Conduction and Convection Heat Transfer

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Lecture – 21
Review of Fluid Mechanics – II

In the previous lecture, we were discussing about the Cauchy Theorem which us a frame work of representing the traction force on any arbitrary oriented surface in terms of the stress tensor components. So, we will write the corresponding expression which is already written here.

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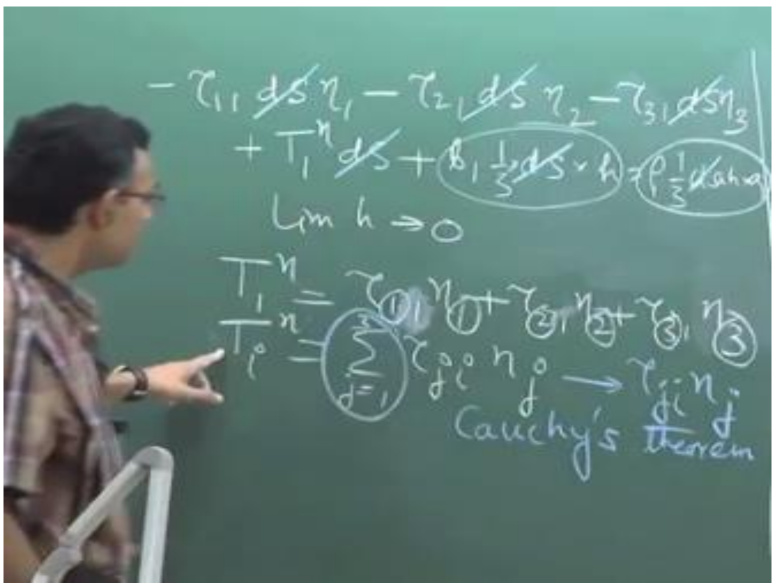

T with subscript i, subscript i is the direction of action of the force and superscript theta which is a direction normal of the surface arbitrary oriented surface in terms of sublimation of Tau j i eta j. So, we will be writing that now in a matrix form.

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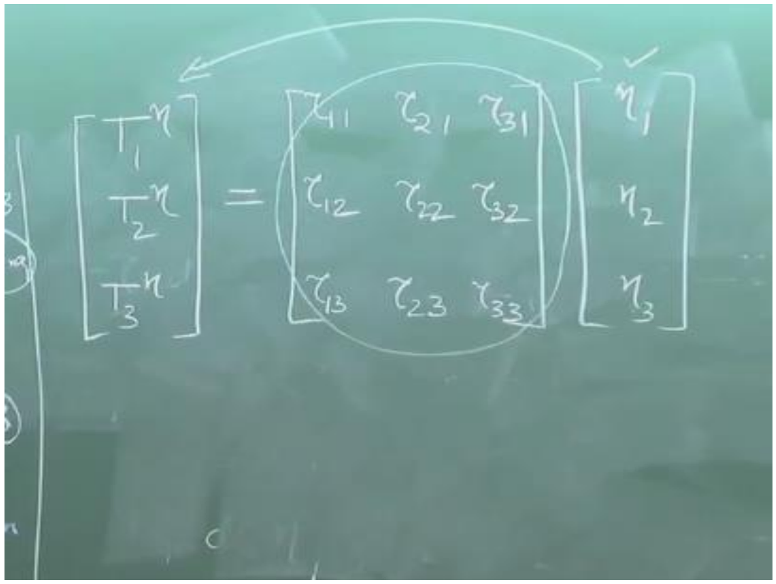

So, we can write the components. The first component you see T1 is equal to tau 1,1,1 eta 1 plus tau 2, 1 eta 2 plus tau 3, 1 eta 3. So, the first row of this multiply with this column vector. Similarly, T 2, the second row of this times the column vector and T 3 is the third row of this times the column vector. So, you can also represent the stress tensor components in the form of a matrix. It is another way of way representing it.

So, here you can see that this is a second order tensor we had already discussed and we had discussed the second order tensor map a vector on to a vector and that you can see from here. This is a vector this is map on to another vector by means of this second order tensor. If this is the case, then we say that the tensor is a second order tensor. It is capable of mapping a vector on to another vector.

Now the question is how many components are here total three plus, three plus, three, total nine components. But out of this total nine components, all are not independent and how they are dependent or how they are independent we have to figure out from a consideration which is conservation of angular momentum. So still now, we have used conservation of linear momentum we have partially done it.

We have not yet written a final form of the equation but we have never used the angular momentum conservation so that we will do in a moment.

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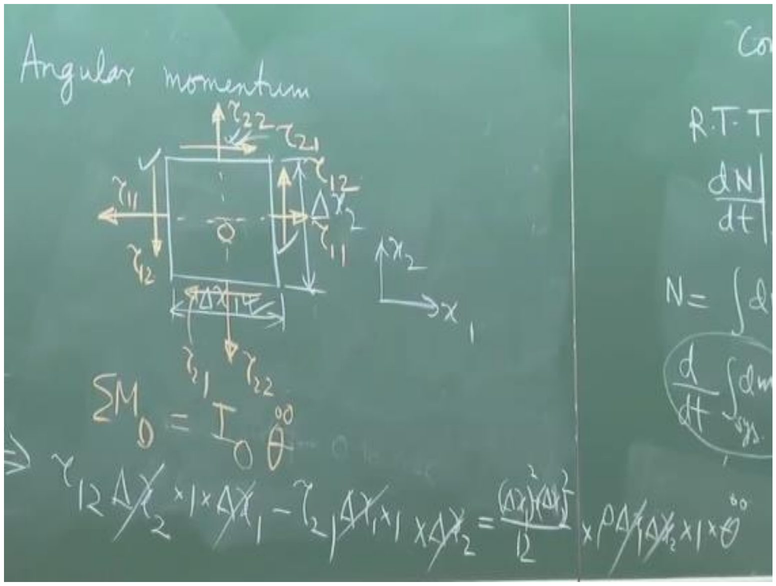

So just like linear momentum relates to translation angular momentum relates to rotation. So, we will consider a simple example when we consider rotation in the x, y plan that is rotation with respect to the z axis. So, let us say that delta x1 in the dimension along x1 and delta x2 is the dimension along x2 of a fluid element. We have taken a two-dimensional fluid element purposefully to simplify the situation.

Because any rotation will always be having a plain of rotation and that plain of rotation let us say this x1, x2 plain. So, let us represent the forces. Let us say the center of this element is O. On this four faces, we have shown the relevant normal and the shear components of the forces. The tangential components are also known as the shear components which you know from basic mechanics.

What we are studying here is actually not anything different from what you have learned in basic mechanics but here we are learning the grammar of basic mechanics. Which is very important see I mean when you start learning how to sing you do not look into the notation you just simply spontaneously learn how to sing but if you want to become a very professional and learned singer or a learned musician then you have to at the end understand the notations.

So, without notations you cannot go further forward. So, what we are trying to do is trying to bring everything in mechanics that you have learned in the perspective of momentum conservation but in a bit of structured manner. So now we will write the equation of resulted momentum of all forces with respect to the axis, perpendicular to the plan passing through O is equal to the moment of inertia with respect to the same axis times the angular acceleration.

This is like equivalent to Newton second law for rotation. So, clearly you can see when you are talking of moment of forces the normal components of stresses will not matter because they are all passing through O. Only the share components will give rise to moments. What kind of moments? So, if you take these two share components they will form a couple. Therefore, their moment is couple moment so what is the couple moment of these two?

Tau 1 to, one is the force, tau is not the force. Tau 1 to times delta x to times lets us say width of the elements perpendicular to the plain of the figure is one. So, this is the force times the length of the couple arm is delta x 1. This is a clockwise couple or anti-clockwise? Anti-clockwise couple moment is there so positive. Then what about tau 2, 1? This one and this one they form a clockwise couple moment.

So minus tau 2, 1 into delta x 1 into 1 this is the total force times delta x 2. This is the couple moment. So, we have represented all the couple moments equal to moment of inertia. So, moment of inertia will be proportional to mass into some length square. So, by sum 12 into the mass, what is the mass? Mass is density times the volume. So, in place of delta y it is delta x 2, in place of delta x it is delta x 1.

So, delta x 1 plus delta x 2 square by 12 into rho into delta x 1 into delta x 2 into 1. This is like m into A square plus B square by 12 something like that. Whether it is by 12 or by whatever it does not matter actually for these derivation. Because we will see that these terms will be zero at the end so this into the angular acceleration. Now you take the limit as delta x 1, delta x 2 tends to zero because these are tending to zero and not equal to zero.

You can cancel it from both sides. This is the first thing. The second thing is that because delta x 1 and delta x 2 are tending to zero this term will into zero. So, from here we are left with one very important thing that is tau 1 2 is equal to tau 2 1. Or tau i j is equal to tau j i in general. Now here once we have derived this you can see that out of this nine components now you can say that tau 3 2 and tau 2 3 are equal, tau 1 3 and tau 3 1 are equal and tau 1 2 and tau 2 1 are equal.

So how many independent components are there in the stress tensor, six. So, you have six independent components in the stress tensor. So now the Cauchy theorem is also you can write as tau i j eta j. Now, one important thing is that while deriving this equation we have assumed that there is no body couple that means there is no couple which is acting on the volume of the fluid.

That is true in general for almost every fluid but there are exceptional fluids which can sustain body couples. One such type of fluid which is used in the research analysis is known as micro polar fluid. Micro polar fluids are certain types of fluids in which you have particles in the fluid which can induce certain rotation in the flow I mean not just in the flow but the fluid can sustain body couple.

Having a rotation in the flow does not mean that it can sustain body couple but the particular medium ensures that there the fluid can sustain somebody couple. So, if the fluid can sustain body couple then tau i j is not equal to tau j i. But here we are not considering that type of special fluid but in general fluid cannot sustain body couple. So, you will have tau i j equal to tau j i with this understanding now let us come back to this equation of motion.

Or conservation of linear momentum that we discussed in the previous lectures.

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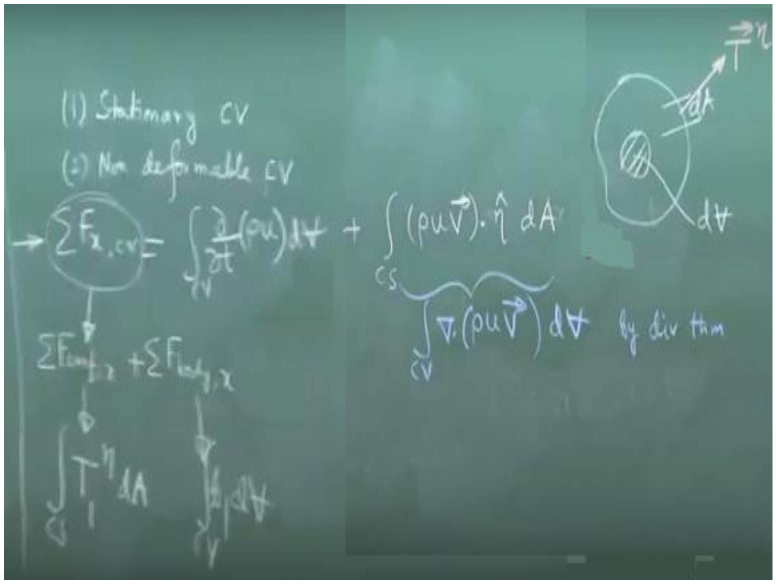

So, if you can recall that with the assumption of stationary control volume and non-deferrable control volume we arrive at this form of the Reynolds transport theorem. Now what we will do is we will write the left-hand side the resultant courses. The resultant force is summation of surface force along x plus summation of body force along x. How do you represent the surface course along x? So, you consider

An arbitrary surface dA then the force per unit area on this arbitrary surface is what? So, you can write the surface force-- why this subscript 1 because we are interested for component along x. And what is the body force? Let us say you take a small volume and let us say b 1 is the body force per unit volume. So, b 1 is the body force along x 1 per unit volume. Now you can write T1 because Ti is tau i j eta j so T1 is tau 1 j eta j.

So, this we can write as tau 11 i plus tau 12 j plus tau 13k dot with eta 1 i plus eta 2 j plus eta 3. We had made a vector artificially made of course of vector like this which we represent in this way. Let us give this name tau 1. We can give any name just let us say we that we give a vector this vector I named tau 1. So, this is tau 1 dot eta. Why we have given it in this way is that in this term you are able to write this as f dot eta da so you can convert it into divergent f db.

So that you can convert it into volume integral. So, if you see right hand side both the terms are converted into volume integral left hand side body course is automatically volume integral. Only the surface force has to converted into volume integral and for that divergence theorem has to be used that is why we have manipulated this in this way so that it is expressed as a dot product. So that you can use the divergence theorem on this dot product.

So, you can write what we will do is we will write the right-hand side of this equation in the left hand and left-hand side in the right hand. So that is what is the standard way of writing, it in fluid mechanic.

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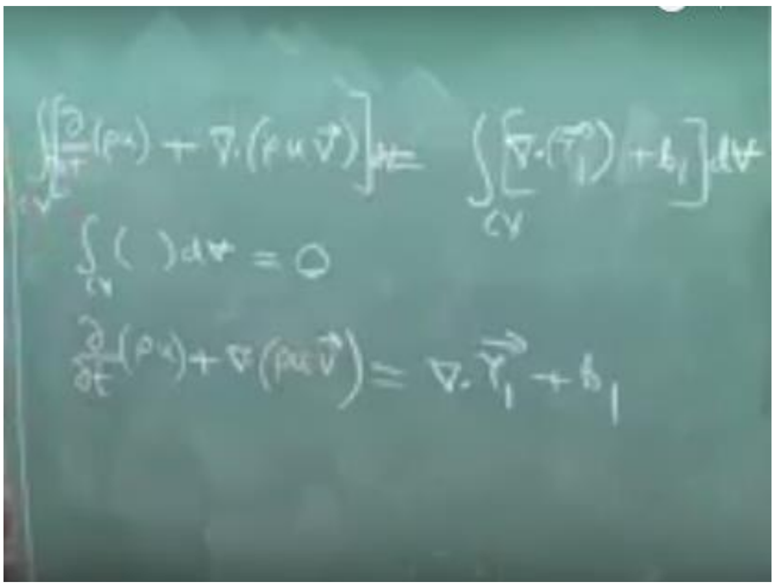

So, we will write integral of delta/rho t. So, this term is tau 1 dot eta dA it is divergence of the tau 1 dV that is what we have written here. So, this is as good as writing something dv, control volume is equal to zero. Again, the choice of the control volume is arbitrary that means this something must be zero. So, you have –so see we have derived the differential equation for fluid motion starting from the integral form.

It is written with vector as one of the essential components but we will use the index notation to write it in an alternative index notation so to do that –so see divergence of F if you write so what is divergence?

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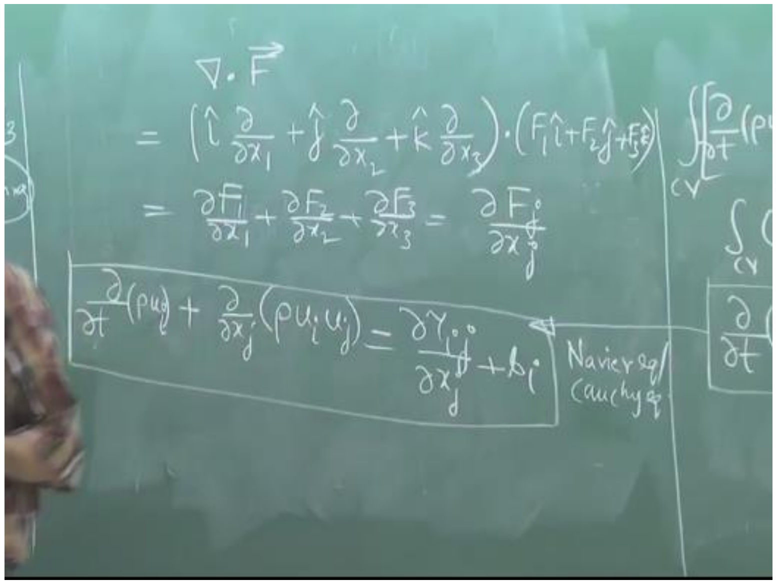

So, in index notation delta Fj/delta x j. So, we will use this term, use this equation right the index notation version of it. So, let us say that we are interested about before writing this equation let us say instead of x component of velocity we are interested about the highest components of velocity then u will be y and in place of tau 1, it will be tau i, instead of b 1 it will be b i. Instead of the x components, if you are interested about highest component just replace one with i that is all. If you add the index one, that was representing x component.

If you are interested about the highest component replace the index one with i that will give you the highest component. So now, if we are using this definition of divergence. This is alternative index notation of writing the same equation. This equation, I mean either the vector form or the index form this is known as Navier equation or Cauchy equation. What are the assumptions under which this equation is valid?

We will look into this equation very carefully but first what are the assumptions under which it is valid? See while deriving this equation first of all this is equation of what? Conservation, of what? Like continuity equation is conservation of maths. This is conservation of what? Again, I'm giving you four options linear momentum, angular momentum, both linear momentum and angular momentum none of the above?

Both linear momentum and angular momentum because we started with the linear momentum but we substituted tau i j equal to tau j i within the derivation that comes from angular momentum. So, it looks like a linear momentum conservation but angular momentum conservation inbuilt within the linear momentum equation. So, this we call as normally momentum equation in terms of differential calculus based representation of equation of fluid motion.

So, momentum equation but what are the assumptions that we had made? So, we had made two assumptions, two major assumptions one is stationary control volume another is none deformable control volume but both of them can be relaxed. Why both of them can be relaxed? Because if you considering a none stationary control volume and if you considering a none deformable control volume both of these can be manifested in terms of an extra body force that can be shown.

So, eventually stationary and non-deferrable control volume also can be relaxed because that will give rise to additional body force. For example, if you have an accelerating control volume that may give rise to centrifugal force so you can put that in the terms of body course. So, pretty general but one very important assumption there is no body couple. Because if there is any body couple then tau i j is not equal to tau j i.

So, the only and the most important assumption that goes behind this equation is that there is no body couple. Other than that, it is pretty general and why it is so important is because it does not refer to what is the fluid. It does not refer to Newtonian fluid, none Newtonian fluid whatever. So, you can use it for all class of fluid so long as you do not have to consider body couple. Now we will see that what are the different terms?

What are the different terms meaning here? So, to understand that let us simplify the left-hand side of the equation.

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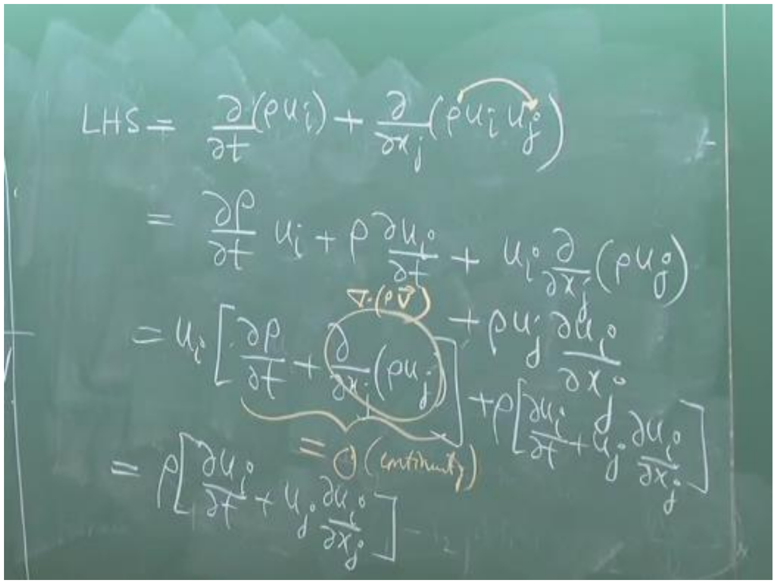

So, what we have done is we have clubbed this rho u j and u i separately and have used the product rule for differentiation for the second term. Rho and u j together and u i separately. So now you can write this as—what is this? By continuity this is equal to zero. This is delta dot rho v. In index notation, this is divergent of the velocity vector. So, this is zero by continuity. So, the left-hand side becomes rho. So, what is there inside?

So, this is like if it was x component it is like delta u/ delta t plus u, delta u/ delta x plus b delta y plus w delta z that thing is written in an index notation. So, what is this? This is the total derivative of the velocity or the acceleration along i. So, the left-hand side is what? Rho into acceleration along i. So, the left-hand side is rho into acceleration along i that means mass into acceleration per unit volume.

So, the left-hand side is mass into acceleration per unit volume. So, what is right hand side, right hand side is the force per unit volume. This is the surface force per unit volume. This is body force per unit volume. So, this is like Newton second law of hot fluid. So, this is mass into acceleration per unit volume right hand side force per unit volume. Now let us look into the situation that how many equations and how many independent equations.

And how many unknown, do we have? Do you have sufficient numbers of equations to solve the velocity? See in fluid mechanics one of the primary matters of importance is to solve the velocity to obtain the velocity field. So how many equations and unknowns do we have? Let us see.

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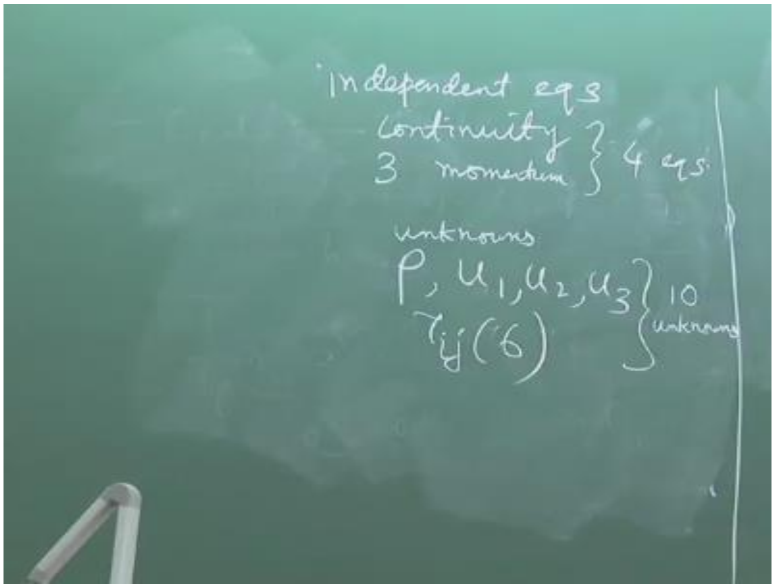

Independent equations, how many equations one is continuity equation. The second is you have this equation for three components so three momentum equations. So total four equations. Unknowns, what are the unknowns? In general, you may have rho as an unknown, density may be unknown then u 1, u 2, u 3 then six components of tau i, j body forces usually prescribed it is know because you are interested about the motion given a particular body force.

So how many are there? One, two, three, four plus six, ten unknown so you can see that whatever equations we have derived that is not a closed system of equations. Because you have four independent equations, with ten unknowns so you require further simplification and for further simplification you need to treat the fluid as a special type of fluid. So far, we have treated the fluid as a general fluid so it can be used for any type of fluid.

But to make further simplifications we have to consider fluid by fluid separately. So, that means when you consider fluid by fluid separately of course that means that may be some particular type of oil or water or plant whatever different types of fluids have to be given due consideration for their constitution and this is what is called as constitutive behavior of the fluid. So that means how does the fluid respond to the given stress that is unique for different types of fluids that needs to be recognized.

We will come into that but one final point which I intend to mention here is that it is an illusion that this rho when it comes out of the derivative it gives an elusive understanding as if rho is a constant. This rho has come out of the derivative not because rho is a constant but because of simplification using the continuity equation so this rho can still be variable. So, rho outside the derivative does not mean that rho is constant.

It has come out of the derivative by virtue of simplification using the continuity equation. Normally, this form we use for computational implementation of the problem or for CFD solution. The computational fluid dynamic solution this is called as conservative form and this form we commonly used for analytical solution of the problem and this is called as non-conservative form. Why this is conservative form?

Because this form is derived directly from the conservation equation and this form is a simplification of that using the continuity equation. So now we will go to the constitutive behavior that is;

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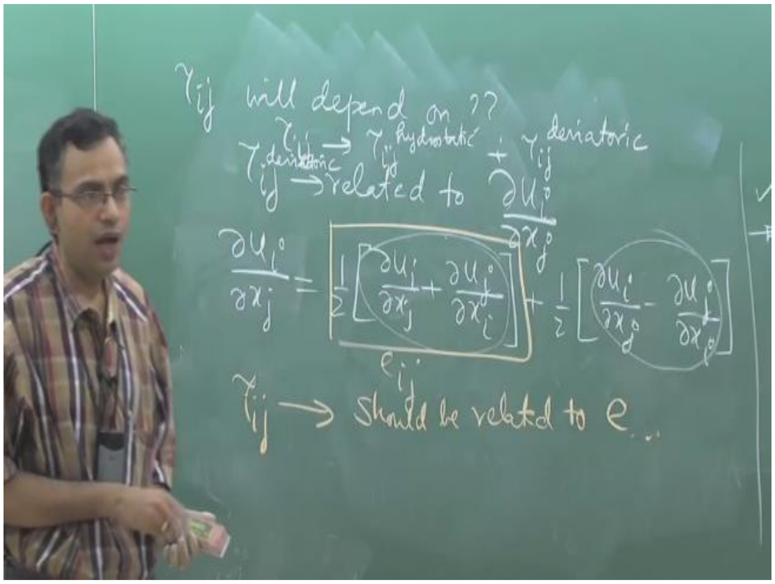

That is tau i j will depend on what? In fluid mechanics, stress is related to rate of strain or rate of deformation and rate of deformation by using your knowledge in fluid kinematics you know that rate of deformation is related to the gradients of velocity. So, tau i j is related to –now you can write –but you have understand one thing very carefully that if a fluid is at rest then there is no gradient in velocity then there no stress.

There is always some stress even when the fluid is at rest. What is the difference between the fluid at rest and fluid at motion? Fluid at motion will be subjected to both shear stress and normal stress whereas fluid at rest it cannot sustain any (()) (39:18) It will be subjected to only normal stress and that is called as hydrostatic stress. So, we will divide tau i j into two parts, one is tau i j hydrostatic and the remaining we call as tau i j deviatoric.

Deviatoric component is a component other than the hydrostatic component. Hydrostatic component is the limiting case when the fluid is at rest so then the deviatoric component is zero. But when the fluid is under motion you have both hydrostatic and deviatoric component. So, tau i j deviatoric is related to delta u i/ delta x j. Now you can write –this is nothing but a equal to half of a plus b plus half of a minus b and it is no new equation.

So why we have written this because there is physics that is associated with this equation. What does it represent? This is shear deformation like delta u delta y plus delta b delta x, right. Shear deformation and what does it represent? Rotation. So out of shear deformation and rotation which will give you stress. You have to keep in mind that this is (()) (41:47) shear deformation but when i equal to j this gives linear deformation.

So linear stroke shear deformation and this is angular sorry this is rotation. So out of linear deformation, shear deformation and rotation which will not give you stress. Rotation will not give you stress. It is like a rigid body type of behavior. So, you can say that you give this a name e i j then tau i j should be related to this e. This is also a second order tensor; this is called as rate of deformation tensor because it uses two indices for its specification.

Now the question the big question is how is tau related to e? How is tau related to e it depends on what type of fluid you are considering if you are considering Newtonian fluid then tau is linearly related to e.

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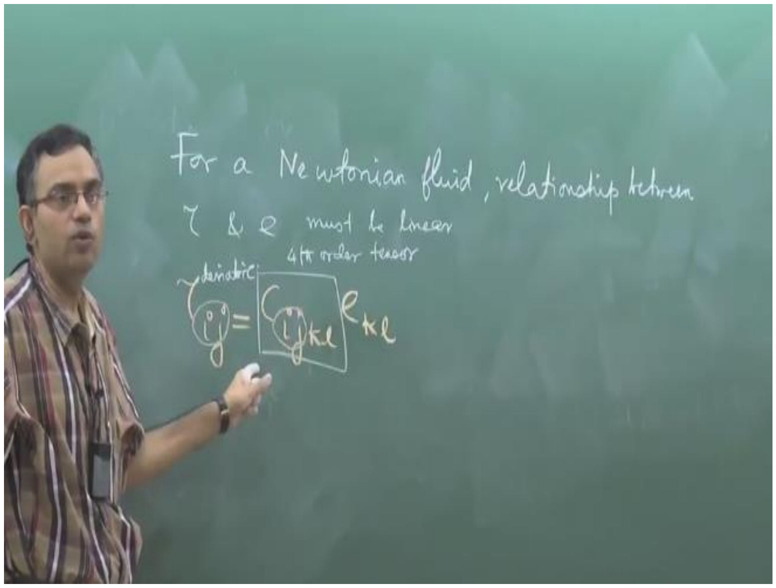

For a Newtonian fluid, the definition is that the tau is linearly related to e. So, for a Newtonian fluid relationship, between tau and e must be linear. So, what we mean by a linear relationship? Tau, what is this tau is a second order tensor. e is also a second order tensor so you must find out what is a linear mapping that maps the second order tensor on to a second order tensor. For example, when we discuss about the Cauchy theorem, we had the normal vector and the traction vector.

The normal vector was mapped to the traction vector by means of the second order tensor. So, a second order tensor map a vector on to a vector similarly there is something which will map a second order tensor e to the second order tensor tau and that is a forth order tensor. So, you can write –this. So, fourth order tensor, this is a forth order tensor. So, a forth order tensor will require four indices for its specification.

You can see in the left-hand side you are free indices i j so in the right-hand side also you have free indices i j there is summation over here and summation over l. Now what does it mean? It means that you can write the rate of stress in terms of the, this is tau i j deviatory if you must remember. This is not the total tau. This is tau i j deviatory. The deviatory component of the stress is related to the rate of deformation through the fourth order tensor.

That means if you know this tensor then given the velocity gradient you know what is the stress?

But how many components of C i j k l you require? See each of this i j k l vary from one to three. So, three into three into three into three so 81 constants are in general required to describe the property of the fluid but I mean this will quickly come down not to 81 but to 36 because you have tau i j equal to tau j i.

So that will simplify it a little bit but still when we are describing properties of fluids like water air etcetera we do not use 36 material properties. We use possibly the most important property which is a viscosity. So, the number of properties as larger this is not used for solving a practical problem. So, to solve a practical problem we can come down to less number of properties necessary and that requires the consideration of a special type of fluid known as homogeneous and isotropic fluids.

So, we will discuss about the consequence of what is a homogeneous fluid? What is an isotropic fluid and how can be write C i j k l for homogeneous? isotropic and Newtonian fluid that we will take up in the next lecture. Thank you.