Conduction and Convection Heat Transfer
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Lecture – 22
Review of Fluid Mechanics – III

In the previous lecture, we were discussing about the constitutive behavior of a fluid and we tried to generalize the concept. But at the same time, we tried to invoke a special consideration of linear relationship between the stress tensor and rate of deformation tensor and that kind of linear relationship is valid for special types of fluids know as Newtonian fluids. So, we will start with that.

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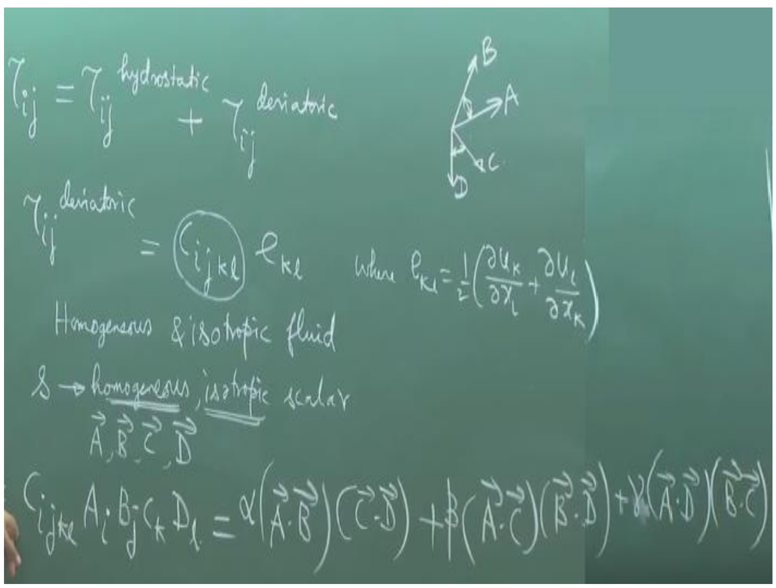

So, we had tau i j equal to tau i j hydrostatic plus. And tau i j deviatoric, we express as C i j k l which is a (()) (1:24) tensor times e k l where e k l is half delta u k delta x l plus delta u l delta k. Now, as we discussed that if we want to describe the behavior of a fluid in this way it will require three into, three into, three into, three into total 81 constant to specify the property of the liquid or fluid. But in general, we do not require that much number of properties.

One possible way by which we reduce the number of propertied required is by utilizing the symmetry of the stress tensor that is tau i j is equal to tau j i. In addition, we also make use of homogeneous and isotropic fluid. So, homogeneous and isotropic fluid. What is a homogeneous fluid? A homogeneous fluid is such a fluid whose properties are especially uniform, right. So, homogeneous means it does not depend on position.

So that means, if you have a property of the fluid that is same at all locations and isotropic means it is independent of direction. That means if you have a co-ordinate system and described by some co-ordinate axis. Let us say you rotate the co-ordinate system. For example, rotation, I am just talking about one type of transformation. It could also be translation, reflection these type of transformations.

So, if you rotate the co-ordinate axis the property description with respect to the rotated co-ordinate axis will also be the same because it is not dependent on direction. So, if that be the case then the number of properties can be reduced further. So, let us try to form a scalar s which is homogeneous and isotropic. Let us say, that our objective is to make a scalar out this fourth order tensor.

And what we have in our hand to convert the tensor to scalar is some vectors. So, how many vectors you require to covert this into a scalar that is the first question. How many vectors you require to convert this fourth order tensor into a scalar? You require four vectors. Why? Because let us say you have the vectors A, B, C, D. Now a vector is specified by one index. So, if you now write C i j k l into Ai Bj Ck Dl. This stands for vector A. This stand for the vector B.

This stand for the vector C and this stand for the vector D. Now, if you write in this way it means that there is an invisible summation over i because it is repeated. Invisible summation over j from 1, 2, 3 summation k and summation over l. Finally, it will have no index so it will be scalar. So, this is a scalar but we want it to be not just any other scalar but homogeneous and isotropic scalar.

So, first of all we will look into isotropic. So, isotropic scalar means what is important. Let us say you have vectors say A, B, C, D. When you have isotropic behavior that means if you now rotate this vector system by some angle then there will be no change in the property description that means that the property will depend on that angle between the respective vector which is preserved during rotation but not the absolute orientation of the vectors A, B, C, D.

But their relative angular position because rotation preserves the relative angular position. Whatever was the angle between A and B before rotation the same will be the angle between A and B after rotation. So that means that this will depend on only the angle between A and B. So, angle between A and B is given by the dot product of A and B. So, you can write this as a combination of –if you pair A with B then you will pair C with D.

How many such combinations are possible? You can pair A with C and then B will pair with D you can pair A with D and then you can pair B with C. I have not written the full expression I have just written what are the terms which will be governing the behavior. So, these are the only combinations possible see A dot B, and B dot A are the same. So, it's not six but three combinations which are possible.

Now, it will depend on this combination so we can say that because of the linear behavior it is proportional to this times a coefficient alpha which you have to determine. This times beta and this times gamma. Where alpha, beta and gamma are some coefficients which because of homogeneous nature do not change with position. They are not position dependent. Now, we will be write this expression in index notion.

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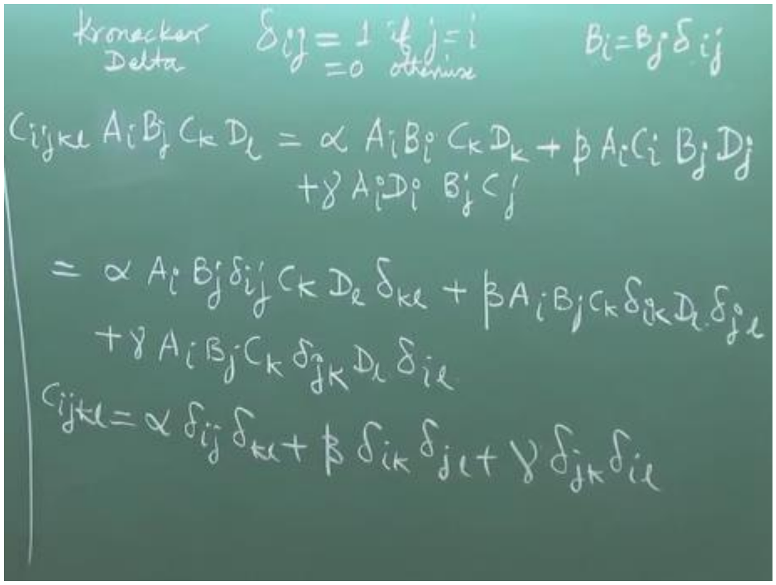

So, C i j k l is equal to, oh sorry, C i j k l into Ai Bj Ck Dl equal to alpha Ai Bi, A dot B you can write Ai into Bi, right then Ck Dk last beta Ai Ci into Bj Dj plus gamma Ai Di into Bj Cj. Now, I will show you a very interesting way of switching the indices. This I j k l you can beautifully switch. You can switch from i to j, j to k, k to l you can play with the indices. How you do that you use an operator delta i j which is called as chronical delta.

So, what it does delta i j is equal to one if j is equal to i and is equal to zero otherwise. This is called as Kronecker Delta. So, you can write Bi is equal to Bj into delta i j. Why? Because delta i j is equal to one if j is equal to i when j becomes i dj will become Bi and this will become one. So, you can see this is how you switch index from one to the other. So, using this, we will write alpha Ai Bj delta i j Ck Dl delta kl.

We are trying to write for all terms Ai Bj Ck Dl because that is what is there is in the left-hand side. So that we will get a expression for C i j k l. Then plus beta Ai Bj Ck delta what will be this? i k then Dl delta j l plus gamma Ai Dj Ck delta what will this? jk. Then Dl delta il. So, from here what we can write? We can write C i j k a l equal to alpha delta i j delta k l plus beta delta i k, delta j l plus gamma delta j k delta i l. Now, how many properties have come?

Magically from 81 properties, you are having three properties alpha, beta and gamma. Now, we can reduce the number of properties even further by noting that Tau i j deviatory is equal to Tau j i deviatory.

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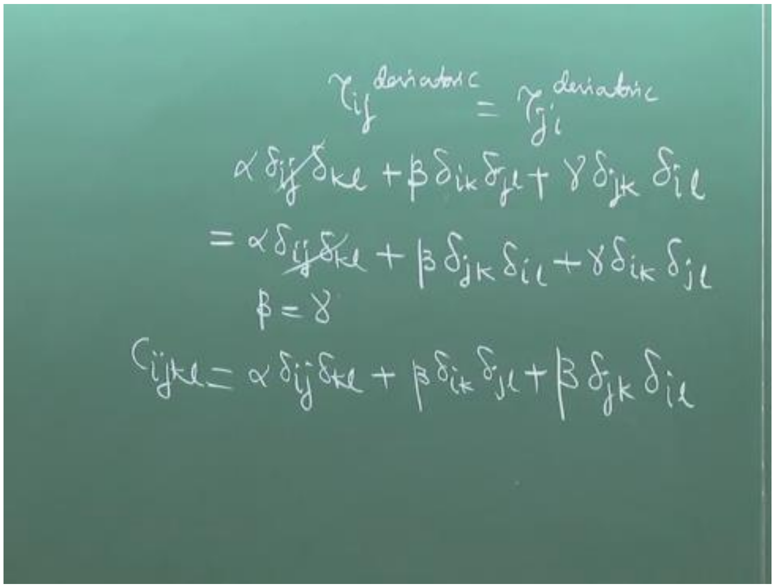

Because the stress tensor is symmetric the individual hydrostatic and deviatoric component are also symmetric. So, we will swap i and j. So, first we will write C i j k l so alfa delta i j delta k l plus beta delta i k delta j l plus gamma delta j k delta i l is equal to now we swap i and j. So, if we swap i j this will become alpha delta but delta ij and delta j I are the same. Delta is a symmetric tensor.

So, you can write delta i j again delta k l plus theta delta j k delta i l plus gamma delta i k delta j l from here what we can conclude. We can conclude, that beta is equal to gamma, right. You can see that it will be beta minus gamma into delta i k delta j l minus delta jk delta il equal to zero. So that means you have beta equal to gamma. So, you can write C i j k l equal to alpha delta i j delta k l plus beta delta i k delta j l plus gamma will be equal to beta delta j k delta i l.

So now how many properties are there? Two properties alpha and beta. We will try to get a physical interpretation of these two properties.

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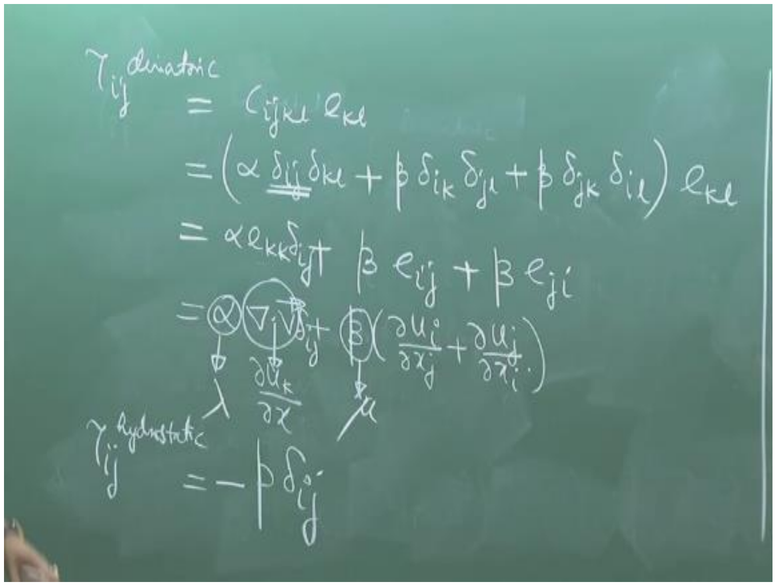

So, tau i j deviatoric is equal to C i j k l into E k l. So, what is that? Now you tell delta i j these are i n j are free indices because it is there in both left-hand side and right-hand side. So, you really do not change i n j because you already have i j in the left-hand side so right-hand side i j is the same as whatever is there in the left-hand side that you cannot change until or unless you change the left-hand side indices. Only indices that you play with are k l.

So, delta k l is equal to zero x except when l equal to k that is the definition of delta. So, when l equal to k this will be equal to 1 so E k l will become E k k. So, alpha E k k plus beta delta i k will become one when k is equal to i. So, when k equal to i this will become i and delta j l will become one when l equal to j. So, one into one into E i j plus beta, delta j k equal to one when k equal to j and delta i l equal to one when l equal to i so it will become E j i.

So now what is E i j this we have defined as half delta u i delta x j plus delta j delta x I. So, what is E kk? E kk is E11 plus E22 plus E33. E11 is delta u 1 delta x 1. E 22 is delta 2 delta x 2 and E 33 is delta u 3 delta x 3. So, delta u 1 and delta x 1 plus delta u 2 delta x 2 plus delta u 3 delta x 3. This in short form we can write delta u k delta x k, right. But what is physically this? This is the divergent of the velocity vector.

So, this as we have discussed earlier relates to volumetric deformation of the fluid. This relates to volumetric strain the rate of change of volume per unit volume. So, alpha is related to the volumetric deformation plus beta. What is E i j and what E j i? See E i j and E i j are the same so E i j plus E i j is basically delta u i delta x j plus delta u j delta x i. j. So just to write the same notation we write here delta u k delta x k.

Now physically what is this? Physically this relates to the relationship between the deviatoric stress and rate of volumetric deformation. So, this is known as volumetric dilation coefficient in most of the books this is given symbol lambda that is why I am just changing from alpha to lambda it does not matter. But since most of the books, used lambda as the symbol that I why I am just changing it from alpha to lambda.

This is called as volumetric dilation coefficient because it is very clear that it relates the stress with the volumetric deformation. Now what is this? Can you recall from kinematics of flow that what is this? This is rate of deformation if i is not equal to j that is sheer deformation and if i equal to j that is linear deformation. But it is in general rate of deformation. So, you have the stress proportional to the rate of deformation.

What is this proportionality constant called? Viscosity of the fluid so this beta physically is viscosity. So, this is tau i j deviatoric. What is tau i j hydrostatic? Hydrostatic stress is due to the pressure acting on the fluid element because the hydrostatic stress is the only component of stress when the fluid is at rest. So, if the fluid is at rest the components of stress that is acting on it is the normal component of stress and that the normal stress is minus p.

Why it is minus p? Because by nature pressure is compressive whereas tensile stress is considered to be positive stress. So, pressure is by nature a compressive stress where by sign convention we take tensile stress as a positive normal stress. So, we write this as minus p into delta i j. Why delta i j? Because this is normal stress this is one only if j is equal to i. J is equal to I means it’s a normal component. When the two indices are the same it is a normal stress.

When j is not equal to i this will be zero. So, you can combine tau i j equal to tau i j hydrostatic plus tau i j deviatoric. One term we have missed I forgot to write this delta i j please correct this. I forgot to write this delta i j here. So, this will be this time delta i j. There was a delta i j here. I just forgot to write it here.

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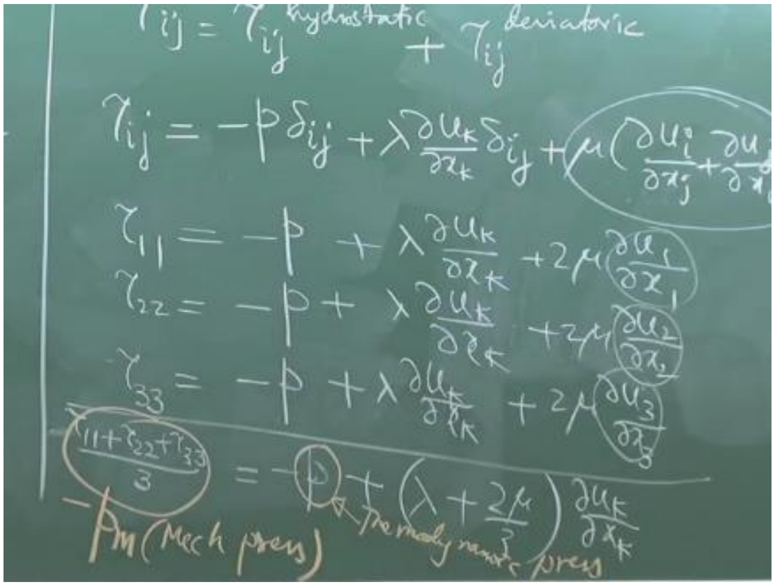

So, minus p delta i j plus lambda delta u k delta x k delta i j plus mu. This is nothing but the Newton's law of viscosity for homogeneous, isotropic and Newtonian fluid. So, at every level whenever you learn certain things you have to understand that like things are possibly much more generalized then what is introduced for the first time. Like for example if you are studying in high school and if you are taught what is a vector? What you will be taught?

Vector is something which has magnitude and direction. Then you go to a little bit higher class somebody will add a vector will have magnitude direction and sense. Then if you go to a little bit higher class somebody will add a vector not only have magnitude, direction and sense. It obeys the commutative law of vector addition either the triangle law of the parallelogram law.

If you go to a class of abstract mathematic in algebra, they will discuss about n-dimensional vector space in a vector in a vector space which does not relate to any magnitude direction or anything. So, the question is which one is the most general one obviously what is taught in linear algebra the n-dimensional vector space is the most general vector space that you talk about. So, always we start with the simplest one.

So, when you first studied Newton law of viscosity, you studied tau equal to mu du dy. Just like for everything there is a version for children, Ramayana for children, Mahabharata for children. So, that is Newton law of viscosity for children. So, when Newton law of viscosity was first introduced that is how it is perceived. Now that can be thought of as a special case of this law. So, what it is basically talking about, it is basically talking about this part.

So, think about a flow when you have say in a two-dimensional fluid is like delta u/ delta y plus delta theta/ delta x. The common symbol that we use this So, if the flow is unidirectional then you have only u components of velocity and then this is not there. So, then that component of tau is mu delta u/ delta y that is what is Newton's law of viscosity that you have learned in your earlier I mean first exposer to fluid mechanics.

But do not think that is the most general form of Newton's law of viscosity. So, for homogeneous isotropic Newton fluids this is what is the Newton's law of viscosity. So, never forget to include this term if you have a two dimensional or three-dimensional flow field. Now we will discuss about the normal components about the stress. So, first tau one so i equal to one and j equal to one so minus p delta one, one is one plus lambda delta u delta k plus two mu.

See about viscosity we have many misconceptions and one very common misconception is that viscosity is always related to sheer deformation. It is not true, viscosity may also we related to linear deformation. You see here this term. What is this? This is linear deformation, right this has nothing to do with sheer but it also has a coefficient which is viscosity. So, normal component of stress may also be related to viscosity.

It is not related to viscosity only when the fluid is at rest. But if the fluid is under motion then the normal component of stress is also related to viscosity. So, it is not true that viscosity is just related to sheer stress. Very loosely we say, that viscosity is related to sheer stress, yes it is related to sheer stress but it may also be related to the normal components of stress. Then tau 22, minus p plus lambda delta u delta k plus two mu delta u 2 delta x.

Tau 33 minus p plus lambda delta u k delta x k plus two mu delta u3 delta x3. So, now if we want to find out what is tau 11 plus tau 22 plus tau 33 divided by 3. The average of the normal component of stress that is equal to minus p this three divided by three plus lambda, three lambda divided by lambda plus two mu into delta u one delta x one plus delta u 2 plus delta u 2 delta x 2 plus delta u 3 delta x 3 that is delta u k delta x k that divided by three.

So, this three added together will be delta u k delta x k. By definition the left-hand side is called as minus of pm where pm is mechanical pressure. So, this is the definition of mechanical pressure, mechanical pressure is by definition the arithmetic average of the negative of the normal components of stress. This p on the other hand is called as thermodynamic pressure. So, what is thermodynamic pressure?

Thermodynamic pressure is the pressure that relates the thermodynamic other parameters through the equation of state. For example, for an ideal gas p is equal to rho rt that means thermodynamic pressure. So, which relates to other thermodynamic properties through the equation of state. So, mechanical pressure, which relates directly to mechanical forces in general may not be thermodynamic pressure.

Now, still there are many fluids for which our general relationship between mechanical pressure and thermodynamic pressure need to be described. And that was described by something called as Stokes hypothesis.

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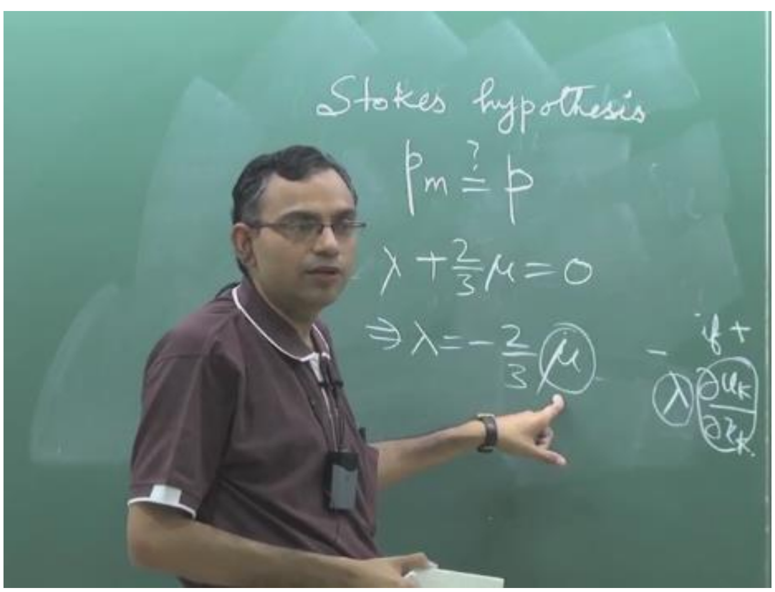

So, just like any hypothesis this does not have a proof but this is found to be satisfied for most of the practical fluids. What is that? The Stokes hypothesis is that the mechanical pressure is equal to thermodynamic pressure. Now, the question is that is it true always that the mechanical pressure is equal to thermodynamic pressure? To, understand that we have to appreciate the molecular origin of mechanical pressure and thermodynamic pressure.

Mechanical pressure talks about the translational degrees of freedom whereas thermodynamic pressure includes all possible degree of freedom. Translational, rotational, vibrational degrees of freedom of the molecule. So, when we say that we change a thermodynamic state what happens? The translational, rotational, vibrational all modes change and they eventually get translated in the form of mechanical pressure.

Now, that takes sometime so that the system gets adjusted to the new change. There are certain types of fluids for which of course you can say that mechanical pressure and thermodynamic pressure is identically the same like dilute monatomic gases.

They do not have rotational or vibrational other degrees of freedom. So, they have only translational degrees of freedom. Therefor for dilute monatomic gases Stokes hypothesis is a law it is no more a hypothesis because it can be proved. That for those, for which there is only translational degree of freedom, monatomic, dilute monatomic gases. But for other substances in general this in not equal. Why?

Because every system when you create a change a system talks a time to adapt to the change, right. It is true not just for materials or fluids but for also us as human beings. So, if somebody wants to change us and we want to change ourselves we cannot do it immediately. Like even if you see in the morning you want to wake up somebody give you a wakeup call or you get the alarming bell still you cannot wake up instantaneously, right.

It takes a time for you to adjust to this disturbance and respond to these changes. This is known as relaxation time of the material. Now typically, the relaxation time is very fast so that the material almost instantaneously responds to the change. But what happens if the material itself is changing at a very high frequency, changing its state at a very high frequency? Let us say that there is a bubble which is expanding and contracting, expanding and contracting very fast at a very high frequency.

Then the bubble when it changes its state from one state to the other before that a new change of state has been imposed. That means the bubbles cannot adjust itself to the change before a new change has come in. So, if you have a bubble which is changing its state very rapidly then if the time scale of the change is faster than it relaxes the time scale. It cannot relax to a new state. So, in that case you do not have mechanical pressure equal to thermodynamic pressure.

So, Stokes hypothesis need not always be valid. But how often, we discuss about such cases of very rapid change possibly very rarely. So, for almost all practical engineering purposes this mechanical pressure is equal to thermodynamic pressure that is why the Stokes hypothesis is such a popular hypothesis. Now for the Stokes hypothesis to be valid, if you have pm equal to p that means, this term must be zero.

Because this is the volumetric deformation this is in general not equal to zero therefor you must have lambda plus two third mu equal to zero. That means lambda is equal to minus two third mu. We know that the viscosity of a fluid is positive therefor lambda is negative. So, what does it mean? What is the physical interpretation of negative lambda? It means that if there is fluid, you see, the stress is what?

The stress due to the component which is there associated with lambda is lambda delta u k/ delta x k. So, if this is positive that means what? If this is positive that means the volume is expanding. Then lambda negative means that the product is negative that means if there is a fluid element which is already expanding then the incremental amount of stress to stretch it further is actually negative.

That means if it is already expanding you do not require a positive incremental stretch to stretch it further. Because then this component of tau i j is coming to be negative. The other important discussion is that there is a special type of fluid for which you do not care whether the Stokes hypothesis is valid or not that is a case when the flow is incompressible. If the flow is incompressible then because of this equal to zero.

You will have mechanical pressure identical equal to thermodynamic pressure. So, then it does not matter it is an identity you do not care about the discussion on the validity of the hypothesis it become an exact reality. So, on one side you have dilute monatomic gases. On another side, you have incompressible flows for which mechanical pressure is identical to thermodynamic pressure you do not have to put any hypothesis to measure its exactness.

So, we have been successful by the Stokes hypothesis to describe the property of the fluid by just one property which is the viscosity. We had two properties lambda and mu but we have shown that lambda can eventually be written as a function of mu. So, only property of the fluid viscosity is good enough to describe the state of stress of the fluid. So, with this understanding we will now get back to the Navier equation.

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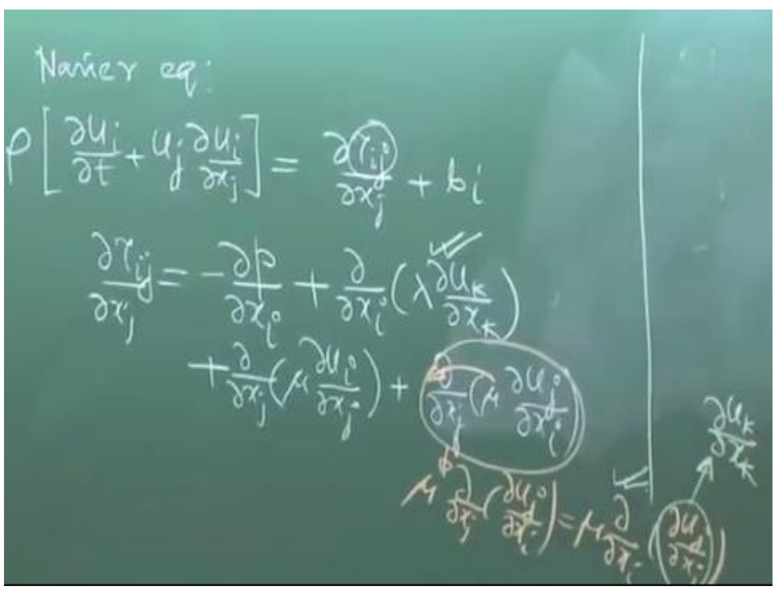

So, this we derived in the previous lecture. This was the Navier equation and we discussed that this is a very general equation only issue is that there should be no body couple but the problem with handling this equation is that this has too many numbers of unknowns as compared to the numbers of available equations. So, now we will use the Newton's law of viscosity in place of this tau i j. So, what is this tau i j?

Tau i j is equal to this one with lambda is equal to minus two third of mu. So, we will substitute that here so we will write delta y j/delta x j. What is delta y j/ delta x j? Go to the first term, partial derivative of this with respect of x j see delta i j equal to one when j is equal to i so x j will become x i so it will become minus delta p/ delta i. Next term plus delta/ delta x i of similarly lambda delta u k/ delta k plus delta/ delta x j of mu delta i/ delta x j plus delta/ delta x j of mu delta/ delta x j.

Now, we will make a little bit of simplification to this term. What is the simplification first simplification is, it is a homogenous fluid we have considered so mu is not a function of position? If mu is not a function of position this will come out of the derivative. So, you can write this as mu delta/ delta x j of delta u j/ delta x i. If the second order partial derivative is continuous then this is as good as mu delta/ delta x i of delta u j/ delta x j.

We swap this x i and x j because for continuous, partial derivative it does not matter with respect to which you differentiate cost. And then this can be written as delta u k/ delta x k. This is a dummy index, this j. So instead of j you can write whatever k l it does not make any difference. So, what we are writing it k here you can see own that this term can be clubbed up with this term. Because here also lambda, it is not a function of position.

So, you take lambda out of this, so delta/ delta x i of delta u k/ delta x k plus delta/ delta x i of delta u k/ delta x k. Here the coefficient is lambda and here the coefficient is mu. So, with this we can write the next step.

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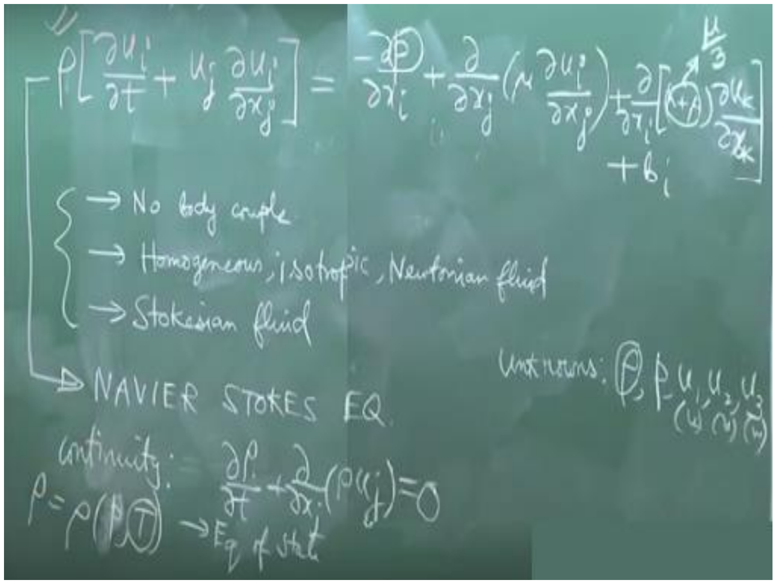

So, rho delta u i/ delta t plus q j delta u i/ delta x j and because lambda and mu are not position dependent we can take them out and inside the derivative without any problem. So, if you now club up these two terms together it will be. So, this lambda plus mu, Lambda equal to minus two third mu so this will become mu/3. Clearly, the last term is important only for compressible flow. Because for incompressible flow this will be zero.

So, this equation what are the assumption associated with equation? So, first we use the Navier equation there the only assumption was no body couple then so no, body couple of course you also have to add up body force that is there so let us write that no body couple. Then the constructive behavior for that we assume a homogeneous isotropic and Newtonian fluid and finally we have assumed that the Stokes hypothesis is valid.

So, a fluid for which the Stokes hypothesis is valid is known as Stokesian fluid. So, with these assumptions we arrive at this equation of motion which is the celebrated Navier-Stokes Equations. So again, one of the physical implication of various terms. The left-hand side is mass into acceleration per unit volume. So, right hand side is force per unit volume. This is force due to pressure gradient.

This is the viscous force and this is the force due to volumetric dilatation combined with linear deformation. So, for compressible flow this term is already there but for incompressible flow this term is not present and this is any other body force. There are special cases when the left-hand side is equal to zero that means the fluid has no inertia that means the fluid is not accelerating then that equation is known as Stokes equation.

So, this the problem when a scientist does too many things. So, you have Stokes law, Stokes hypothesis, Stokes equation these are all different. Stokes law you have learned in high school physic that if you have a sphere which is in a terminal motion at a very slow velocity in a fluid then what this terminal velocity. How do you calculate the terminal velocity the drag force and so on?

That is Stokes law. Stokes hypothesis, mechanical pressure equal to thermodynamic pressure and Stokes equation is the Navier-Stokes equation with the left-hand side equal to zero. So, all these three are different things. Now let us see how many unknowns and how many equations you have? So, coupled with that you have the continuity equation. So now you tell how many unknowns are there? Unknown?

This equation basically, there are three components I equal to 1, 2, 3. So, how many unknowns? What are the unknowns? Rho, p, u1, u2, u3 these are like basically u, v, w. So, you have five unknowns. How many equations you have? Here you have three equations and you have one, four equations. So, these also does not close the system. In a special case when it closes the system it closes the system when the density is the constant or given.

It may be a constant may not be a constant but some given function. So, if rho is given or rho is a constant as a special case then this system is closed. So, then you can solve for the velocity field by using the continuity and the momentum equation, momentum equation eventually becomes Navier-Stokes equation. If rho is not given, then rho is a function of what? rho is function of pressure of temperature through the equation of state.

So, rho is a function of pressure and temperature this is equation of state. But have increased another equation you have got another equation but you have increased another unknown the unknown is temperature. So, you wanted to solve for rho but to introduce rho through the equation of state you involved and unknown which is temperature and to solve for the temperature you need to solve for the energy equation.

You get a governing equation of temperature which is the energy equation in convection. So, you require basically the equation starting from fluid flow to the equation of state and the energy equation and sometimes the momentum and the energy equation may be directly coupled because the body force may be a function of temperature and that happens in natural convection and free convection.

So, you can understand that what is the broad system of equations that we need to solve we will come to the derivation of energy equation later on after we complete our discussion on the fluid mechanics relevant to convective heat transfer. Now finally, what are the challenges associated with solving this equation? If of all this equation is nonlinear partial differential equations. Why nonlinear?

Because of these kind of terms, this is actually the nonlinear coupled system of partial differential equations. This is not actually one equation these are three equations i equal to 1, i equal to 2, i equal to 3. So, nonlinear coupled system differential equations this is point number one. Point number two is you have pressure as a variable but you do not have a separate governing equation for pressure.

So, you need to play certain tricks, you need to relate the pressure with the continuity equation to somehow get a governing equation for pressure and that not very straight forward you have to do it through some numerical technique. So, normally because of this nonlinearity and because of this particular typical nature of pressure for which there is no separate governing equation you cannot solve this equation analytically.

But for some special cases for some special problems which we will discuss in the next lecture it will be possible to solve this equation analytically. But otherwise for a general situation you need to use computational fluid dynamics or CFD to solve this problem numerically. So again, this course is not a computational course but if you are interested to know to learn how to solve the

Navier–Stokes equations in a computational platform.

Please go through any standard text book on computational fluid dynamics or you may also look into my NTPL video lectures on CFD or computational fluid dynamics. The final note with which we will conclude todays lecture, this is index notation but how to write this equation in a normal x, y, z system because for problem solving normally do not use index notation but the normal x, y, z system. So, we will write the equation in Cartesian system.

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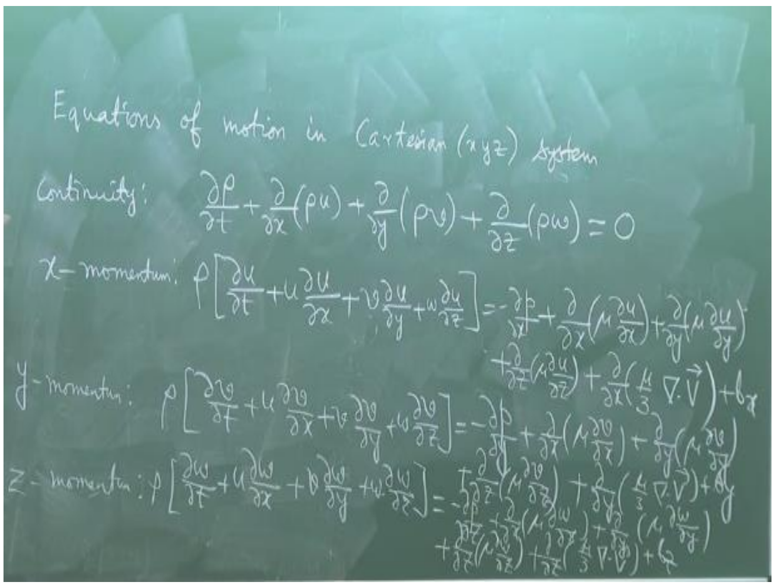

What I am doing is I am just writing this equation in normal x, y, z notation. So, this is the x momentum. Similarly, we will write the y and z momentum. So, you see the elegance of using the index notation is that such big equations you can summarize through the index notation in this very compact form. So, to summarize we have through these introductory lectures on convective heat transfer we have related the fluid mechanics with heat transfer qualitatively.

And then we have derived the basic equations of fluid mechanics which will be necessary for analyzing convective heat transfer. We will stop here today. Thank you very much.