no flow, no other energy process to the boundary, so therefore it is the change. Within the control volume, it is the change of the internal energy, so this is the left-hand term. So, therefore, this equation, you know, you are familiar with, general energy equation or heat conduction equation, but now the question is that this is in frame of Cartesian coordinate system, x, y, z.

How do I get the same equation in cylindrical coordinate system or cylindrical polar coordinate system. There are two ways of doing that. The most simple way of doing this, that is just transform this equation in vector form. You write this in general vector form del T/del t is equal to what is these del/del x of K del T/del x, del y, this is divergent of K grad T plus qG.

And you know in Cartesian coordinate system, grad T is what, i, j, k being the unit vector along x, y, and z direction, i del T/del x plus j del T/del y plus k del T/del z and divergent is a vector operated, which i del/del x plus j del/del y plus k del/del z, this is again a school level thing and if you make this scalar product, which the operator and the exact K grad T, so if you convert this.

Then your job will be only to expand this term in different coordinate system, not only cylindrical, even the spherical polar coordinate system, what is the expression of divergent K grad. This is only the spatial derivation, which change with respect to the coordinate system. That is all.

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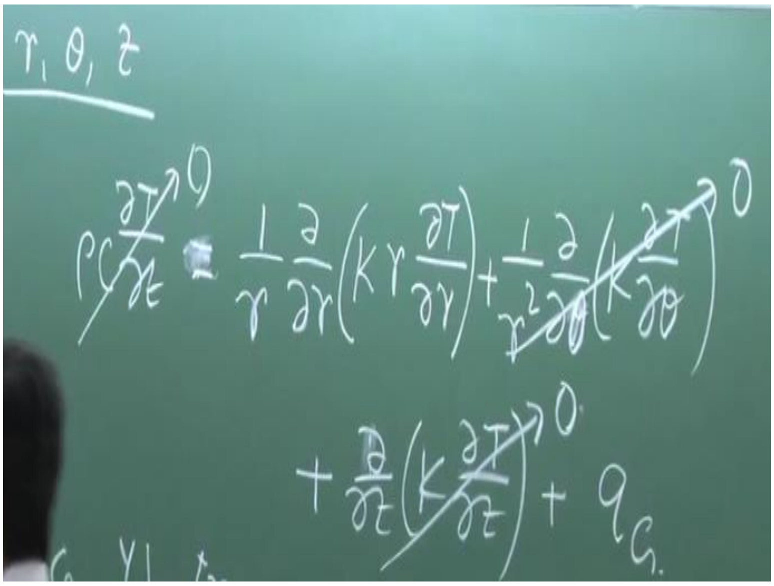

So, in a cylindrical polar coordinate system, r theta z. If we write this divergent K grad T, then it will be like this, rho C del T/del t is equal to 1 by r, I think you know this thing, del/del r of Kr del T/del r plus 1 by r square del/del theta of K del T/del theta plus del/del z of K del T/del z plus qG. That means, in a cylindrical polar coordinate system, if we define a cylindrical polar coordinate system like this, in x-y plane.

If I define this the point r and its azimuth of this and this is the location in three-dimensional cylindrical polar coordinate system, from the origin by the radial coordinate arc, azimuthal coordinate and the axial z coordinate. So, in that r theta z if you expand this divergent K grad T you get this. That means this is the counter part of this and here, for one dimensional definitely you just ignore this and steady state, steady state one dimensionally 1 by r, that means if you take out this becomes the same expression.

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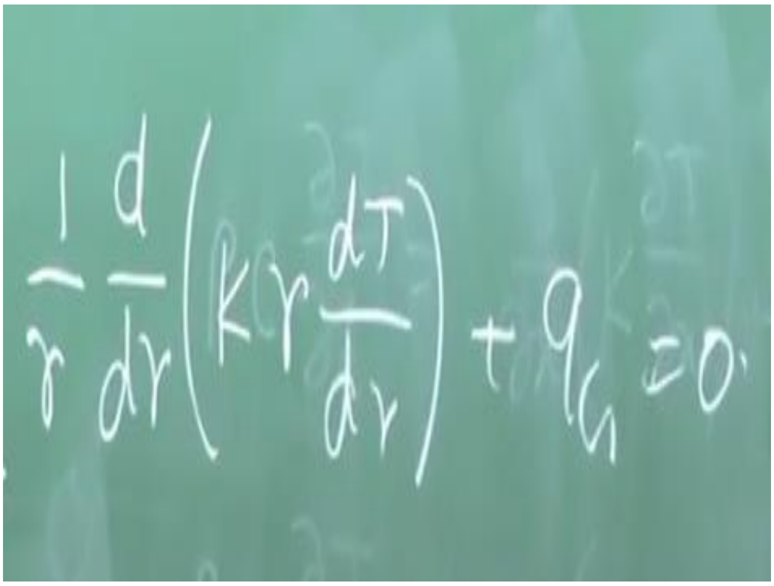

That means this equation becomes 1 by r and del/del r, you write d/dr, that means from the general heat conduction equation we can also get the same expression that is 1 by r d/dr of Kr dT/dr plus qG is 0, so this is my one-dimensional steady heat conduction in a cylindrical wall. It is as simple as that. Another way of deriving this equation is the same. This is the most simple and intelligent way.

Another way of deriving this equation is the same the way you derived the heat conduction equation in Cartesian coordination. What you did? You took an element or controlled volume in the conducting medium whose surfaces are parallel to the coordinate planes or edges are same thing parallel to the coordinate axis and which becomes a parallel with pipe x for a Cartesian coordinate system.

Similarly, you have consider an element of controlled volume whose planes are parallel to cylindrical coordinate system. That means parallel to r theta, r z, z theta plane and then recognize the heat flux flowing across this plane and take a balance of the total heat flux coming into the control volume, which is change in energy, the way it has been done for Cartesian coordinate system.

And it is a routine matter and it is done in any book you can see that it is not being done in the class, but most easy way is to convert this into cylindrical polar coordinate system by changing, expanding the divergent K grad T in the respective cylindrical coordinate. Now, after this my job becomes much more simple mathematics at school level to integrate this equation and before that, I consider another simple case that K is constant.

We consider that K is constant. Now when K is constant, this expression comes.

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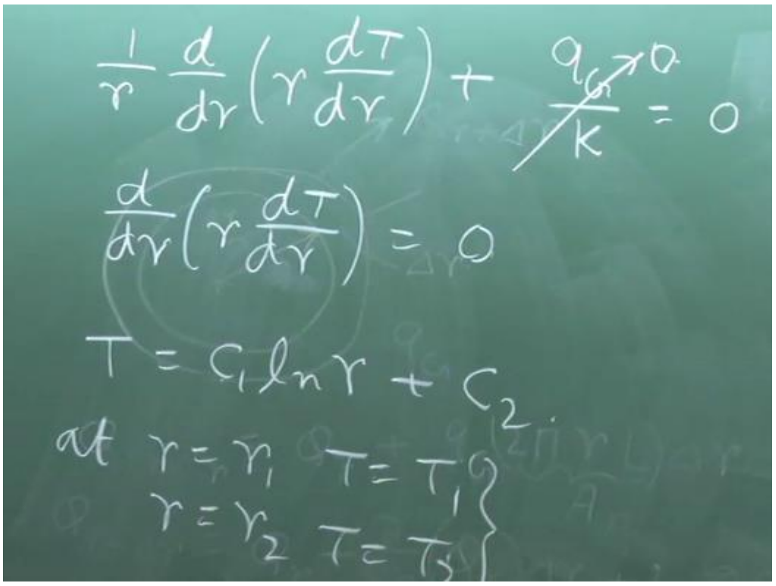

1 by r d/dr of r dT/dr plus K is constant, so if you take K, plus qG by K is 0. Now next you consider without heat generation, that means q 0, that means without heat generation steady state constant thermal conductivity, the expression is 1 by r d/dr of r dT/dr, which means d/dr. Now this becomes so simple that it is a mental problem without paper, we can solve this. The solution of this is T is equal to some constant lnr plus C2.

r dT/dr is C1, it is constant and then again integrate C1 lnr dr/r plus C2, okay. What are the boundary conditions? Boundary conditions are this and there are two constants, two boundary conditions also given in the problem at r is equal to r1 and T is equal to T1 and r is equal to r2 and T is equal to T2.

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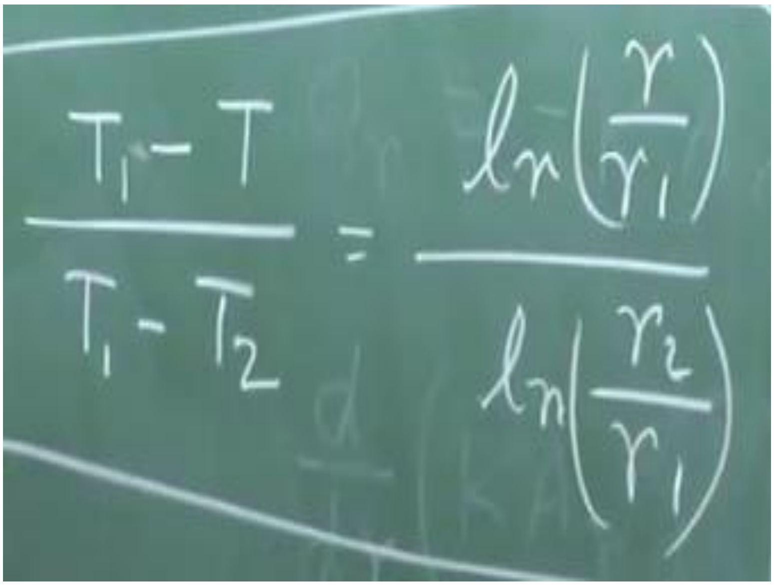

If you use these two boundary conditions, you get a profile temperature distribution T1 minus T divided by T1 minus T2. T1 is the inner surface temperature, which is higher than T2 is equal to ln r by r1 divided by ln r2 by r1. It is very simple that you substitute this to find out T1 and T2. T is equal to T1 ln r1 plus C2 and T2 is T1 ln r2 plus C2 and you can find out T1, T2 and finally you get this as the logarithmic distribution.

This is the temperature variation with r. It cannot be linear, because the area is parallel. In a way that it is directly proportional to the r. Now, if you find out the Qr at any section, at any arbitrary location that already we wrote the expression, minus K into ar, that is twice pi rL into dT/dr.

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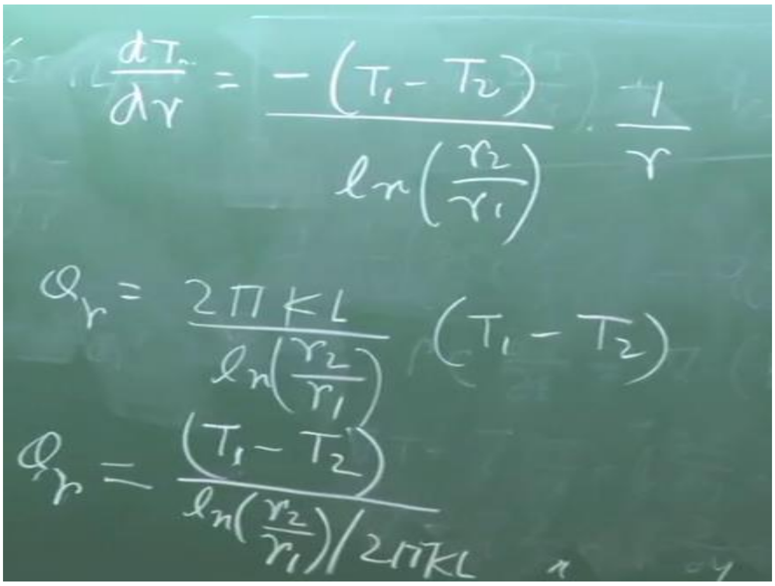

What is dT/dr? dT/dr is minus dT/dr minus T1 minus T2 divided by this is constant ln r2 by r1 and this differentiation r1 by r and again 1 by r1, that means 1 by r and if you multiply here that r cancels. That means Qr becomes, and minus minus is plus, so therefore twice pi KL divided by lnr2 by r1 into T1 minus T2. So, it is found that with this temperature distribution, heat flux at any radial location r is independent of r.

Obviously because I have found the temperature distribution from the steady state constant without heat generation, Qr plus del r has to be equal to Qr, we have made Qg 0. So, therefore it has to be like this. It is proved that it is okay. So, this is the heat flux and this heat flux in the similar way can be expressed as T1 minus T2 in the numerator, Qr, divided by ln r2 by r1 divided by twice pi KL and this acts as the conduction resistance.

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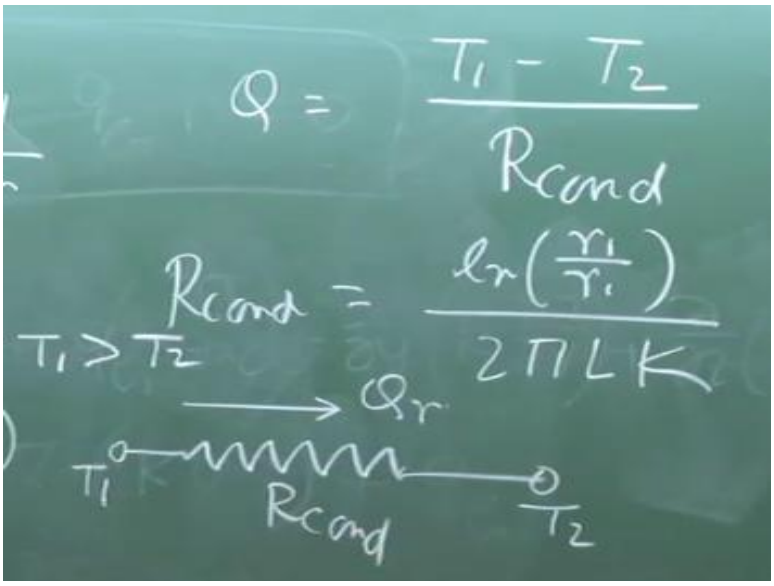

So, therefore if I write Q is equal to T1 minus T2 divided by conduction resistance. This conduction resistance is equal to ln r2 by r1 divided by twice pi LK, that means this can be represented by electrical analogous circuit like this, that this is the Qr radial direction, this is T1, this is T2, 2 potential T1 greater than T2, T1 greater than T2 and this is the R conduction, which is in case of plane surface it was L by K.

In case, cylindrical wall it is ln, or Q by r1 or what nomenclature you use or O by ri from outer radius to inner radius, that means ln times the ratio, that means ln of the ratio of the Q radius and then, r2 by r1, twice pi LK, okay. Now this thing can be also be deduced in a very simpler way, how? The same thing Q is twice pi KL, T1 minus T2 divided by ln r2 by r1 twice pi KL.

I am telling you another method, which you see often in a book, because this is a very generalised approach to find out the temperature distribution and like this, but another method is sometimes used.

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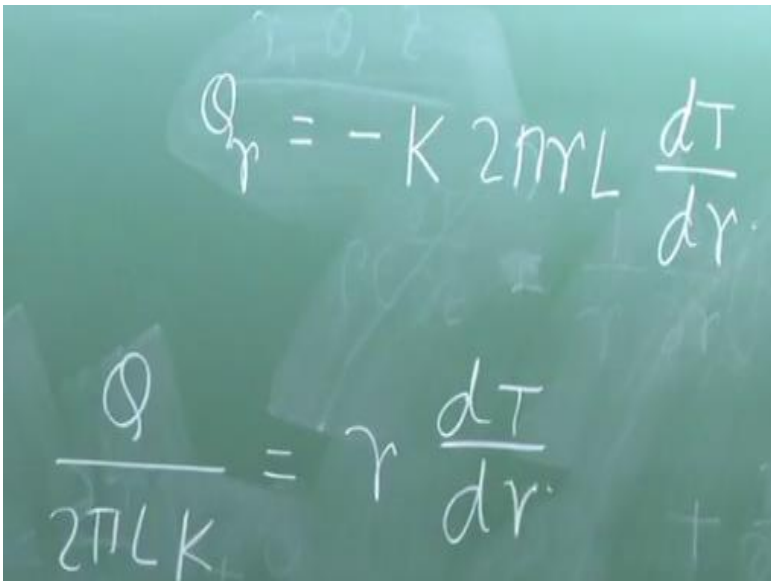

That has any radial location Qr, I know the temperature distribution is written like that, heat flux is K into twice pi r L dT/dx. There is no assumption, Qr is minus K 2 pi r L dT/dx, this is an analog of heat conduction. Now if I neglect heat generation, steady state, we did so many things, but it is a quick approach, that Qr, that is, across any cross section, at any R is constant, that means Qr itself is constant.

Then you can write as Q itself. That means I can write Q, and I can make like this twice pi LK equals to, pi R, pi LK equals to R dT/dr, and next we can write Q by 2 pi LK dr by r is dT.

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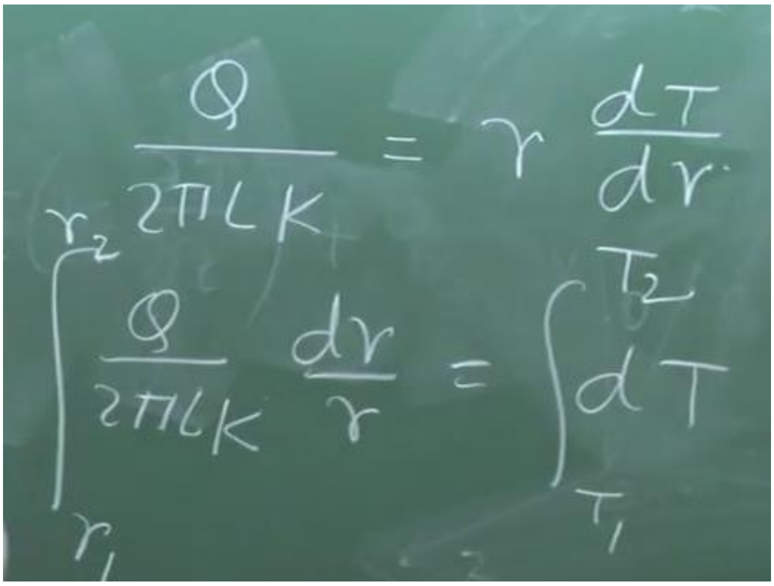

Now if I integrate this from r1 to r2, inner to outer, and a variability T1 to T2, and with this consideration, that Q is constant, which is not varying in the radial direction, because for a steady state heat transfer without heat generation same heat has to pass through each and every perception. In the similar way, I told for plane wall also it can be done like that, in many books it is done like that.

If you integrate this taking this out, this will be simply T2 minus T1, you straight get the equation. Q is equal to Qr or simply Q whatever you call this now you can think of as Q, so Q is equal to that heat flux already in the radial direction there is no point of writing in the radial direction Qr, Q is equal to this, automatically you get this.

Now the question comes here, by this I arrived an expression of heat transfer in terms of the terminal temperature difference, but where is the temperature distribution, very good. Then you can take the arbitrary r not the r2 the outer radius, then you take the T, then you get an equation Q, heat flux in terms of T1 minus T, and here it is ln r by r1, becomes a mental problem. And divide one by another, you get the expression like this.

So, this is the most easy way, also to find out the temperature distribution. That you write the heat flux, connect from one to two integrate, you get the heat flux in terms of the total potential difference, or you take any arbitrary location r, which a temperature T there. Then you get an expression Q in terms of T1 minus the temperature T, where this will be r instead of r2, and divide one by other you can get.

This is the easiest way of finding out that, but always it is better to derive an equation under certain special or complicated cases, you must have the practice of generating the basic equation from an energy value for small element, of the heat conducting medium, which is always good. So, that you arrive at and solve the temperature distribution and then write the periodic conduction equation to get that thing.

And this is one of the easiest way most of the book write that taking Q to the constant, that of all section to get this. That is all, now next similar to the plane wall, we can have a convective boundary conduction, means that this if I draw this conview that will be in arctic or plane.

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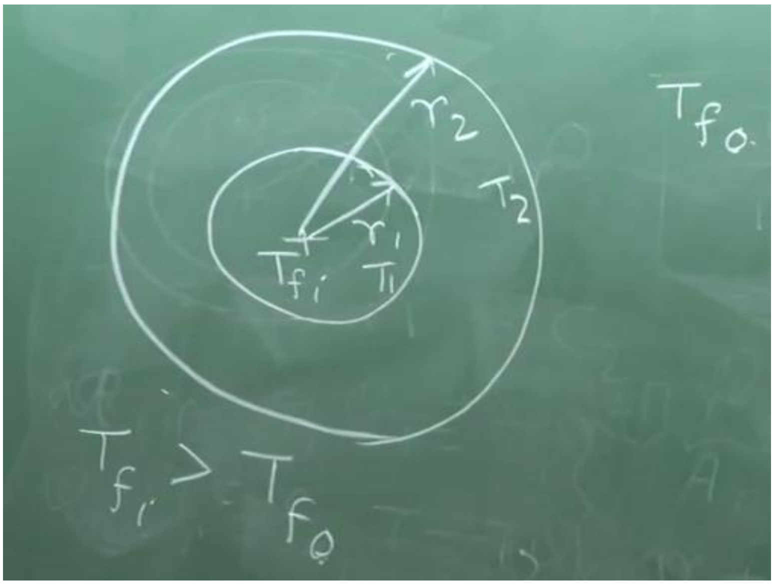

Instead of prescribing the inner surface than outer surface temperature, we prescribe the fluid temperature, inner fluid temperature which may be Tfi and outer fluid temperature which may be Tfo. That means the heat comes from the inner fluid through this, let us consider Tfi fluid temperature inner is greater than Tfo. Then first by convection heat will come to the inner surface and it will have a temperature T1, which is not prescribed.

And heat will flow by the conduction to the outer surface, which will attain some temperature T2, and from T2 by convection it will go to outside fluid Tfo. For example, heat is lost from a hot fluid flowing through the pipe, flowing through the pipe. So, in that case T1, if T1 and T2 are not prescribed only Tfi Tfo, then what we can do we know that conduction heat transfer between T1 and T2 are the inner and outer surface temperature, which may not be prescribed but I take as temperature denoted as T1, T2.

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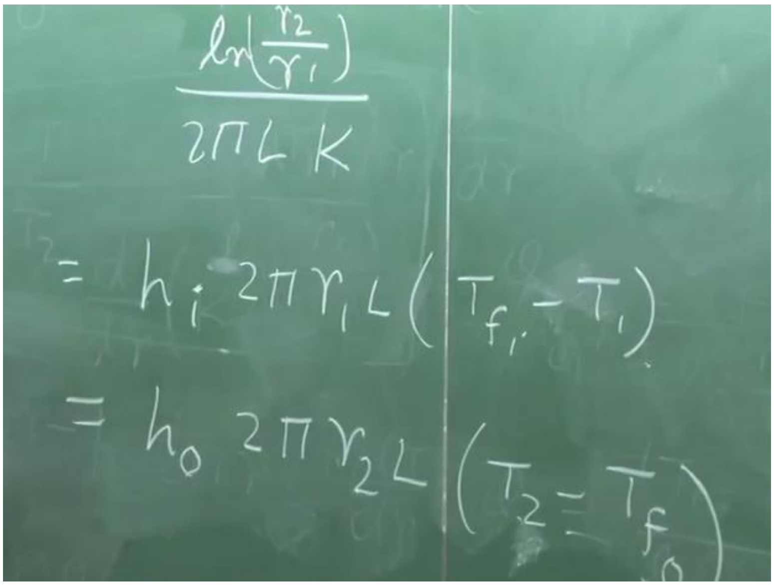

I already know that this is equal to ln r2 by r1 divided by 2 pi LK, and same heat Q is coming through, because of steady state the same thing is flowing by convection from the inner hot fluid to the surface, and if I prescribe HI as the heat transfer coefficient of the inner fluid and HO that of the outer fluid, then by our definition of the heat transfer coefficient in convection heat transfer.

It can be written as heat transfer coefficient into area, twice pi r1 L, inner surface area into Tfi minus T1 and that is same as that those out from the outer surface to the outer fluid environment, which is taken to the cold here less, therefore this will be twice pi r2 L into T2 minus Tfo. So, in the similar way as we did earlier, we can express the T1, T2, Q times this. Tf minus Ti is Q time this, Q divided by this, and T2 minus Tfo is Q divided by this, and this will sum it up.

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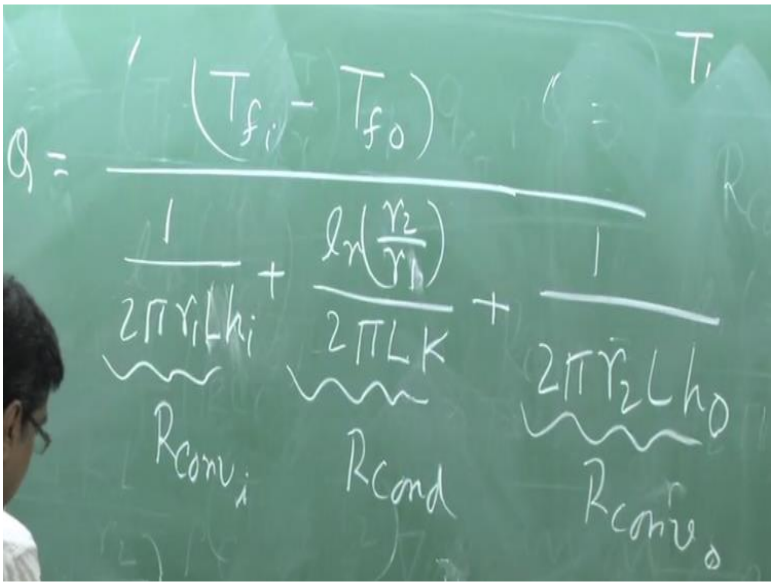

Then we get Q is equal to in terms of the extreme temperature difference, we here prescibe that is Tfi minus Tfo divided by 1 by 2 pi r1 h1 plus ln r2 by r3 divided by twice pi LK, r1 l, sorry twice pi, I am sorry, twice pi r1 L hi plus 1 by twice pi r2 L ho, extremely simple. So, it is the r1 and it is r2, ln r2 by r1, I have written r3, sorry, very good. Now this is same as the flat plane thing.

That means it is the sum of these three-series resistance. This is convection resistance, heat convection, inner surface i, this is R conduction, and this is R convection outer surface.

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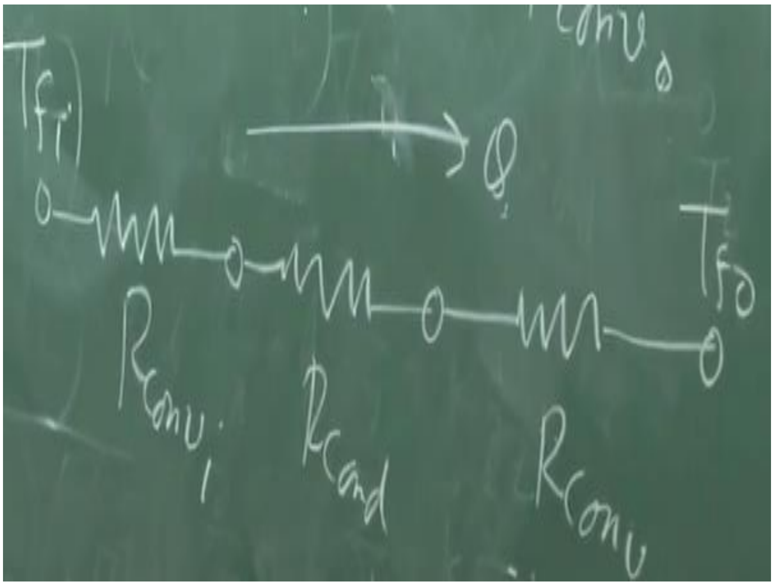

That means extremely simple, series resistance are in parallel, through which the same heat is flowing, and the electrical analogous circuit is like this, Tfi and Tfo. The same heat is flowing Q, with this is R convection i, whose value is 1 by 2 pi r1 L h1, and this is conduction and this is R convection, same thing. We can also think of composite cylinder or composite cylindrical wall, which may be composed of different thermal conductivity.

Then, we will have the different conduction resistance in series, different conduction resistance in series. So, it will be nothing great if you solve problems, then you will see the application of this. That means instead of one cylindrical wall, we may have a composite cylindrical wall. That means we have another cylindrical layer of material, of different thermal conductivity.

Then we will be adding another conduction resistance like this, that will be from r2 to r3. That means if we have another, just like this I give you a picture like this and it will be very simple to deduce.

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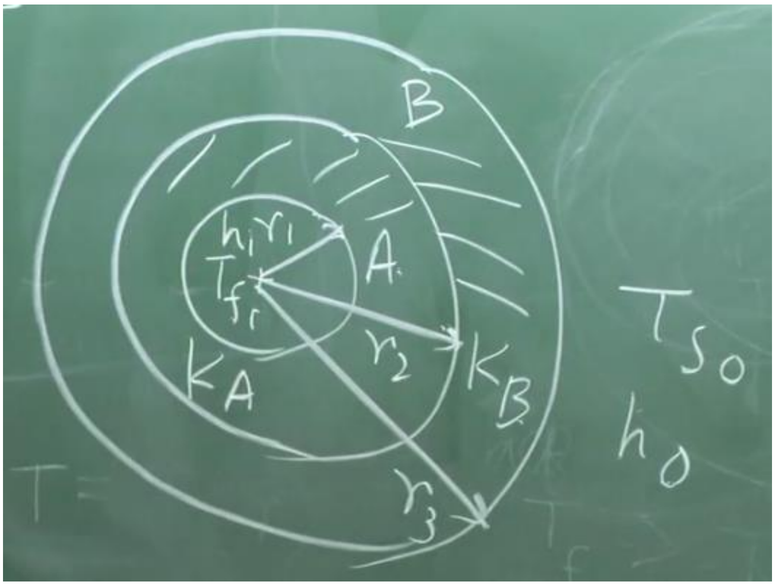

It will be very simple to deduce, like this if we have two cylindrical wall, the radius is r1, r2 and r3. Then r1 to r2, this material, this is A, thermal conductivity KA, and this r2 to r3, this material is B with thermal conductivity KB and if we have similar Tfi and Tfo, with ho and hi, heat transfer coefficient, then your circuit will be the same with another added resistance, which will be very simple to conceive and we can draw the network like this.

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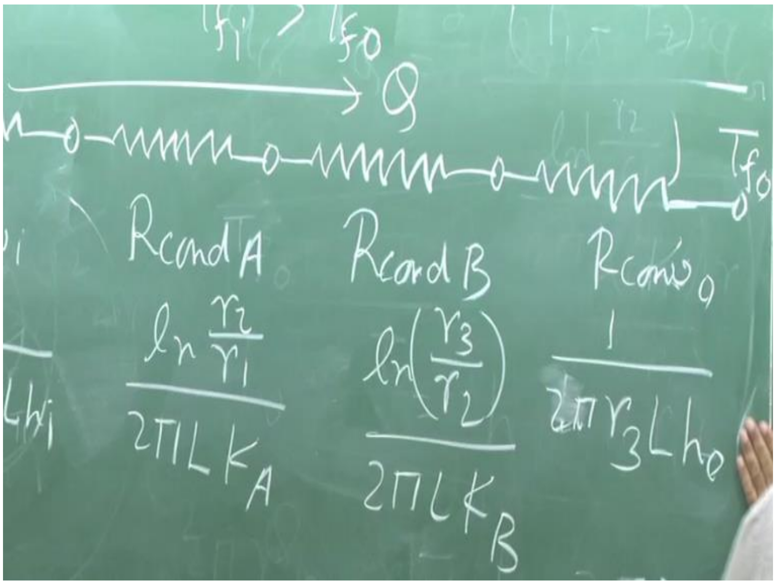

One convective resistance, then two conduction resistance and another one convective resistance, so that the two terminals have the extreme potentials Tfi, Q flows like this. Tfi is greater than Tfo and we get this convection. What is that? 1 by twice pi r1Lhi then this is r conduction for material A, that is Ln r2 by r1, outer to inner radius divided by twice pi LKA. Similarly, this is R conduction B, which is ln r3 by r2 outer to inner radius of this cylindrical wall divided twice pi LKB.

And this is finally R convection o, that is outer 1 by twice pi, this I write first r3, the area, L into ho, so this is so simple. Therefore, this is the thing, that means composite cylindrical wall with convective boundary condition are tackled like that. How do we get this distribution? We solve the temperature distribution by developing an expression from the energy balance that is by heat balance, energy balance in the conducting medium by taking a small elemental volume.

Or you can start with the general heat conduction equation as I have told, then you solve it for the special case. If there is heat generation, then heat generation format have to be taken is dependent with the spatial coordinate have to be taken if thermal conductivity depends on temperature, then the dependent has to be taken, and the problem becomes mathematical that means whatever is involved is mathematics.

So, there is no other heat transfer concept, but one very important case, which sometimes many books forget to tell that for a variable area, plane area problem, if you start from the general energy, general heat conduction equation, you will be lost. You have to develop that equation by taking the, that I told in the last class, energy balance. Because the concept of variable area is not there.

Because it is integrated over a cross section to get a new temperature effect or the temperature being uniform, but area is varying that part will not be manifested. If you step forward take from the heat conduction equation, this is a very general mistake the student does, I tell you. A teacher always tells his experience from students end. The students always jump to the general energy equation.

He finds that okay, we make steady state, del T/del t 0, we make Q 0, we make everything 0, then you get that d square T/dx square 0, T has to be linear in x. Unfortunately, in a one-dimensional heat conduction is an approximation of four, for a varying area T is never linear. It is A into dT/dx, it is constant, so dT/dx is inversely proportional to A, that constant has to be cleared, but in cylindrical coordinate system.

Because of the coordinate system itself that area, it is inherent to the coordinate system that the area normal to heat flow, for example in the r direction heat is varying with that, it is directly proportional with r, so therefore from the general heat conduction equation, it is x a special, it will be the same as we derived by taking a simple element one dimensional Qr and Qr plus del L like that. Both the things are same.

I think for you people these two things have to be kept in mind, otherwise you will be in problem. That even a variable area, you will draw a linear temperature profile, then come to the teacher and tell why in a steady state one dimensional constant thermal conductivity, temperature is linear, why I have not got the marks, so this is very important. So, today I will stop here.

Well, this is little before the time, but the next thing is the critical thickness of insulation. That I will take in the next class. Now, I tell you, just wait. I give you a clue now before starting the next topic that when this heat flux is given by this, one thing you see that if you increase the outer radius, then two things happen in contrasting nature. We increase the outer radius, the conduction resistance increases, but the convection resistance decreases.

This is because the area, surface area of heat transfer increases. You understand, which is not the case for a plane area. If there is a heat transfer from a wall, if you go on adding material to increase the wall thickness in the direction of heat flow, you are sure that you are putting more conduction resistance and the heat flux will be arrested. That means, it will be reduced, obviously, because the dT/dx is getting reduced, but in a cylindrical geometry to increase the area.

If you increase the radius by more material, that means you are increasing the radial pump for conduction and conduction resistance increases, but at the same time mathematically the convection resistance decreases means, the surface area, from which the convection heat transfer takes place increases and convection is directly proportional to surface area. Do you understand me?

So, therefore the two counteracting heat results in a very interesting problem as critical thickness of insulation, that means if you instill it a cylindrical pipe by giving insulating material, adding insulating material, the thermal conductivity is relatively much lower than common conducting material, does not always mean that we are going to reduce the heat loss. Ironically, you will see that you are increasing the heat loss.

Because they are two contradictory things. Conduction resistance increases, but the convection resistance decreases, clear, so with this clue, next class, I will tell you the critical thickness on insulation and the expression for heat transfer in cylindrical wall with heat generation.