Conduction and Convection Heat Transfer
Prof. S.K. Som
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology- Kharagpur

Lecture-26
Review of Fluid Mechanics- VIII

Next, we will discuss about momentum integral method. When we learn any technique, there should be a purpose and it is important that we first we discuss about the purpose of these, that why we are going to learn these? Before understanding that what actually the method is? See as I told you a few times before in engineering, it is important to evaluate sudden parameters of engineering interest.

Once such important parameter is the wall shear stress for interaction between the fluid and the solid. Because if you want to pump a fluid over the solid, you have to overcome the wall shear stress. So, that is the cost that you have to pay to maintain the flow. Therefore, it is important that we know how to calculate the wall shear stress. Now to calculate the wall shear stress, of course you can calculate wall shear stress by using this method.

But do we have seen that this method is little bit more involved in a sense that it requires the solution of the third order ordinary differential equation which is not having any analytical solution. So, we are trying to look into a little bit more simplistic approach, yet not so in accurate approach of solving the same problem of calculating the wall shear stress. So, how to do that? The name itself suggests, what we are going to do?

We are going to integrate the momentum equation, the momentum equation is nothing but the boundary layer equation here, I mean one of the boundary layer equation is the momentum equation and other equation of course is the continuity equation.

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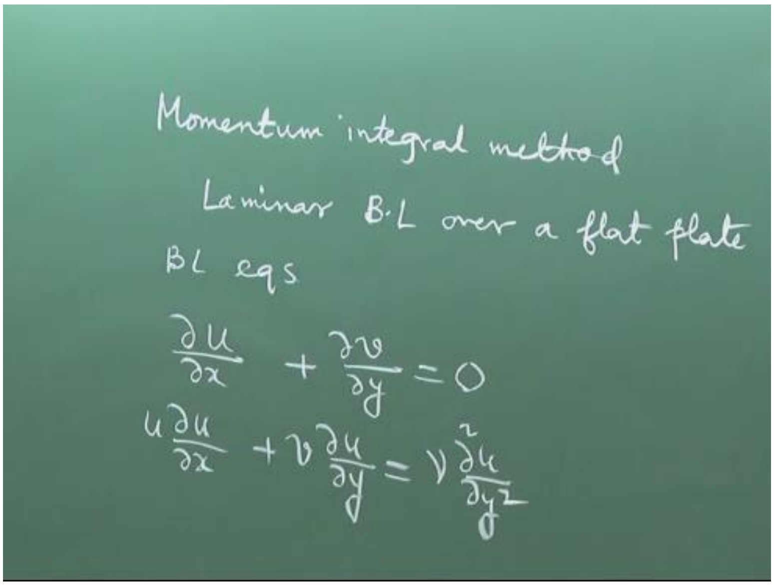

So, we will integrate both the continuity equation and the momentum equation across the boundary layer and that is why it is called as momentum integral method. So, let us write, so we consider laminar boundary layer over a flat plate. So, we will write the boundary layer equations; these are our boundary layer equations we derived in the previous lecture. So, what is the first step is, we will integrate with respect to y from 0 to delta. Okay.

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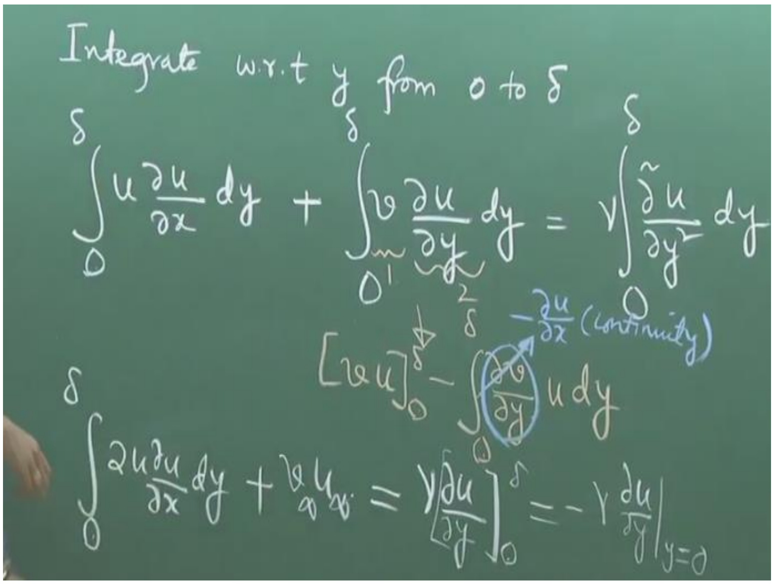

So, that is the first step, then what we do? well we simplify the second term for that we integrate by parts, so this is the first function, this is the second function. The reason is obvious. Because the second function is having a higher order derivative, integral of the second function will be the, what we evaluate during the integration by parts. So, one order derivative will come down.

So, this will be the first function into the integral of the second – integral of derivative of first x integral of the second. Okay. Now we can write that this del v del y is nothing but, - of del v del x, Why? This is because of continuity. Del v del x + del V del y= 0, okay. So, these term is, it becomes + u del u/ del x, then there is another term + u del u/ del x. So, basically you come up with integral 0 to delta, 2 u del u/ del x +; what is this? What is v at y= delta?

Whatever, let us give it at v infinity. We have clearly seen that; it may be small but it is not 0. So, v infinity x u infinity – v at 0 x u at 0. Okay. So, v at 0 is 0, because of no penetration, u at 0=0, because of no sleep, very interestingly the result of these method does not change, if there is a sleep at the wall, the reason is even if there is a sleep at the wall, that is, u at the wall is non-0, still we had the wall is 0, because of no penetration. So, the product will be 0.

Then these =integration of these is, what is del u/del y at y= delta? What is the value of del u/del y at y= delta? 0. Because at y=delta u, becomes uniform, after that you does not change, so the derivative of u with respect to y will be 0. So, it will become – nu del u / del y at y=0. See one of our objectives was what? To calculate the wall shear stress and this is nothing but tau l/ rho. Because tau l is nu del u del y at y=0.

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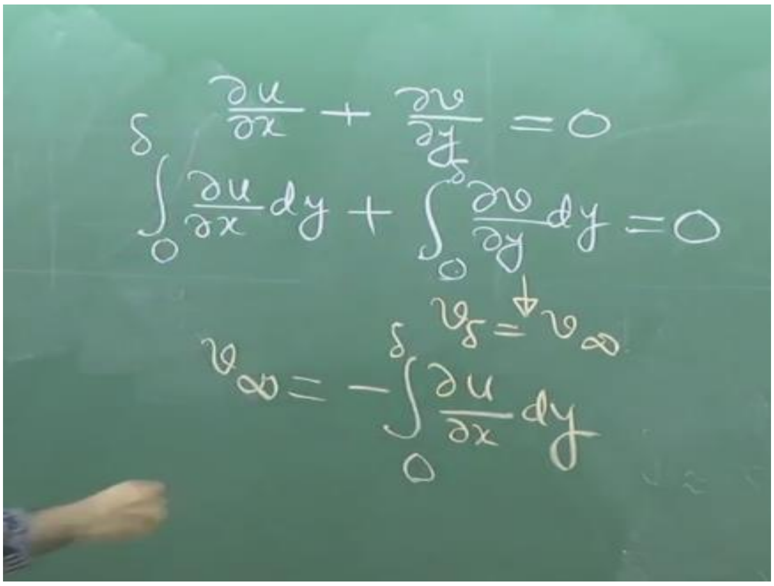

Now there is still unknown in these equation which is v infinity, so we will try to obtain a v infinity by using the continuity equation. We will integrate this equation with respect to y. So, what is this? This is v at delta - v at 0, -v at 0 is 0. So, v at delta is v infinity. So, v infinity= - integral of del u/ del x. So, this v infinity, we can substitute in this equation. So, in this equation, we will substitute the v infinity, which we got from the continuity equation.

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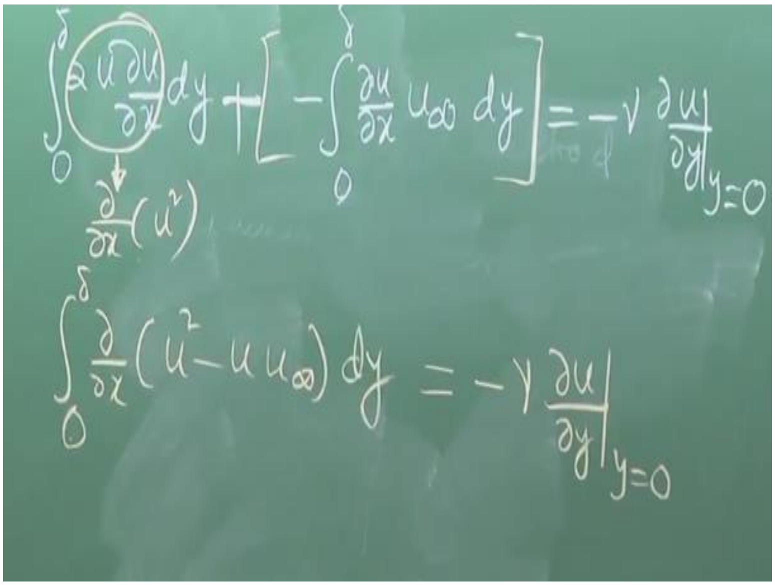

So, that will become integral of 0 to delta, 2u del u/ del x dy+ in place of v infinity, we will write –del u/ del x dy. So, this can be written as del del x of u square, right. So, if you now consider these two terms together, it will be integral of 0 to delta, del del x of u square – u u infinity dy, okay. Now the next question and this is the very important question, can we bring these derivative out of the integral.

That means can we write these as, say del del x of integral 0 to delta u square – u u infinity dy. Yes, or No? See first of all we have to understand that by our wish or whimsies, we cannot take any derivative inside and outside the integral as per our like. There is a rule, which is called as Leibniz rule, which talks about differential under integral sign. Let us look into that rule and try to apply that rule here.

So, let us say that fxy is function of x and y. So, when these are integrated with respect to y, there is no more dependence on y, there is dependence of x only, that is why it has become d/dx.

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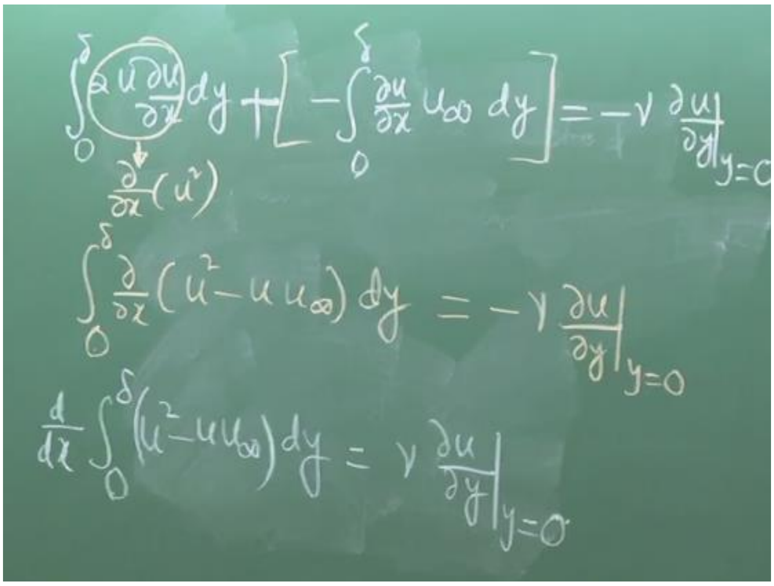

Okay. So, now this derivative, it can be taken inside the integral but with this correction terms. This is the Leibniz rule. See, in engineering or physics or mathematics, there are so many beautiful things that we often overlook and it is one of such very beautiful examples, where the mathematics in some way talks about very interesting physics.

We have discussed about Reynolds transport theorem, the rate of change with respect to system= rate of change within the control volume+ outflow –inflow. So, this is like rate of change with respect to system, this is like rate of change within control volume, this is like outflow- inflow. So, it is like a physical situation being represented by a mathematical equation, which may be derived independent of that physical situation.

Anyway, that is not our objective ideas to made a passing remark concerning this one, what is f in this example? U square- u u infinity, what is ‘a’? 0 and b is delta, which is a function of x. So, let us look into the correction terms, f (x, b) dv/dx, what is f? x, b at b or b means delta, at delta this becomes u infinity square – u infinity square. So, this is 0 and da/dx if x (da/dx), anyway is 0. Because a is 0, so da/dx is 0.

So, you can see that the correction term is 0, right. So, sometimes actually ignorance is the blessing. So, if you do not know the Leibniz rule, you straight away say, I will take the derivative outside the integral. You save a lot of time and this how many times, I mean, students are tutored for competitive exams. I mean, you are trained to use certain formula for solving certain problem.

The formula may be correct but the way of an arriving at the formula may not correct. So, there are many books which immediately after these step, we will write the derivative outside the integral. So, it will give you, create an impression in your mind, as if that is the rule that has to be there always. But this is the rule fortunately the correction term is 0 in this case, okay. So, you will get; see now what should be the strategy of calculating the wall shear stress.

You will be able to know the wall shear stress if you know how u varies with y. because then you do this integration and then differentiate this, right. So, question is we do not know how you varies with y? But what kind of variation is expected? See this is the integral that we are going to calculate, u x u-u infinity, right, dy of course, that I am writing.

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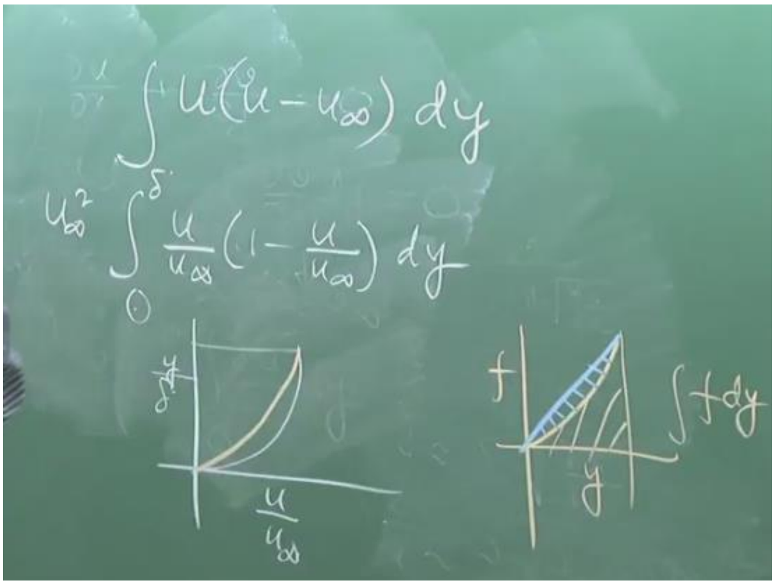

So, if you write these as u infinity square you take outside the integral, then it is u/u infinity x 1-, okay. So, u/u infinity is a function of y/delta, what kind of functions? Something like this. Let us say you do not know these, so instead of these, you make an approximation say some approximation, say it may be linear, it may even be sinusoidal, it may be quadratic, it may be cubic, some kind of function you approximate.

So, clearly this is erroneous, but see the very interesting thing, it is not of importance to us, what is the arid in the function. It is important for us, what is the arid in the integration. Now integration yc, if you have integral of say fdy, say this is f, as a function of y. So, integral f dy is the area under the curve, right. Now instead of these if you approximate it by this curve, you make an arid in area which is these, right.

The first observation is that the relative arid in this area is much less significant as compared to the arid in the function. So, integration in some ways smoothens out the error, the second observation is that, the here the function is multiplied by its compliment, 1- the function. So, if there some positive error in calculating integral of fdy, that is nullified by the negative error in calculating 1- fdy.

So, the net effect is that, even if you make some very poor approximation in u/u infinity as a function of y/delta, the net effect of that in calculating this integral is not that severe. So, with that understanding, you can actually use any approximate velocity profile but with the satisfaction of the proper boundary conditions you have to do that. So, that, you are able to calculate this parameter.

So, the, use of the momentum, this is known as Von Karman's momentum integral equation. Of course when Karman derivate the momentum integral equation, he derives for a general case, not flow over a flat plate. So, there are dv/dx (()) (25:42), is also there, this is the special case of momentum integral equation, where dp/ dx =0 for flow over a flat plate. Now let us take an example. Let us consider a polynomial, say up to cubic polynomial value.

You can also take higher order or lower order polynomials, more number of constants, more will be the constraints that you can put as boundary conditions.

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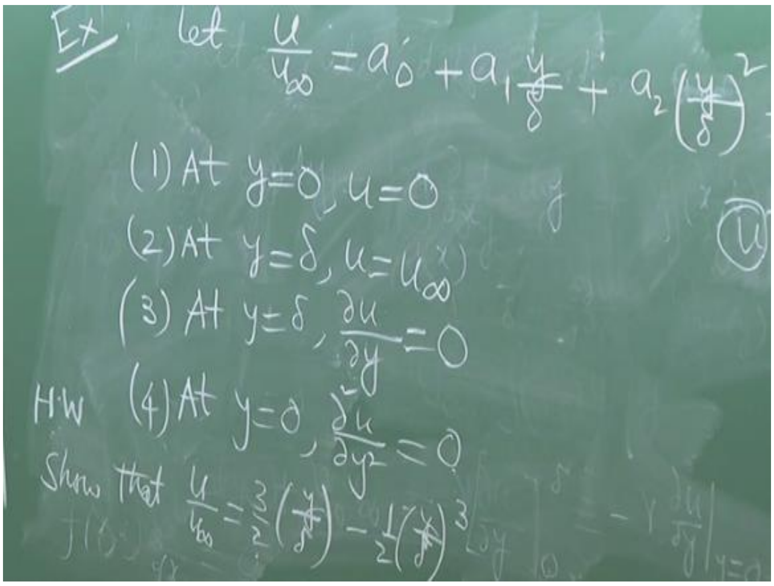

So, now, what are the boundary conditions that need to be satisfied? Most essential boundary condition, say you had taken u/u infinity =0 +a1 y/delta, had you taken that? There would be 2 constants, a0 and a1. So, you could impose 2 boundary conditions. What are the 2 boundary conditions? You must impose, first is at y=0, u=0, no sleep boundary condition and the second is at y=delta, u= u infinity. These are the basic boundary conditions.

So, let us write that. Here you have 4 constants, right. Because you have 4 constants, that means you have to put 2 additional constraints. What are the additional constraints that you can put here? If you want to give a priority that is, let us say, you have up to the quadratic term, then what is the next, these are the 2 high priority boundary conditions, these 2 must be satisfied. The third one, see what will happen?

At the edge of the boundary layer, the velocity does not change any further, so that means, that y = delta del u/del y=0 and then what is the fourth boundary condition? Because these 3 boundary conditions more or less come directly from physical intuition but the fourth one is not something which will come directly from physical intuition but you have to give a little bit more thought into that.

So, let us write the boundary layer equation somewhere, u del u/del x+ v del u/del y= nu del 2 u/ del y 2. So, now with this hint you tell what could be the boundary condition? If you apply this equation at y=0, at y=0, this =0, this is =0, therefore this must be =0, right. So, at y=0, so now it becomes a well-defined problem with 4 constants need to be evaluated from 4 constraints. “Professor - student conversation starts” I will not go through the algebra; I will leave it on you as homework. “Professor - student conversation ends”

So, you show that u/u infinity=3/2 y/delta-1/2 y/delta) cube. This is if you take this example, if you take other velocity profiles of course there will be different coefficients. So, with this velocity profile, our next objective will be to calculate delta as a function of x, the wall shear stress as a function of x and I will illustrate this velocity profile with other velocity profiles, like linear velocity profile, quadratic velocity profile, sinusoidal velocity profile.

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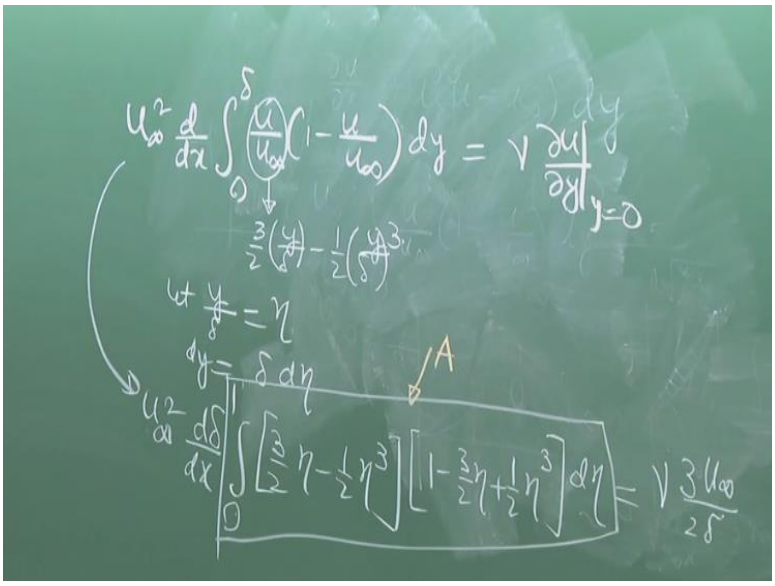

The answers are given in any textbook on fluid mechanics and you can check your answers with those books. So, we will continue with this and use that in the momentum integral equation. So, u infinity square d/dx of integral 0 to 2 delta u/u v infinity x 1 -, there is a – here, right; so, I have, what I have done is I have (()) (33:42), to these term. These term are I have made the first term and these term second term and divided by u infinity.

So, because I have swab the term, these – have become +. So, now you write u/u infinity is (3/2 y/delta-1/2 y/ delta) cube. Okay, that velocity profile. So, y/ delta, let us call y/delta= Eta. So, dy = eta d delta, sorry, with respect to y, delta d eta, right. Because when you are differentiating y at a given x, this is the variable that is changing, eta is changing. So, this becomes u infinity square d/dx of integral.

If you change from delta to eta, this is 0 and this is eta = what? u/u infinity is 3/2eta -1/2 eta cube 1-u/u infinity 1-3/2 eta +1/2 eta cube, dy is delta d eta, so that delta you can bring here because delta does not depend on eta. What is del u del y at y=0? At y=0, del u/del y is 3/2, 3 u infinity/ 2 delta from the velocity profile. So, I will not waste any time in evaluating this integral.

This is- this may be very tedious but there is no special trick which you can play, you have to just simply multiply and integrate these polynomials.

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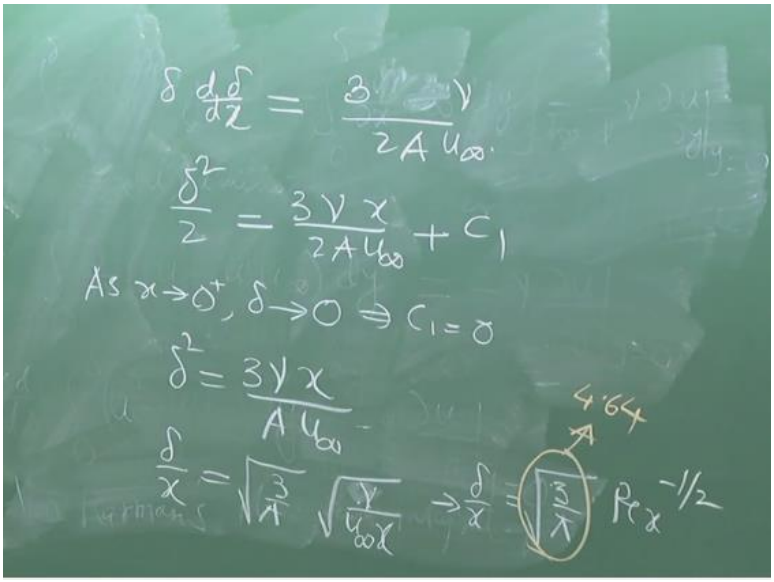

Let us say that this is = some number a, okay, so u infinity square, delta, d delta/dx =3u infinity nu /2a. So, 1 u infinity will, let us write u infinity here in the denominator, so delta square/2 =3 nu x/2a u infinity+ some constant c1, okay. Now as x tends to 0, delta tends to 0, as I told you earlier at x=0 delta is not defined. So, it is x tends to 0+, that will mean c1=0. So, delta square=3 nu x/a nu infinity, that means delta/x. Alright.

This if you evaluate, this will come to be 4.64. These are the classical results in fluid mechanics. Okay. See, what is the difference in the result from the similarity solutions, similarity solution is a correct solution, there it was 5. Now instead of 5, it is 4.64, it may be appeared to be quiet in accurate but our ultimate objective is not to calculate delta but to calculate the wall shear stress.

So, let us see that how does it (()) (40:52) in calculating the wall shear stress. So, what is toe wall? Toe wall is this, right.

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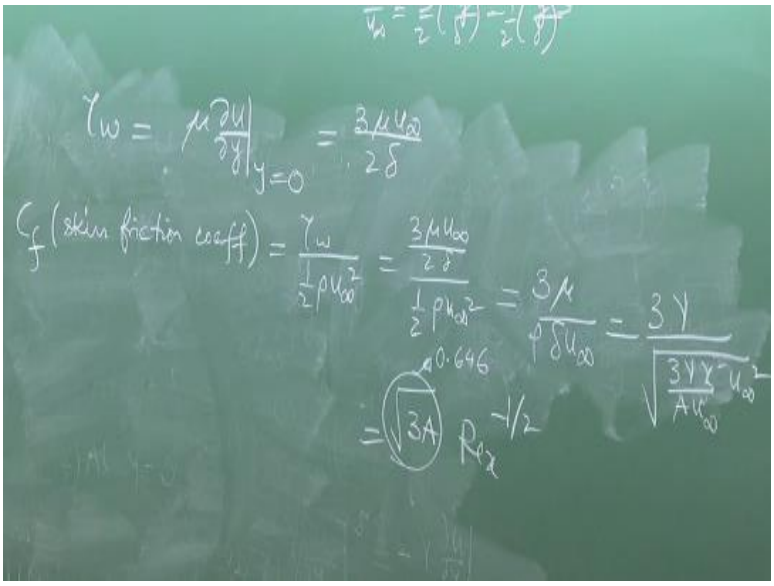

So, that is remembered u/ u infinity is (3/2 y/delta-1/2 y/delta) cube. So, toe wall is 3 mu u infinity/2 delta. Normally, the wall shear stress is expressed in a non-dimensional form by using these non-dimensional parameters cf, which is called as skin friction coefficient. This is toe wall/ 1/2 rho u infinity square and delta is, what is delta? Delta is square root of 3 nu x/au infinity.

So, this becomes, if there is any algebraic mistake, please let me know but I am trying to do it generically. Because this integration will different for different choices of velocity profiles, so given that integration, how you proceed with calculation of the remaining thing, so that you know the procedure and you can apply it to any velocity profile that eventually you have to consider.

So, this will be typically be 0.646 and the drag force, how do you calculate the drag force on the plate, which is the ultimate engineering importance. How do you calculate drag force on the plate?

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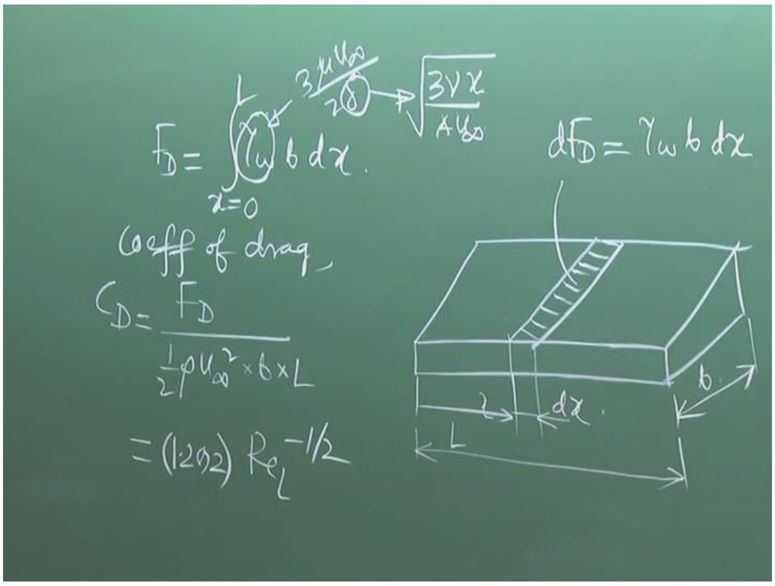

So, let us draw the plate like this, so at a distance x, you take a stripe of width dx. What is the force on this? Let us say that the width of the plate is b, and the length of the plate is l, so what is the elemental drag force on this? First why do we have to choose an element? You have to choose an element because the wall shear stress continuously varies with x, so you have to take a small element or which the wall shear stress is like a constant.

So, what is the wall shear stress here? Let us say toe wall, so toe wall x b x dx. Okay. So, what is the total drag force? Now what is the drag coefficient? Coefficient of drag, cd, this is the non-dimensional drag force. So, this is FD/ 1/2 rho u infinity square x b x l. So, now you can substitute the values, you can write toe wall as a function of x, toe wall is what? toe wall is 3 mu u infinity/2 delta.

And these delta is a function of x, delta is square root of 3 mu x/au infinity. Okay. So, it is finally a polynomial function of x, that we integrate with respect to x and do it. If you do this, then you will come that these = coefficient x Reynolds2 to the power of -1/2, their coefficient is typically double of that, that is 0.646. If you do this problem by or if you work out these problem by the similarity solution, this will be like typically 1.33, 1.332 like that.

So, see the error is really very very small, instead of 1.29, it is 1.3 or 1.33 and this is the main dependence, Reynolds2 to the power of -1/2. So, it shows one very interesting thing that although there is error in the choice of the velocity profile, but that error is somehow smoothened out when you calculate the wall shear stress, non-dimensionally or the drag force non-dimensionally in terms of drag coefficient, then the error is not that much.

So, you can use these velocity profiles, whatever approximate velocity profiles to first calculate delta is a function of x. See, delta is a function of x is the main thing. Because once you know delta is a function of x, that wall shear stress or the drag force all are functions of delta. So, if delta is the function of x, you can evaluate the local friction coefficient or the overall drag coefficient by integrating the delta as a function of x.

So, knowing delta is a function of x is important and we can use the momentum integral method to evaluate the delta as a function of x in a somewhat accurate manner. So, to summarise, what we learned so far? We have learned the hydrodynamic boundary layer. What is the hydrodynamic boundary layer? What is the boundary layer theory? What are the boundary layer equations? And 2 techniques for addressing the boundary layer equation.

One is the similarity solution technique; another is the momentum integral method. Now we have to keep in mind that at the end our objective is not just to study fluid mechanics in the course of heat transfer. We want to utilise these understanding of fluid mechanics to solve problems in heat transfer and for example, when we studied flow over a flat plate, we want to next study heat transfer at a heated flat plate or a cool plate. How do address that?

We have to utilise or we have to derive equations which, just like Navier-Stokes equation for fluid mechanics will be useful for solving the temperature distribution in heat transfer and that is nothing but the energy equation. So, in our next class, we will derive the energy equation for convection heat transfer.