Conduction and Convection Heat Transfer

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Lecture - 03
Heat Conduction Equation

So far as Prof. Som has discussed on some of the introductory issues related to various modes of heat transfer and we will next proceed towards mathematically describing the equations for conduction heat transfer that is the agenda of today. Now why we require all these, so let us say, I mean let us think about a practical problem.

Let us say that there is a piece of metal, which is acted upon by a heat source and there is some change in temperature within the metal, because of the change in temperature, there will be a temperature gradient and the temperature gradient will in turn dictate the microstructure and the microstructure will dictate the mechanical properties of the material. So, it is very important to analyse that how the temperature varies with position and time within the material.

So, this is one example, I mean we can give many examples. I will just give another example. Let us say that there is some medical doctor, who is doing surgery with a laser, so when the medical doctor is doing surgery with a laser what the medical doctor is essentially doing, the medical doctor is trying to say kill or destroy some tumours. So, this is an example of course laser can be used for several medical purposes, this is just one example.

So when this is happening then one of the big objective is to make sure that only the diseased cells are killed but the normal cells are not damaged, because it is a very common side effect that after a treatment when the disease cells are killed, the normal cells are also destroyed because of the heating effect in a surrounding tissues and then what happens is that basically because the normal cells are also damaged it can give rise to weak immune system of the patient and the patient may suffer from serious side effects of the treatment.

So now how do you know that what will be the extent up to which that effect of the laser treatment is failed. So, then you have a laser source then you have to find out the temperature within the tissue as a function of position and time and then figure out that which are the damaged tissues or which are the damaged cells which are cells not yet damaged and one can essentially figure out a suitable parameter of the laser.

For example, laser power, what should be the radial laser power and all these things, so that the correct treatment can be made. So, there are several situations in which conduction is a dominating mode of heat transfer, number one and number two it is important to find out how the temperature within the material, the material may be a physical material. It may be also a biological material whatever.

The temperature within the material varies as a function of position and time and to solve this problem one has to look into the problem from a mathematical perspective. One can of course do experiments but when you do experiments it is not possible to figure out at each and every point how temperature varies at a function of position and time. But if you numerical stimulations or if you analytical solutions, if analytical solutions are possible then you can figure out that at each and every point how the temperature varies with position and time.

Now before going for the analytical and numerical treatment, you have to understand that what equation you should solve and we will derive that equation today that is the agenda of today's lecture. So, when we are thinking of the equation that governs the temperature as a function of position and time. So, the kind of equation that we are looking for is the differential equation, because essentially a differential equation can only give up point to point variation.

Now if you are interested about a global balance, global picture you can use some equations which are also known as integral equations. In this particular course of conduction and convection heat transfer, we will also discuss about many situations where we will be using the integral equations but today we will start with a differential form of equation for heat conduction.

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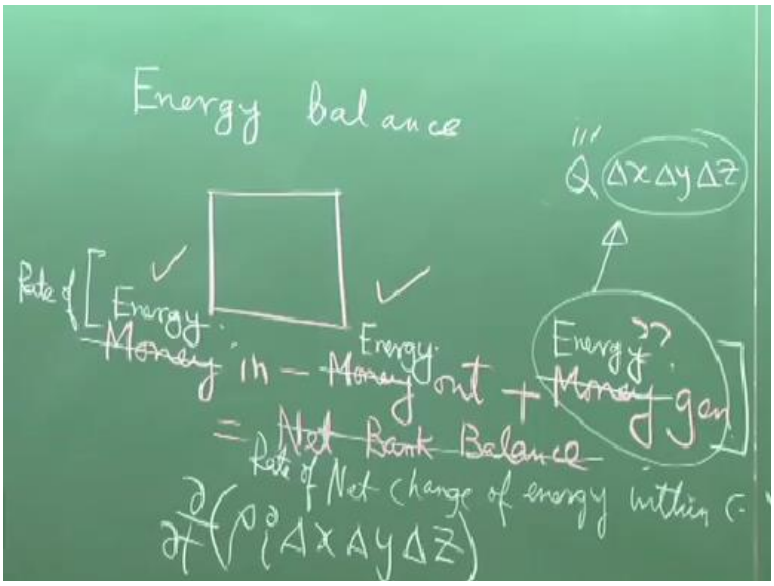

So, to understand that so the agenda is basically to write and expression, a mathematical expression that talks about energy balance. Now energy balance is something which is very obvious, and we will start with a description of energy balance, but I know that right we are having a lecture when it is quite late in the evening, raining outside, there are like compelling situations, which can make you feel distracted from the subject.

So, if we start with energy balance principle and all those things, you may feel oh this is not interesting. So, I will start with something, which is not energy balance, but which is money balance and money balance can make you feel excited at this end of the day. So, we will start with balance of money rather than balance of energy. So, I will start with the story of a good old day when people had to go to a bank to open a bank account.

Say you have got a new, nowadays you do not have to go to a bank to open a bank account. I mean you may have several online resources to do that. The bank person can individually come to you and help you to open the bank account. What is the good old day when we were very young or our fathers were even young in those days people had to go to the bank to open bank account, so let us say that somebody has got a new job.

He has gone to the bank to open a bank account. So, once you get a job and you open a bank account of course you have to deposit some money to the bank. So, when the money is deposited to the bank, your account gets activated. Now every month your salary is being deposited in the bank account. Now someday you have got into the bank and you realise that you have to withdraw some money from the bank.

This is the good old day time so no ATM business. You have to physically go the bank and withdraw the money. But before withdrawing the money, you must know that you are having the correct amount of money in your bank or you have the bank balance to withdraw that money. So, you can check your passbook and then what you do is you are withdrawing some money from that. Now when you are withdrawing some money from that you mentally try to do a balance.

What balance, so some money has flown into your bank account this black box is like a bank account. So, money in how was money in input to the bank account. You first deposited some money and then your employer has deposited your salary in the bank account so money in - money out. Money out is what you are we have withdrawn from the bank is it your net bank balance. This is not your net bank balance.

You see your bank balance is little bit more than this. It is more than this because bank is giving you interest. So, money generated. Money cannot be generated right. It is very difficult to generate money. I have never been successful in generating money in my life. But I mean money is generated because of some issues in economics which is not the agenda of heat transfer. So, we will not get into that.

So, money generated, this is what is net bank balance right. This is the summary of the story of money balance in a bank account. Now think of instead of a bank account you think of a control volume and energy transfer is taking place across the control volume.

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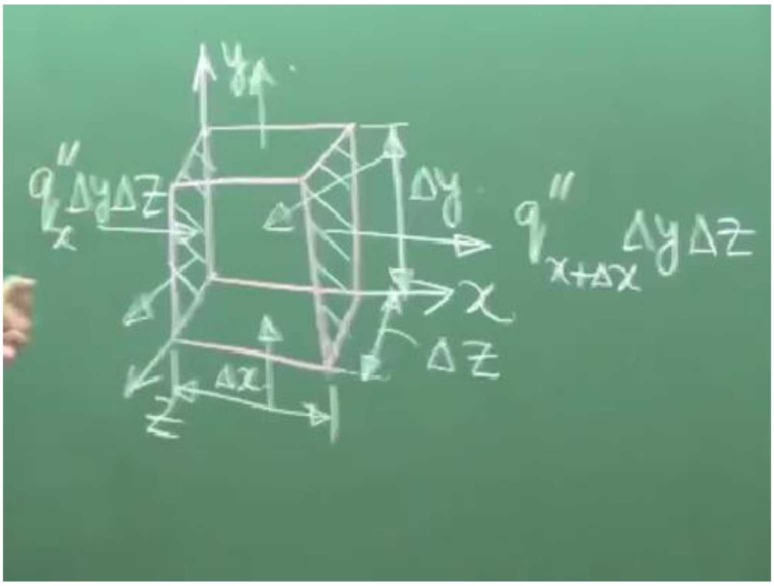

So, this is the control volume and energy transfer is taking place across this. So, when energy transfer is taking place across this you have some energy coming in so this is a control volume. I believe all of you understand what is a control volume right. A control volume is a fixed region in space across which mass and energy can flow.

So as if you are sitting with a camera, focussing your camera over this region and across this region there is some transport of energy and mass. So, when you do that so instead of money in, now you can write energy in - energy out + energy generated. How can energy be generated again like it is questionable term, it is a very loose way of stating what is happening because as we all know from our school days that energy cannot be created or it cannot be destroyed.

So, it is basically some mode of transfer so what is happening. Let us say that in a system there is electrical wire and current is flowing through the electrical wire so when current is flowing through the electrical war there is a heating effect because of dual heating right, I 2 R. So that is something what by an example of what we mean by energy generation or in a combustion reaction there are reactants and the reactants combine together to form a product and then there is a release of thermal energy.

So that is another example okay or there may be nuclear reaction so there are various possibilities by which you can have energy generation. So instead of bank balance now you have net change of energy within the control volume. So, energy in - energy out + energy generated is net change of energy within the control volume. Just it is like net change of money within the bank account instead of money it is energy.

The bank account is similar to a control volume okay. So now because in heat transfer all things are more or less expressed in terms of a rate process so it is often written in terms of rate of energy in - energy out + energy generated is equal to rate of net change of energy within the control volume right. This is a statement of basic energy balance and believe me or not if you have understood it correctly.

And if you know how to represent it mathematically you know 50 percent of heat transfer. So, we will see that how we can write this expression of balance. This is a physically intuitive expression right. Even if we do not know much of a science this is something which comes from physical intuition in - out + generated equal to change. So, we want to write this in terms of mathematical expressions. So, this is a qualitative physical expression.

Next our objective will be to write it mathematically. So, we will proceed towards that. So, to do that, let us say that we have a control volume like this okay. So, this control volume, let us say this is x axis, this is y axis and this is z axis. Now we have chosen such a control volume is because we want to derive the equation in terms of the Cartesian coordinate system but you could choose any other shape control volume.

There is no restriction on the shape of the control volume just for simplicity of using the Cartesian system we have taken such a control volume. Next, what we will do is we will account for the rate of heat transfer over here. In heat transfer, just like in mechanics you have two types of forces one is the surface force another is a volumetric force or a body force.

Similarly, in heat transfer you have either surface heat transfer or volumetric heat transfer. So, this rectangular shape body or control volume which has dimension say delta x along x, delta y along y and delta z along z okay. Now what we do is we will consider all the phases, how many phases are there, you have six phases in the volume. So over each we are interested to write a description of the heat transfer and the description of the heat transfer.

For example, if we consider this phase we will represent it by a quantity which is called as heat flux right. So, this heat flux, this is what rate of heat transfer per unit area normal to the direction of heat transfer. Rate means time rate. That is what is expressed per unit area. So, heat transfer per unit area normal to the direction of heat transfer per unit time, so that is this, this is a typically notation by which we write that.

So, this is the heat flux, what is the total rate of heat transfer through this, this heat flux times the area of the shaded phase. So, what is the area of the shaded phase, Delta y into delta z. Just consider the opposite phase, this phase. We will first investigate what is happening along x and similar things will happen along y and z.

So, we have considered two phases which have normal along x and similar two phases can be considered which have normal along y and other two phases which are normal along z. So, what is heat flux here, so if it is q double prime x, this is q double prime at x + delta x right. Same heat flux, this is at x, if this is x equal to x, this is x equal to x + delta x. This times, see this is energy in, this is energy out right.

So similar thing from the bottom and from the top and from the back and from the front, I am not writing all those clumsy. So, what I will do is I will just write one of the terms which is governing the behaviour along x similar terms will come along y and z.

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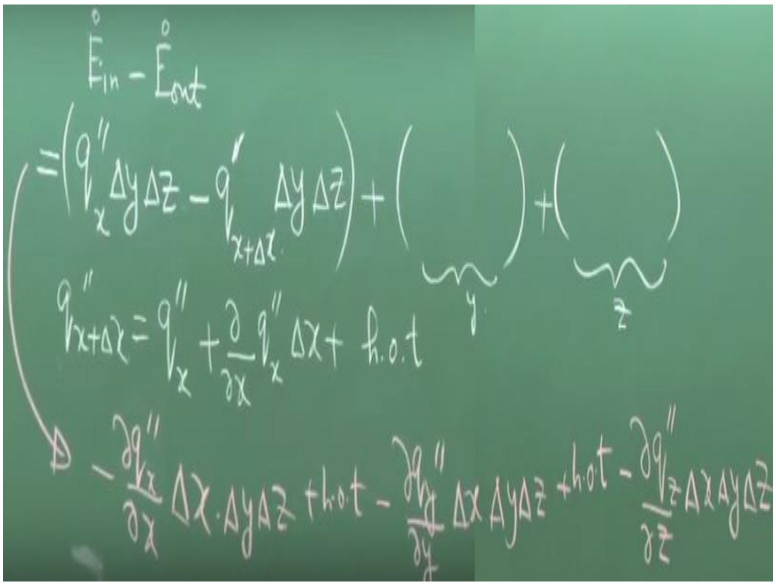

So, rate of energy in - rate of energy out, this dot means rate. So, what is this, this is along x + similar terms along y and similar terms along z. We will write those terms very soon and we will write those terms by looking into the analogy within this. Now you see here that basically we have to write the difference between q double prime x and q double prime x +

delta x.

So, what is q double prime, x + delta x. You can write it in terms of q double prime x by using a Taylor series expansion right. So, q double prime at x + delta x is equal to q double prime x + some higher order terms which involved delta x 2, delta x 3 like that. So, you can simplify this term and write q double prime x - q double prime x + delta x that is equal to this times delta y into delta z + higher order terms.

So, this delta x multiplied with delta y, delta z becomes delta x, delta y, delta z. So, if this is the term along x what will be the term along y. Similar thing and along z also similar term + higher order terms of course are there. So we have written energy in - energy out that means we have written these this term and this term. Then we will write energy generated.

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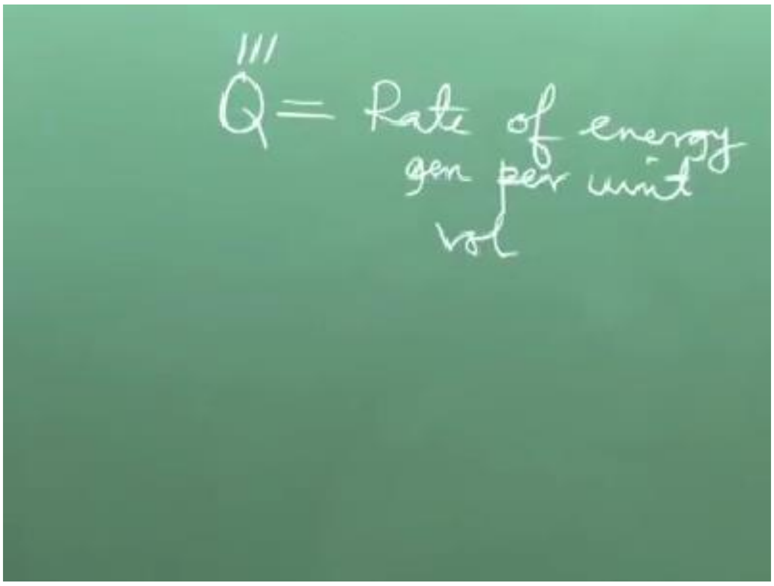

So, for energy generated, we typically consider this as rate of energy generation per unit volume. So, what is the rate of energy generation this term it is simply Q triple prime into delta x, delta y, delta z. This is per unit volume. This is the total volume of the control volume. What is the net rate of change of energy within the control volume. How do you write it, we have written the left-hand side of this. Now we are interested about the right-hand side.

So how do you write the net rate of change of energy within the control volume yes. So that is the in terms of thermodynamics, what is the energy within the control volume which form of energy is representative of the energy within the control volume. It is the internal energy.

Internal energy is not necessarily CP into T or CV into T whatever. We will discuss about that clearly.

So, we call internal energy. What is internal energy, how it should be described and how it should be brought in the context of this equation, we will discuss about that. So, let us say in thermodynamics we have used a symbol U for internal energy. In heat transfer we will avoid that because we will preserve the symbol U for velocity because we have a whole length of discussion on convection where the fluid mechanics will interface with heat transfer.

So there the velocity field will come. So, we will not use the symbol U for internal energy better we will use a symbol I for internal energy. Small i is the small u that you have learnt in thermodynamics for internal energy. So, this is what, this is a property per unit mass. What is the mass of the control volume, the density Rho into delta x into delta y into delta z, okay?

So, it is not this, it is the rate of change of this. So, rate of change of this means this one. Very important, this is not ordinary derivative. This is partial derivative. The reason is pretty clear that when you are considering the net rate of change of energy within the control volume you are fixing up the position and at a give position you are finding out how the internal energy is changing with time.

So, it is a rate of change with respect to time keeping the position fixed. That is why partial derivative and not ordinary derivative.

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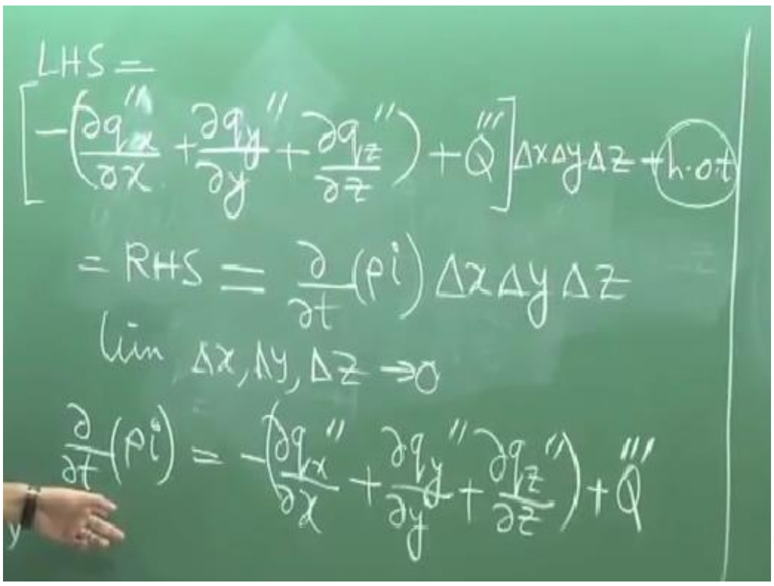

The next job will be to now combine the left-hand side and the right-hand side. So, the left-hand side, because the delta x, delta y, delta z all are constant so we have taken that out of the time derivative okay. So now see, our objective is to derive a differential equation and differential equation means the equation which is valid at a point. At a point in space as well as at a given instant of time okay.

So, you have to basically shrink this rectangular volume to a point. How do you shrink the rectangular volume to a point. You take the limit as delta x, delta y, delta z all tend to zero. So, take the limit as delta x, delta y, delta z tends to zero because these are all tending to zero these are individually not equal to zero so these get cancelled from both sides and the higher order term will tend to zero.

Because the higher order term will involve what either delta x 2, 3 like that or delta y 2 3 like that or delta z 2 3 like that. So, when you take this limit that term will tend to zero. So, then you have left with this equation. Now this is a pretty generic equation so far, we have not yet introduced the implication of conduction in this equation. Now what can be the basic modes of this heat flux.

The basic modes of this heat flux may be one of the basic modes of heat transfer. So, we have not committed that it is conduction. Now we will commit that it is conduction. So, we will write this keeping in mind that the mode of heat transfer that we are considering is conduction that will lead to the basic heat conduction equation. So, if we write the heat flux as a function of a parameter which is directly measureable for heat conduction how can we write that.

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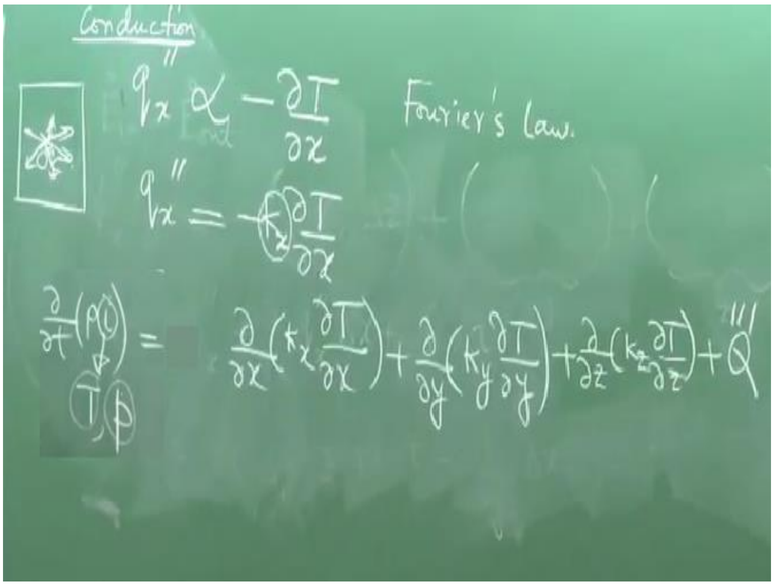

So, let us say that heat flux along x. So, heat flux along x how do you write it in terms of a measureable parameter for conduction, so heat flux along x is proportional to the negative of the temperature radiant along x. This is nothing but Fourier's law of heat conduction. Now there are many situations in engineering when the Fourier's law is applied or is applicable but there are many situations if not many at least quite a few situations when the Fourier's law is not applicable.

So, it is very important to discuss the various considerations based on which we can use the Fourier's law before simplifying the equation further. So, the Fourier's law, see one of the very, this is called as a constitutive law, it somehow relates the cause with an effect. The effect is a heat flux and what is the cause, cause is the temperature gradient because heat always gets transported from higher temperature to lower temperature.

So, the positive heat flux will be along the direction of negative temperature gradient and to make an adjustment for that you have a negative sign here. So, this is the cause effect type of relationship and this is a linear relationship as you can see that this type of relationship is a linear constitutive behaviour.

That means this is linearly related to the temperature gradient. Now this assumes that we are considering a situation when physically if you create a disturbance in temperature at a point that propagates to all possible directions at infinite speed that is the basic physical premise of the Fourier's law of heat conduction. So that means what this is something like this. Say it is a domain.

Let us say everywhere in the domain the temperature was atmospheric temperature. Now suddenly you have make this point zero degree centigrade by bringing in contact with eyes. So, this temperature disturbance will be instantaneously propagated in all directions at infinite speed. So that all points in the domain will immediately know that there is a temperature change I have to respond to the change.

This message, so there is a messenger within the material which propagates the effect of temperature disturbance by virtue of heat conduction and that messenger is nothing but the thermal conductivity of the material. So, in our subsequent discussions we will see that a more effective parameter to describe the efficacy of this message propagation is not thermal conductivity but thermal diffusivity. We will discuss about that in a moment.

Now see if whenever we learn sine formula we also have to know that when it is applicable more importantly when it is not applicable because many times when it is applicable we apply it is fine. Many times we tend to apply it when it is actually not applicable. So, when you say that we are using this Fourier's law there are typical situations when the Fourier's law is not applicable.

Let me again give an example of say a treatment of a biological tissue with a femtosecond laser. See a laser with a pulse of femtoseconds, very short pulse. Now the laser pulse will be coming, it will be on and off in a very short span of time. So, let us say the medical treatment is going on with a laser, the laser is falling on the tissue for a very short period of time it is acting on the tissue then it is going off.

So, what is happening is when the laser is acting on the tissue. The tissue takes on time to adjust to the change in temperature brought in by the laser but before the tissue adjust to itself immediately the laser parts is switched off and by the time it gets readjusted another new pulse has come.

So that means that the time interval over which the change takes place is shorter than the relaxation time of the material. The material takes the time to adjust to the change. Like all of us cannot adjust to a change instantaneously materials also do like that. So, you apply a heat transfer to the material, the material cannot instantaneously adjust its temperature to take care of that. So, it will take a little bit of time.

Now if your time duration over which you impose the disturbance is still shorter than the time then what happens is that the temperature disturbance cannot propagate at an infinite speed but it propagates at a finite speed. It is very similar to the manner in which a disturbance propagates to a medium and that speed you know from basic physics is known as the sonic speed.

So there are various possibilities when the temperature disturbance within the material propagates by a finite speed and then Fourier's law is not applicable. So those types of situations are called as non Fourier heat conduction. It is not within the scope of this particular course discussed about that, but I have just brought that situation out to give you an example that we should not take it as ritual that the Fourier's law is always valid.

But we will proceed with the case assuming that the Fourier's law is valid. Now the constant of proportionality is replaced by an equality where this equality is called as the thermal conductivity of the material. So, we have given a subscript x because thermal conductivity is a property which can vary with deduction. So, if you have anisotropic thermal conductivity then kx, ky, kz these are all different.

But most of the times we are dealing with materials for which kx, ky, kz all are same at some constant k. So, if that be the case then you can write. Now just out of curiosity you might observe a very interesting thing. See fluid mechanics and heat transfers of course these are two different subjects and that is why you are starting these as different courses in your curriculum.

But there is a whole lot of similarity in terms of mathematical approach of fluid mechanics and heat transfer and there are various fundamental foundations behind that. When you are talking about this Fourier's law can you think of an equivalent law in fluid mechanics. The Newton's law of viscosity right, the Newton's law of viscosity is like for our flow which is taking place along x and with velocity gradient along y, you write Tau is equal to Mu du/dy right.

So, it is like a velocity gradient related to the shear stress. Chemical engineers often call shear stress as momentum flux because the shear stress is brought about by a disturbance in momentum like let us say there is a flat plate and fluid is flowing over the flat plate because of the frictional or the physical effect of no sleep at the wall what is happening is the fluid molecules at the wall are having zero velocity if the wall is having zero velocity.

But the fluid molecules which are next to the wall will not directly feel the effect of the wall, but how it will know that there is a wall, there is an invisible messenger within the fluid that transmits the momentum flux or momentum disturbance and this invisible messenger is viscosity. So, it is very much similar to thermal conductivity.

So, this viscous effect is because of momentum diffusion and thermal conduction is because of heat diffusion, both are diffusion phenomena and you also have a similar effect in mass transfers which is called as mass diffusion and that is governed by another law called as fix law. So, we will not come into that because of mass transfer is not within the scope of our present course.

Now let us say you want to solve this equation. Now are you in a position to solve this equation. See you have temperature as one of the variables now here you have internal energy so these two variable are not the same so you must close this system with an expression that relates temperature with internal energy okay.

So how you relate temperature with internal energy, so even from practical consideration you must express it in terms of temperature or pressure whatever something which is measureable right. If you are making experiments then you have devices which can measure temperature, you have devices which can measure pressure. Now there are devices like thermometer but there is nothing called as internal energy meter.

So experimentally to measure something and to relate that measured parameters with what you predict from the theory you need to have this internal energy expressed as a function of measureable parameters. What are the measureable parameters that you require, see it depends on, first of all how many measureable parameters are required to express internal energy as a function of those parameters? How many parameters.

It depends on the physical state of the system and what kind of system it is so in thermodynamics there is something which is called as a simple compressible substance. So, a simple compressible substance is a substance for which pressure, volume, temperature changes are much more important as compared to like electrical effect, magnetic effect other normal thermal effects.

So, we are mostly concerned about simple compressible substances that is number one, number two is we are concerned about something which is a pure substance so a pure substance is a substance which is chemically homogeneous. So simple compressible substance and pure substance, these types of substances if you consider then you require two independent intensive thermodynamic properties to describe the state of the system.

This is known as state postulate in thermodynamics. So basically, you require two independent properties. So, can we write those two independent properties as temperature and pressure, so what we are trying to say is that if we have a simple compressible pure substance we can describe any intensive thermodynamic property in terms of two independent intensive thermodynamic properties.

So, can those two properties be temperature and pressure, yes or no, not always necessary why the requirement is that these two properties have to be independent. Can you describe a situation in thermodynamics when the temperature and pressure are not independent; they are dependent on each other, change of phase. Let us say that liquid water is getting converted into water vapour.

So, the phase change temperature and phase change pressure are dependent on each other. So, when we are claiming that we are writing internal energy as a function of these two, we are precluding the case of phase change. So, we are not bringing the case of phase change in the analysis.

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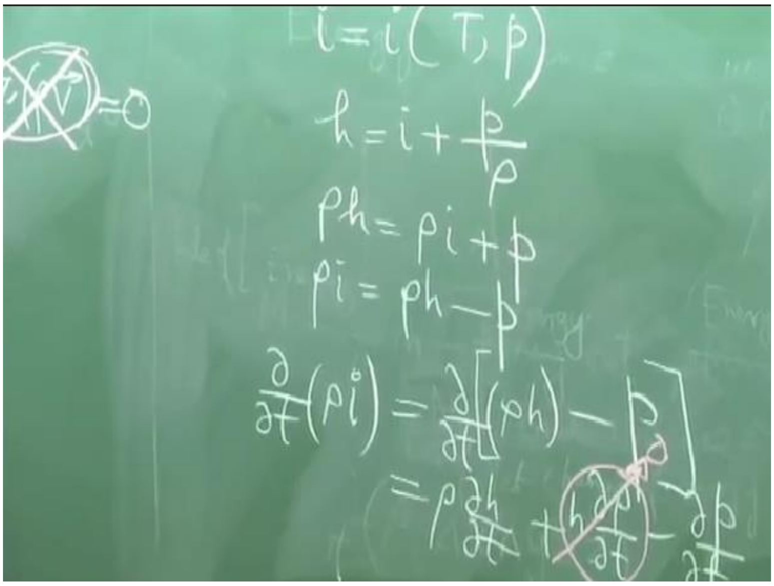

So, the internal energy we are writing as a function of temperature and pressure with an understanding that we are not addressing the problems of phase change. So sometimes, instead of internal energy we express it in terms of enthalpy so the enthalpy is, this is like h is equal to u + pv. The specific volume is one by density instead of u the symbol is i okay? So, Rho h is equal to Rho y + p.

So, Rho i is equal to Rho h - p. so we are basically interested to simplify this term. So, what is this del del t of Rho i that is del del t of Rho h - p all right? Now I will ask you a very simple question, can you tell what is the value of this term, remember we are addressing the case of pure heat conduction without fluid flow.

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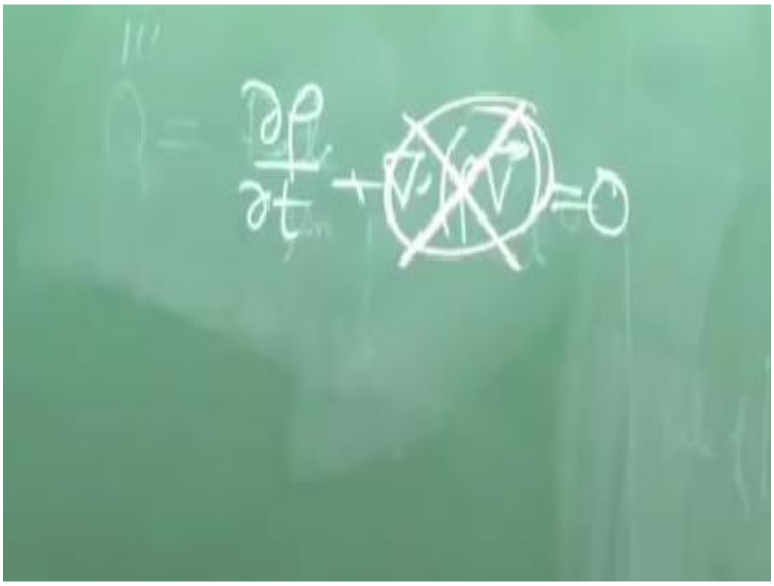

So just think of the continuity equation in fluid mechanics. This is continuity equation in fluid mechanics right. So, we are thinking of conduction that means pure conduction. That means this term is not there. So, this must be equal to zero.

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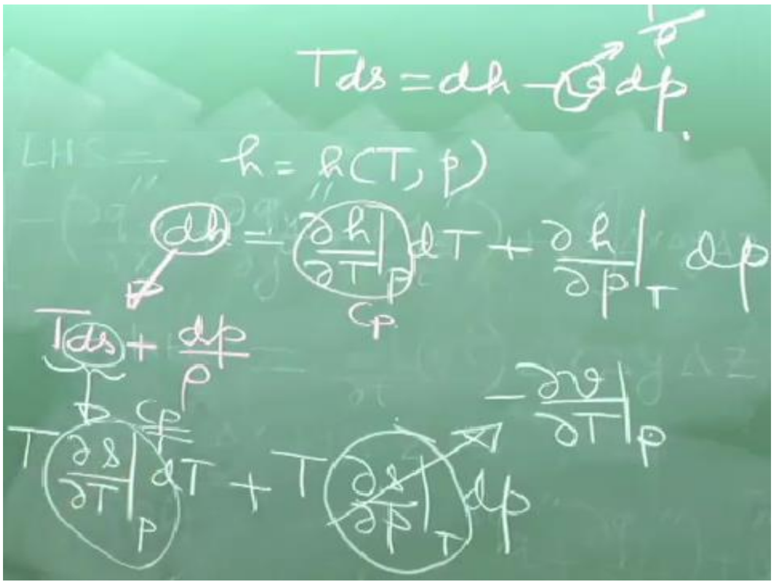

Now we will express this enthalpy as a function of pressure and temperature instead of internal energy we are writing h as a function of temperature and pressure. So, we can write dh, basic rule of partial derivatives. The total change is sum total of the partial change due to change in temperature + partial change due to change in question. By definition you know this is what cp right.

This is the definition of cp. Now we can express these by using one of the Tds relationships. So, in thermodynamics, you have encountered this tds relationship right. Tds equal to dh - vdp. This v is one by the density. So, in place of dh you can write T ds + dp by Rho. Now you can see that right-hand side is expressed in terms of dt, dp, left hand side there is one dp and there is a ds.

So, we can express ds also in terms of dt and dp because we can write entropy as a function of two measureable parameters t and p. So, this will be, now this you can write in terms of other measureable parameters. See this is not a measureable parameter, entropy is not a measureable parameter. So, you can write it in terms of a measureable parameter using this. This is one of the four Maxwell’s relationships in thermodynamics.

Tds using the Tds relationships you can derive this, you must have done this in first year physics or chemistry, then later on in thermodynamics okay. T into - del v del t at constant pressure. Now if you compare this with this, so left hand side this has dt and right-hand side this as dt then what is this, this is cp, this is t into this. So, this is cp by T right. The left-hand side and right-hand side coefficient of dT must be the same right you follow this right.

This is cp, this is T into this, so this also should be cp. So, this must cp by T okay.

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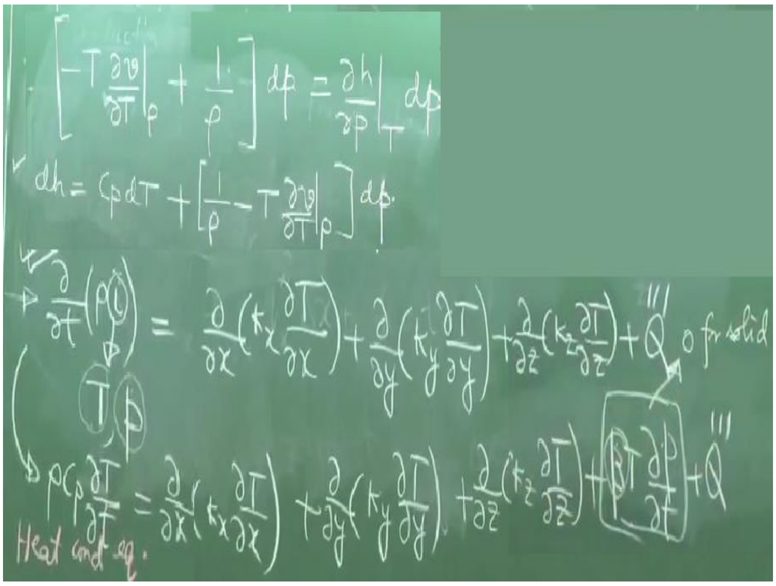

Now we can write, if you now equate the coefficients of dp, so if you equate the coefficients of dp on the left-hand side, the coefficient of dp is here - t del v del t then + one by Rho and the right-hand side, this is the coefficient of dp right. So, you can write dh equal to cp dT, this is the contribution due to pressure. This is the total change in enthalpy, this is the change in enthalpy due to temperature and this is the change due to pressure.

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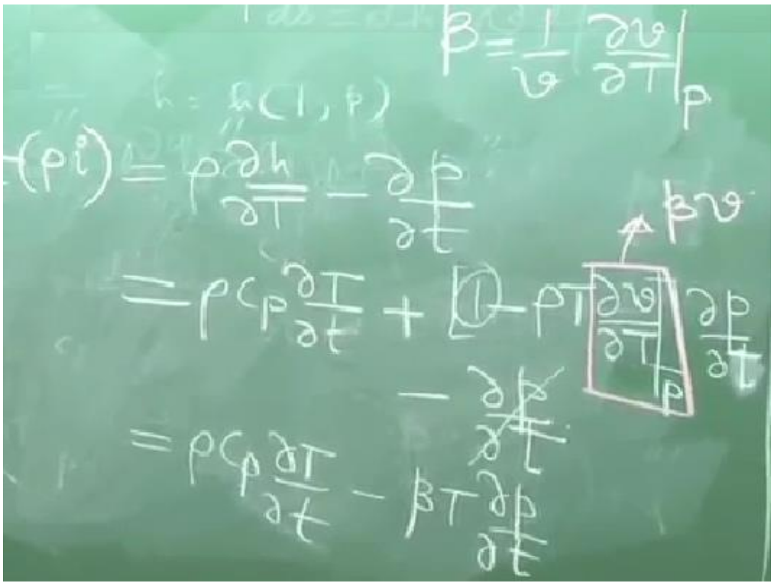

Now look into this expression, so Rho del h del t, we will use this term. So, the first term will be Rho cp. What we have done, we have basically within this as delta h is equal to cp delta T + this into delta p divided both sides by delta t and taken the limit as delta t tends to zero. So that will give rise to this equation okay. This is t time not temperature. Now this first term will get cancelled with this term right. This is del p del t, this is - del p del t.

What is the second term, this, remember, this is del v del t at constant pressure. You have a term which is volumetric expansion coefficient beta. Volumetric expansion coefficient is change in volume per unit volume for each degree change in temperature. That is what is mathematically written in this form. So, you can write this as beta into v where this beta, what is the name of this beta, volumetric expansion coefficient. So, this will be Rho cp okay.

So, this equation now we can write this as. So, what we have done, we have replaced this del del t of Rho i with this expression okay. In place of del t of Rho i we have written Rho cp del t del t - beta t del p del t and that beta t del p del t we have brought in a right-hand side. So, this equation is a general equation for heat conduction. Now typically in undergraduate texts we are interested for heat conduction in a solid and for solids the volumetric expansion coefficient this is very very small.

Not only that you do not have a pressure like quantity varying with time within a solid. So, for all practical purposes for a heat conduction within the solid this term goes away. Typically, when you see derivations of this equation in any book you will find this term is not there but we should not presume that and we should start from a consideration that yes internal energy or enthalpy could themselves vary with both temperature and pressure.

Because solids are not pressure sensitive so eventually it boils down to only temperature dependents and not pressure dependents. So, for heat conduction within a solid you are basically getting this equation without this term and this is known as the heat conduction equation. So, this equation involves temperature as a function of position and time and this is in terms of partial differential equation theory.

This is like a initial boundary value problem where you prescribe the condition at time equal to zero and based on that initial condition the solution will evolve as a function of x, y and z depending on the boundary conditions. So, we need to discuss about what are the possible boundary condition associated with this equation and then we will work out a few very simple problems to illustrate the use of this equation.

I will not show you how to solve this equation because that will be the agenda of the subsequent lectures but I will just show you how to simply use this equation to address heat conduction problems. So, we will take a short break of about five minutes and then we will take another half an hour typically to complete the boundary condition for this and as a tutorial work out one or two or three simple problems to illustrate this concept.