Conduction and Convection of Heat Transfer
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Lecture-04
Heat Conduction Equation and Different Types of Boundary Conditions

In the previous lecture, we were discussing about the heat conduction equation, so write the heat conduction equation assuming that we are now interested for heat conduction within the solid only so, rho Cp. Let us consider an example with Kx = Ky = Kz = K, so then you can write; this in, compact vector calculus notation is known as; it can be written as Laplacian of T or Del square of T, where Del square is Del square/ Del x square + Del square/ Del y square + Del square/ Del z square.

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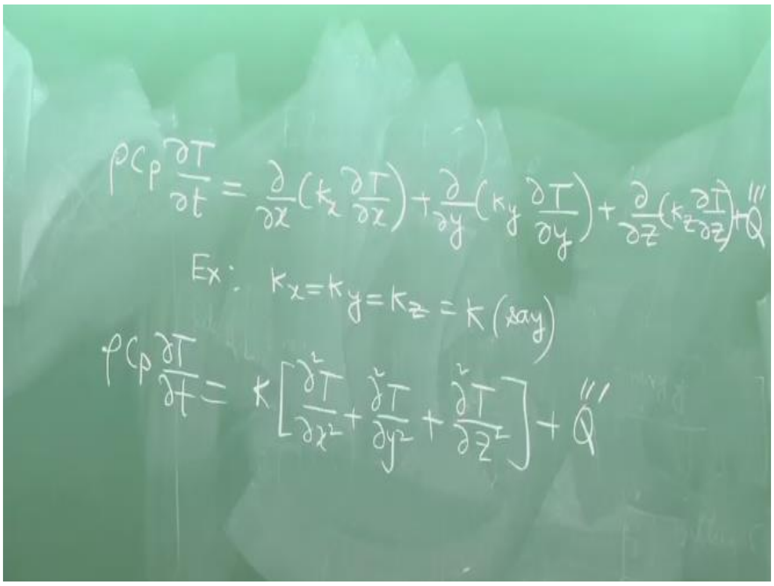

Now as obvious, these form of the equation is valid for a Cartesian coordinate system. Now you can express this form; this particular equation in terms of other coordinate systems like cylindrical polar coordinate system or spherical coordinate system. You just have to adopt the expression for Del square in various coordinate systems. These are given in your text book and I am not trying to copy that material from the text book to just waste time.

But it is a very simple exercise that you; from any engineering mathematics book or even in your heat transfer textbook, you will find an expression for Del square for the cylindrical and the spherical coordinate system and you can substitute that here to get the equations in a various coordinate systems. Normally we expect that for a problem solving, you remember these particular form but the spherical and the cylindrical polar form may not be easy to memorise and reproduce.

For exam purpose, we are not going to test your memory so, if those forms or equations are necessary, we will provide you with those forms. So, do not be bother about remembering those forms as a part of your preparation for the exam. Now, as I told you that this equation is in terms of mathematics and initial boundary value problem so, what kind of problem we are talking about; let us say that there is a rectangular slab, just as an example.

At time =0, the temperature is some initial temperature which is say, some atmospheric temperature and then, some temperature disturbances is applied at the boundary and then, the temperature within these rectangular piece will change as a function of both position and time, right. So, you are applying some boundary conditions, based on the boundary condition only, the temperature within the domain will change the other function of position and time. So, question is, what types of boundary conditions are present?

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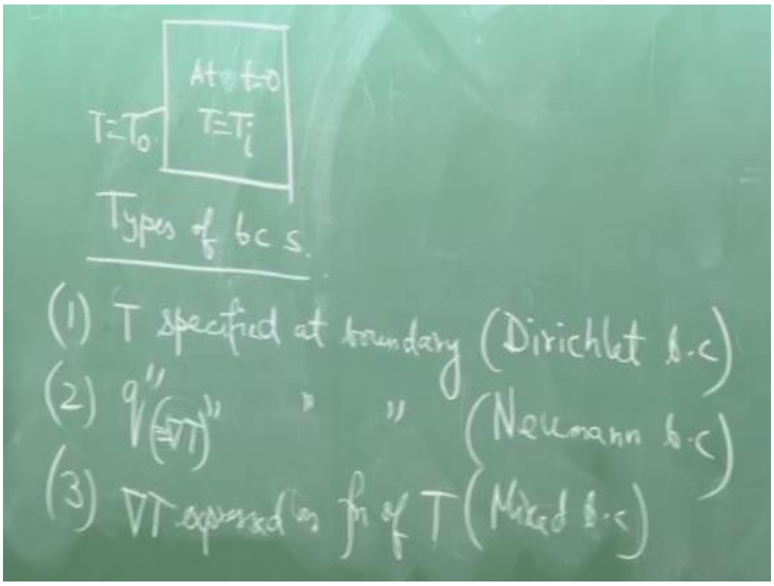

So, the first type of boundary condition, that we talk about is the most obvious one, where at some boundary, we specified the value of temperature. We say this is a temperature= say T0 at this boundary; so, T specified at the boundary. When the temperature is specified at the boundary or the dependent variable in general is specified at the boundary, this in mathematics is called as Dirichlet boundary condition.

Next possibility is that, the temperature is not specified at the boundary, but heat flux is specified at the boundary. Heat flux is specified means; that gradient of T is specified because heat flux is related to the gradient of temperature through the Fourier's law. So, the gradient of T is specified at the boundary, this is called as Neumann boundary condition. At third possibility is that, neither of temperature is specified or heat flux is specified but, one is expressed as the function of the other at the boundary that is called as mixed boundary condition.

So, grad T is expressed as function of T. Now we have to understand a very simple thing or we have to answer a very simple thing. Can we give any boundary condition for any problem? or is any boundary condition out of these three are valid boundary condition? that is something which will follow from the physics of the problem. I will give you an example, let us take an example 1.

Let us say you have a rod like this, this is an one dimensional problem; one dimensional problem means we are having temperature variation only along x, along y and z.

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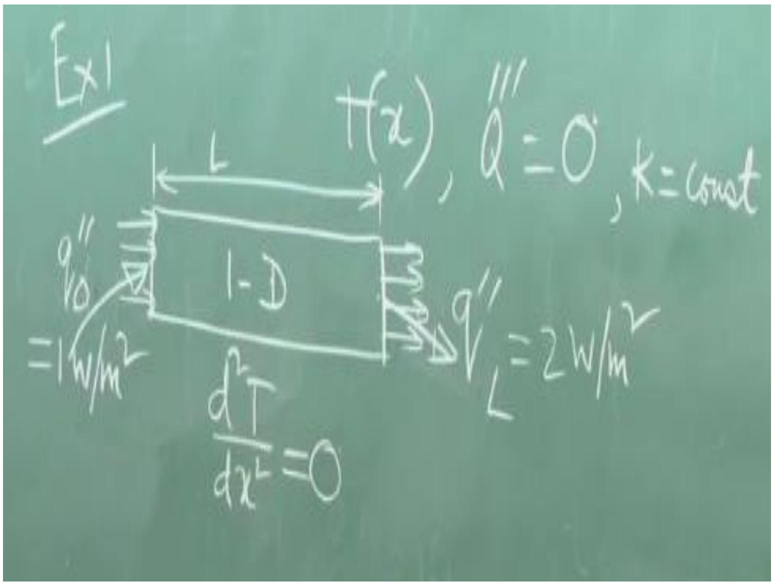

It is very large so, there is very small temperature gradient, so along mainly you have, or the temperature gradient is imposed along x, so there is a temperature difference that is taking place as you are moving along x. So, whatever may be the reason, we say that T is a function of x only. Though if T is a function of x only and Q triple prime =0, there is no heat generation, then what is this equation, if T is a function of x only, these term is 0, T is not a function of time.

So, you will get this term=0, this term=0 and this term=0 and because T is a function of x only, you have; not only this, the other consideration is K=constant. Because if K is not a constant, it will not be d2T dx2 = 0, but d dx of K Dt dx = 0 if K is not a constant. So, K is a constant is also an assumption. Let us say we want to solve this equation. This is very simple, elementary school level not even high school level but it is important that we look into these very carefully.

Let us say, that I tell you that at this point the heat flux is this, at this point the heat flux is, this is the heat flux direction, L is the length of the rod. What type of boundary condition we have given? Neumann type boundary condition, right. So, in a mathematician’s language this is a nicely defined problem, because we have given 2 boundary conditions, valid boundary conditions at the 2 ends.

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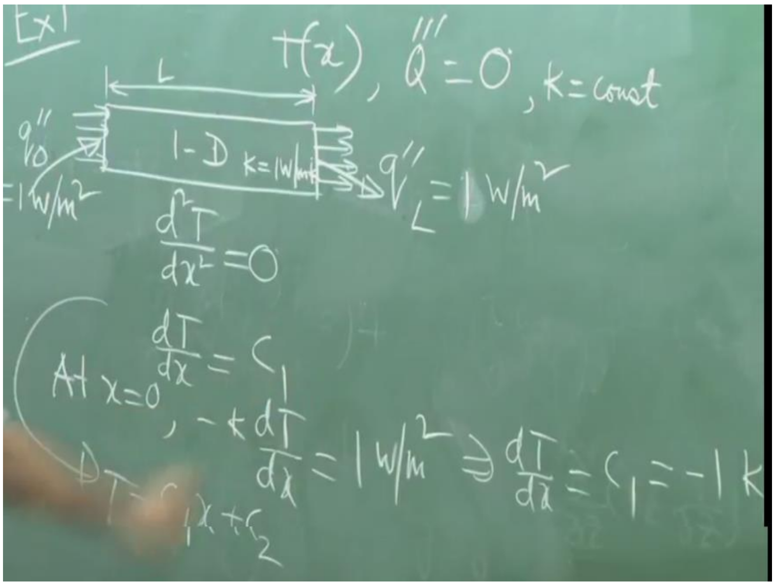

Now can you solve this equation, let us say that this is 1watt/ meter square, let us say this is 2 watt/meter square given. So, this is where, you have to bring your physical insight into mathematics; mathematics is not about integration, differentiation and all. Always try to understand the physics of mathematics that is how, you can enjoy learning mathematics. So, when you look into these equation, you will see that here you have a heat flux, which is 1, here do we have a heat flux which is 2.

Now the rate of heat in, is different from the rate of heat that is going out. When is it possible, when there is some generation inside or there is some change with respect to time, but it is a steady state problem. So, if these term is 0, that is called the steady state problem, so, there is no change with respect to time. So, when there is no change with respect to time and there is no heat generation, whatever comes in, the same must go out, so this is a physically inconsistent problem, ill pose problem.

So, any boundary condition you give is not a valid boundary condition. Now you may say that well I understood it now, let me give these as 1 watt/meter square, then it is physically consistent, now can you solve these problem? let us quickly try to solve these problem. So, you have dT/dx=c1, right, integrating these ones. So, let us say that thermal conductivity of this solid is 1watt/meter kelvin. We just to simplify the calculations.

So, the heat flux at x=o, you have - kdT/dx=1 watt/meter square. So, you will get, what is dT/dx; so that means, what is c1, this is -1 kelvin/meter. Because k is 1 watt/meter kelvin. When you integrate these, you will get T=c1x+c2; right, how will you get this c2? You cannot get this c2, right. So, this problem, so does it mean that this problem does not have a solution? No. This problem has infinitely large number of solutions.

So, what are the possible solutions? So, if you plot T as a function of x, so it is a family of straight lines with the slope of -1 kelvin/meter, so it could be this, it could be this, all parallel ones. Infinite such possible straight lines with same slope but intercepts are different but, if you are practically measuring temperature in a problem. Then at a given point, the temperature cannot be infinite number of possible values, it will be a unique temperature, right.

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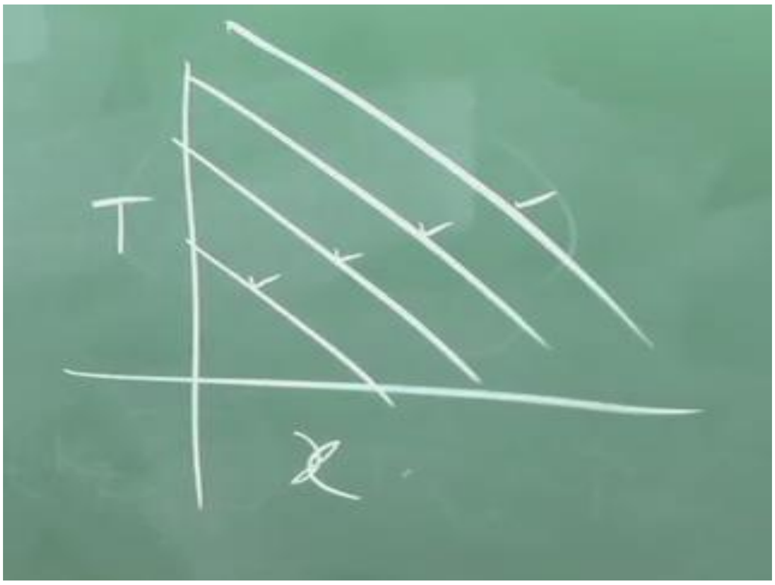

If you put a thermometer or a thermocouple in a physical body, you will get a unique temperature, that means this is not a well posed boundary value problem. So, mathematically these are called as ill posed problems. So, at least at one boundary, you must specify the temperature for these steady state problem otherwise you will not able to identify which of these straight lines to a solutions. So, the moral of the story is, any condition given at the boundary is not a valid boundary condition.

You have to apply your physical judgement before giving the appropriate boundary condition. So, we have discussed about the Dirichlet and Neumann. The third point that we will discuss about, what is a mixed boundary condition? How does it physically come? Before attempting to solve, want 2 or 3 simple problems and then we will call it a day. So, now let us take an example again, let us say that you have a rod like this, wind is blowing with a velocity, some velocity, so that this end of the rod loses heat to the ambient by convection.

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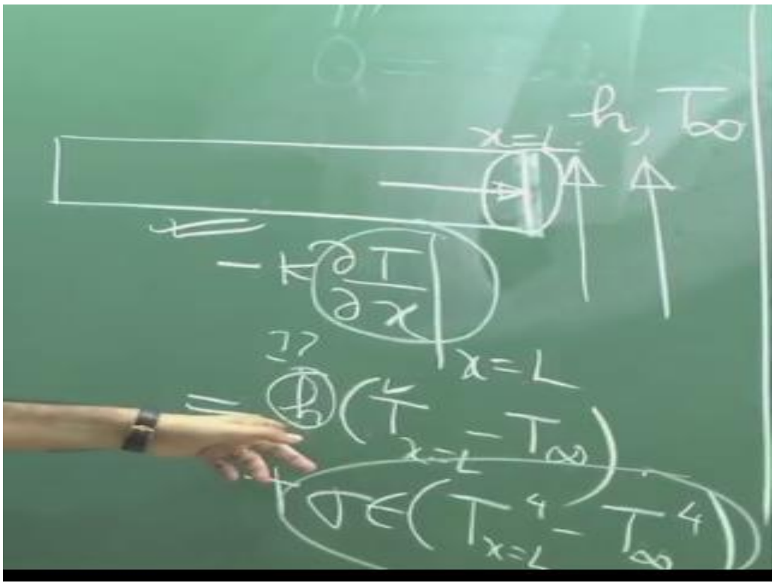

Let us say that the convective heat transfer coefficient is h, and the temperature is T infinity; ambient temperature is T infinity. So, h is the convective heat transfer coefficient and the ambient temperature is T infinity. So, what is the boundary condition here? The boundary condition here is that whatever heat flux reaches here by conduction, thus the same heat flux gets left from here by convection.

So, what heat flux reaches here by conduction that is -k Del T/ Del x, let us suggest as x=L, = what heat flux leaves here? Due to convection and if you in addition taking to account radiation, then for a grace surface where sigmoid the Stefan- Boltzmann constant and epsilon is emissivity. So, because these particular move course is basically on, emphasizing on conduction and convection. For the time being, we will not consider these term.

But you should keep in mind, that if radiation is there, these term is important and for not all practical problems, radiation is important; you can see that the radiation is important only when these temperature differences is very large, because it deals with the fourth powers of the absolute temperature difference. So, if radiation is not important, then you can write these as -k Del T/ Del x at L = h x (T – T infinity).

So, this is what, temperature radiant expressed as a function of temperature. So, this is called as mixed boundary condition. Now, where is the fluid mechanics coupled with these problem? The fluid mechanics is coupled with these problem, because you have to understand, what is the value of h? You cannot just say, let us assume h=1, 2, 100 like that; it will come from the basic fluid mechanics and how does h come from the basic fluid mechanics, that will be our agenda for studying convection.

For the time being, while solving the conduction problems, for taking these are the boundary conditions will assume some value of h. But how does the value of h come? Does it come by magic, does it come by intuition or does it come from fundamental fluid mechanics? that answer will give and we discuss about convection. So, this is a mixed type of boundary condition.

Now let me ask you another question; let us say that instead of a steady state problem, this is an unsteady state problem. So, far like that problem, we are considering as steady state problem, but let us say it is an unsteady state problem. Can we use the same boundary condition for an unsteady state problem that means temperature; unsteady state problem you understand right; temperature also varies with time.

Can we use this same boundary condition for an unsteady state problem also, because what condition we have use that whatever heat, rate at which heat has come here, at the same rate heat has left? Can we use the same condition even for an unsteady heat transfer, that is my question? Yes, or No. So, these days, all of you are habituated with multiple choice answers, so let me give four answers yes, no, may be the third one I do not know, the fourth one.

So, what will be the answer? You have followed my question, right. So, we have used these for a steady state; can you use this for an unsteady state also? Answer is Yes. We can use it for an unsteady state also. Why the reason? The reason is that this is what, this is the surface. Surface cannot store any thermal energy; thermal energy can only be stored by a volume. So, the rate at which heat has come here at that instant, at the same rate, heat should leave.

Because it is, at the end of surface which cannot store thermal energy, no matter whether it is steady or unsteady state, but if it is not heat transfer, but electrostatics, it is not the same thing. Because a surface cannot store thermal energy but, a surface can store electrical charge. So, throughout these course, I will try to give you analogy between one with the other, because engineering at the end is like basic applied physics, chemistry, mathematics.

All the branches of engineering are somehow interrelated by very unique mathematics and if you develop that kind of insight, you can relate practical problems with the mathematics that you learn. So, this is value given for unsteady state. The next point is that, if you write this Del T/Del t, take the rho Cp here, then you can write these as k/ rho Cp Del square T+ Q triple prime/ rho cp.

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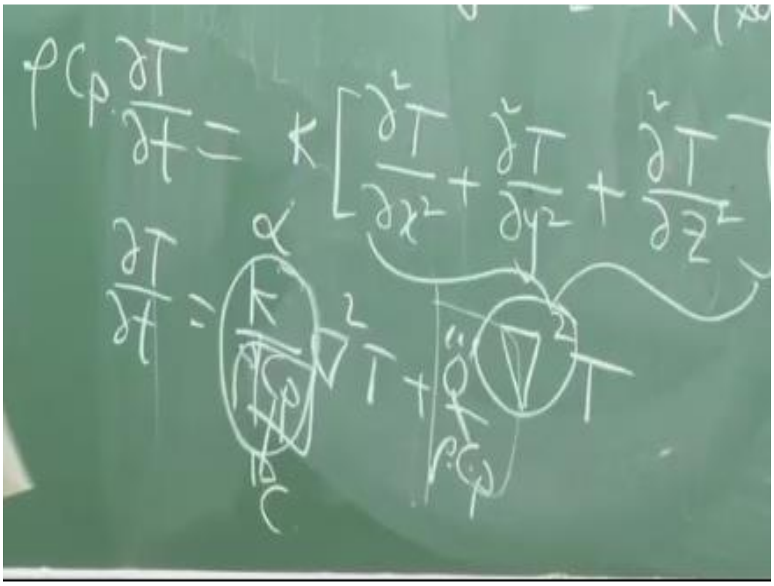

What is alpha, the thermal diffusivity. So, as I told you, the value of k itself is not that important, but k/rho Cp is more important, right. From mathematics, it is followed in these way but, how do you physically interpreted? See what represents k or k represents what physical attribute? k represents the ability to conduct thermal energy and low into C is what, it is like the specific heat, it represents the ability to store thermal energy.

So, it is the relative ability of conducting energy with respect to storing energy, that it is what is important; not the absolute value of conductivity, but the relative ability to conduct with respect to the ability to store. What is the unit of this? This unit is meter square/ second, so and, again the analogy in fluid mechanics, what is the equivalent parameter? kinematic viscosity.

So, kinematic viscosity is also the ability to transmit momentum disturbance divided by the ability to sustain momentum disturbances; sustain momentum. Okay. Now for a solid, it does not matter whether it is Cp, CVS or what; because for solid, Cp and Cv are the same. So, we commonly for solid, you write these as C instead of Cp or Cv, because for solid, because compressibility effect is not there, Cp, Cv, C are all the same; same specific heat.

Normally in mathematics books, you will find that these term is not often consider, so this equation is represented as heat equation, you must have studied in partial differential equation in mathematics and that is this equation. Now we will work out a few problems. All the problems that I will be working out today are from the exercise problems of (()) (24:43) book, which is one of your text books.

So, I am not giving the problem number, because you might be having books of different editions, where the problem number might change, so I will better discuss about the problem and you can identify the problem number, if you are interested by looking into the exercise of the book. So, let us say that there is an infinite medium; this is a problem. So, there is an infinite body in which the temperature is given as a function of x, y, z.

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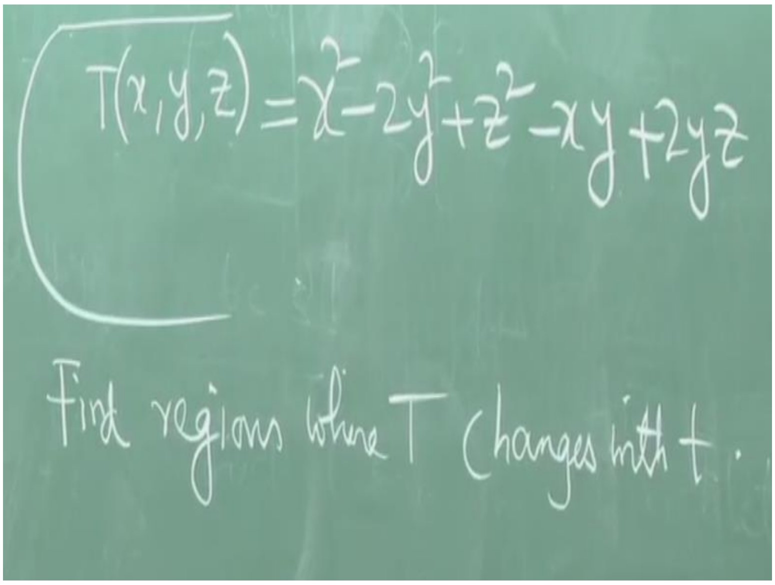

Find the regions where the temperature changes with time. So, we will assume that there is no heat generation, so let us calculate what is Del Square T? because temperature dependence on time is governed by these equation, so if we calculate the right-hand side, we can get the left-hand side. So, the first derivative, let us, so if you add, so what is the answer, there is no region where temperature changes with time.

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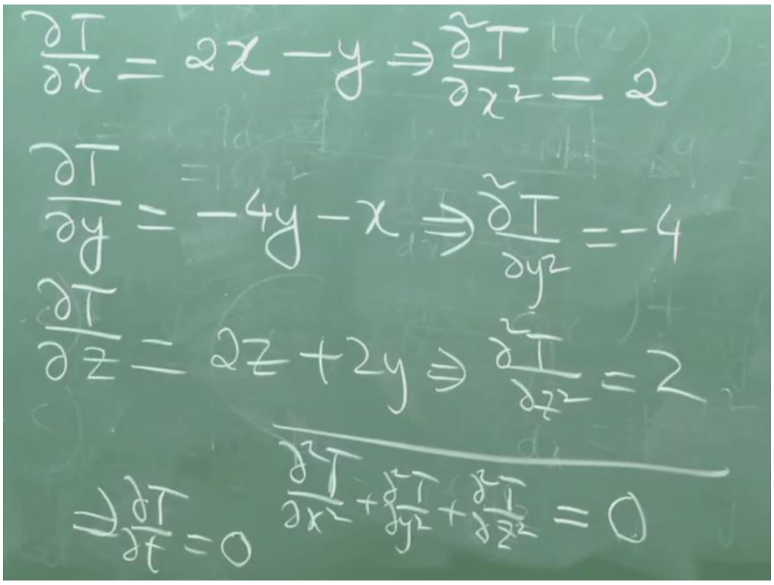

What are the assumptions that are we made for solving these problem? No heat generation, constant thermal conductivity; these two are the major assumptions, that we have taken. Next problem, so you have a one-dimensional problem, where the boundaries are given, boundary temperatures are given and temperature within the slab is also given as a function of x. So, the first part is, find the surface heat flux?

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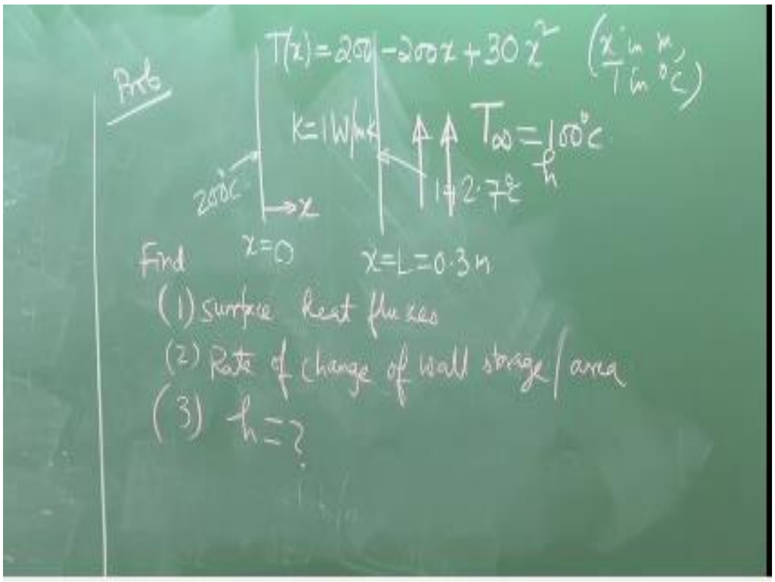

Basically, you use the Fourier's law. The heat flux, here T is the function of x only, so –kdT/dx. I am just outlining the solution and you can plug in the numerical values, I will give the final answer, you can check. So, what is, a heat flux at x=o, this is –kdT/dx at x=0. So, you find out dT/dx, T as a function of x is given and then find out –kdT/dx, substitute x=0. So, this answer this 200 watt/meter square.

This is 182 watt/meter square. So, what is the rate of change of watt storage, is basically the difference, so this is what is entering, this is what is leaving; the difference is the change, so this is 18watt/meter square. So, this is an unsteady situation when there is a watt storage, so it is better not to write, I have written by mistake, because I thought it is a steady state problem, so do not write it as dT/dx, but write it as Del T/Del x, just for fundamental reasons.

The third is you can write; what type of boundary condition it is? Mixed type of boundary condition. So, from here you can find out, what is h? It is 4.3 watt/meter square kelvin. So, you can check by substituting in these equation that you will get a Del T/ Del t term. This right-hand side will not be equal to 0. So, temperature at a given point will change with time. Depending on whether it is positive or negative, temperature at a given point will increase with time or decrease with time, that will come from the calculations.

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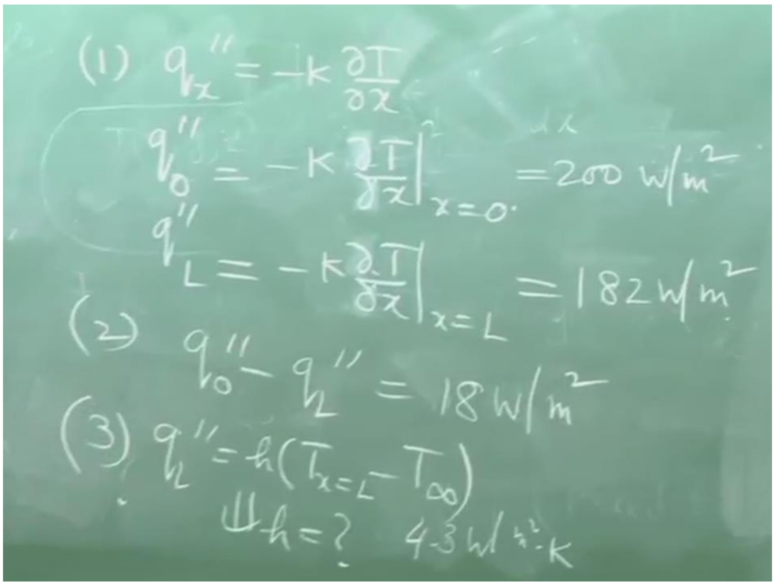

We will work out one final problem from this chapter. Now before working out the final problem, I will try to bring out a relatively apparent but paradoxical situation concerning the previous problem, so you might argue that temperature is given only as a function of x, but how can you say that the temperature depends on time right, and these problem I have not created, this problem is from an established text book and there is a reason behind these.

It is a valid situation, what it codes is, at a given instant of time, it gives temperature as a function of x, right. So, at a given instant of time, it gives temperature as a function of x, so that temperature, as a function of x, does not include the time because it is at a given instant of time. So, at a specific instant of time, you will have say, there was time within the expression, must time=1 second has been substituted.

Once it has been substituted, the time is not appearing so that the temperature has a function of x, is at a given instant of time, that will resolve the paradox, because it will be kind of apparent that why it is so, the temperature is the function of time we have seen but it does not appear in the expression of T, it appears as a function of x only. So, the final problem; so, this is a very practical problem where there is a material and laser is falling on the material.

It may be used for various reasons starting from manufacturing to medical treatment.

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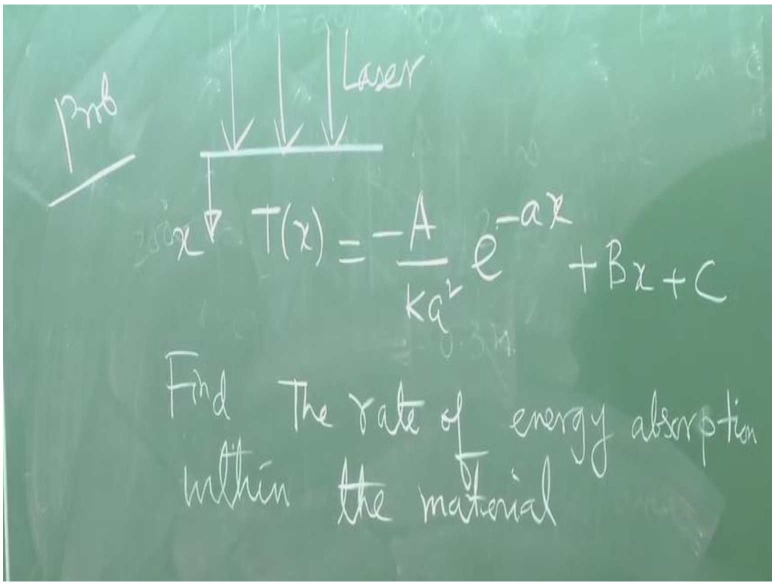

But whatever may be the reason but laser is falling on the material and the temperature is exponentially decaying so the temperature is maximum at the location where the laser is falling and as you move along x, the temperature is decreased exponentially, so it is a attenuation as a function of x, k is the thermal conductivity, abc are some constants. Now when the laser is falling on the material, the material is absorbing the part of the energy given by the laser and that gives rise to the change within the material.

It may be a biological change for manufacturing, it can be a micro structural change. So, it is very important to find out the rate of energy absorption within the material. So, how will you do that? There are many ways of solving this problem, but because we have derived the heat conduction equation today, we will use the heat conduction equation to solve this problem.

So, we make some assumption; constant thermal conductivity and one dimensional steady state, okay.

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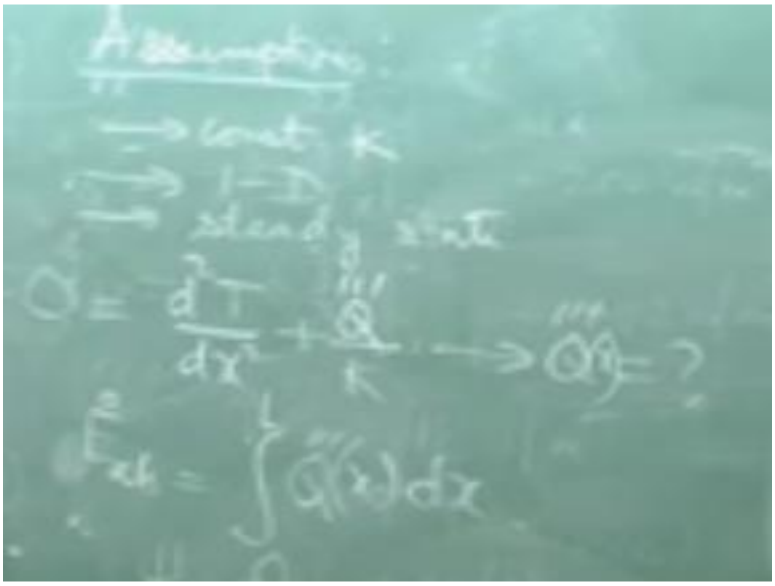

So, the equation will boiled down to 0, right. So, because T is given as function of x, from here you can find out what is Q triple prime? This will be what? This will be a function of x, right. So, what is the rate of energy absorption? L is the length along x of the material, okay. So, you integrate these over the length; you integrate it over volume but one-dimensional problem, length is replacing the volume.

Because the area in the other direction is infinitely large. Basically, this is the representative of the length; representative of the volume but for one dimensional case, any change is taking place along the length only, the cross-sectional area is not of importance. So, that is why we integrate it over the length from 0 to L and let me give you a final expression. So, let us summarise what we have learned today; we have learnt.

First how to physically perceive the energy balance, then how to express the different terms in energy balance mathematically. How to express that in the form of a differential equation that involves a temperature as a function of position of time, how to prescribe the boundary conditions and what are the possible and feasible ways of giving boundary conditions and what are their physical implications. This is all what we have studied today.

Now in the next lecture you will be introduced basically one dimensional steady state heat conduction problems that will be a special case of this equation, where this term will be 0.

This term will have only one component of derivative, that is derivative along x, derivative along y and z will not be there. So, these one-dimensional problems and their applications in engineering will be covered by Prof. Som and then we will discuss, I will come back to discuss about two dimensional and unsteady state problems. Thank you very much.