Conduction and Convection Heat Transfer
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Lecture - 17
One Dimensional Unsteady State Heat Conduction – II

So, we have been discussing about various issues related to unsteady heat conduction and we will discuss about one final issue about unsteady heat conduction before concluding this chapter and that issue is about unsteady heat conduction in a semi-infinite medium. So, let us discuss what it is? So, let us say that

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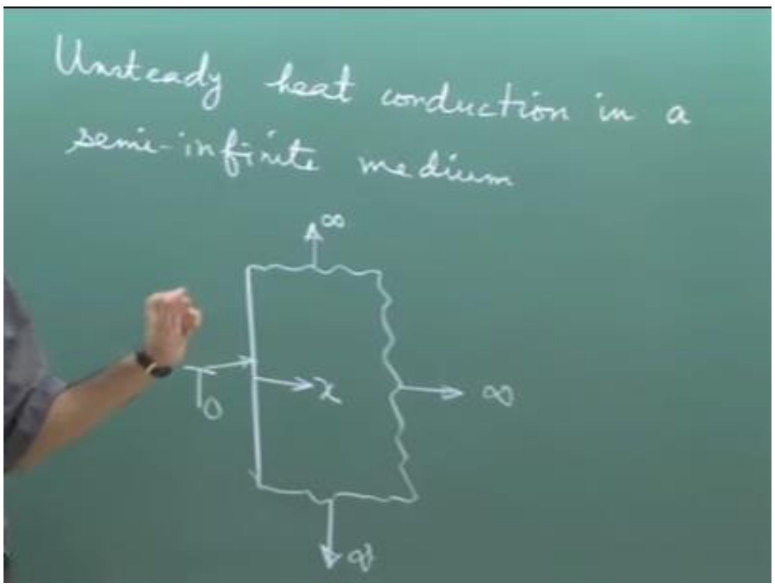

You have a solid medium like this which is also open to this direction which is x direction up to infinity but it is bounded in the other direction. That means, it is not open from minus infinity to infinity but open from zero to infinity. So, that is why semi-infinite medium. Now, question is like whenever we discuss some problem conceptually the first question that should come to your mind is that in engineering is there anything called as infinity.

Infinity is something which is a concept which we borrow from mathematics and basic science but in engineering can you say that let us make a wall of infinite thickness. So, that is not something we do in engineering. However, the term infinite has to be understood from a notional point of view and not from a literal sense. So, what do we understand by infinite or beyond what critical distance, it can be triggered as infinite although it is finite that is something which is a matter of importance.

So, we will see later on that. It is not literally infinite but what it means it that beyond the critical distance the or beyond the critical thickness of the wall the wall thickness can be assumed as infinity. So, we will try to assess that problem. So, just like the other problem these directions are also infinite so you have an one dimensional heat conduction along x. So, let us write the governing equation. Very similar to the problem of infinite plate we had this equation.

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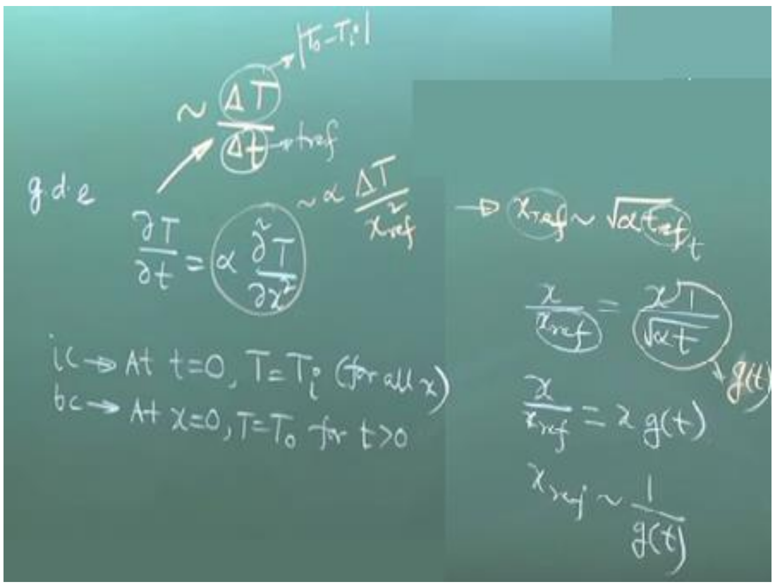

Which property has to be constant for this equation to be satisfied? K, Rho and CP, none of the above, tell? K, right. So, has to be constant. Now, out of other assumption are that the heat conduction in the y and x direction are not important that brings you to the one-dimensional problem. And you have to keep in mind that we are not considering any heat generation in this particular problem.

Now, we will solve this problem so let us write the initial and the boundary conditions as we discussed this kind of problem is called as initial boundary value problem in differential equation where you also have initial condition and you also have boundary condition. So initial condition At t equal to zero. This is the initial temperature and boundary condition. That at x equal to zero, this is x equal to zero, the temperature is T zero for T greater than zero.

So physical situation is there is block. This block was having an initial temperature. Suddenly, this wall is subjected to a temperature which is different from the initial temperature either hotter or colder and then there will be a temperature change within the block as a function of position and time that we have to find out. Remember that this is not the only possible boundary condition, you could have heat flux boundary condition or there are other possible boundary conditionals also.

But this we are solving as a representative problem to get the physical insight. But for other types of boundary conditions the mathematical solution may be different. But the physical insight will be very, very similar. Now can you tell, what is the most important difference between this problem and the infinite plate problem that we did in the previous class? What is the most important difference?

So, if you recall the problem of infinite plate that we did in the previous class this particular dimension we consider a fixed dimension of L the entire block was 2 A, half of the block was L. Now, for this we are not considering any fixed dimension. So, we are not considering any fixed dimension but we are claiming this to be infinite. Question is what possible type of dimension it can take for the claim infinite to be valid.

To understand, that we will do a little bit of order of magnitude analysis or scaling analysis. This term have an order of magnitude of delta T/ the time t. What is this? That in term of order of magnitude this represents a characteristic change in temperature delta T over a time, delta t. Now what is this typically? This is mode of T zero minus Ti. I mean, it depends on which one is higher and which one is lower?

But you have the characteristic change in temperature from the initial temperature to the maximum the boundary value which is provided. What is the order of the magnitude of this? Alpha, delta T/ x reference square, what is x reference? x reference is the distance along x over which the thermal effect is failed that means let us say you make this wall hot. Let us say, initially this block was at a uniform temperature now we make this wall hot.

If you make this wall hot there will be a distance up to which this heating at the wall will be failed. Beyond that distance if I'm standing I will not feel that the wall has been heated because the heating affect is not propagating beyond the critical distance. So, the question is how much the heating affect is propagated. Now, that can be obtained by noting that these two equations sorry these two terms in the equation they are balancing so they must be of the same order.

That means if you equate the order of this two then x reference (()) (10:58). This delta t we call as t reference just reference time of the characteristic time. So, what does it mean? It means that in this problem unlike the previous problem there is no natural physical length scale. The length scale is related to the time scale. This is the new (()) (11:39) that we get in this problem. In the previous problem while we were talking about the infinite plate the length scale was fixed by the geometrical length L.

Now in this particular problem it is not a fixed geometrical length that is the length scale of the problem. It is the length that depends on time that at what time you are considering the problem at that particular time there is a critical length up to which the thermal disturbance at the wall is felt. So that length is square root of alpha into the time that has elapsed. So, this is a variable length scale and the variable length scale depends on the time scale.

Now, none dimensionally it is possible to define a non- dimensional length. Let us call t ref as t, the local time t just symbolically. T reference is time when x is local x. So, this is a natural length scale of the problem expressed in a non-dimensional manner. And we will show later on the solution of this problem uniquely depends on this non-dimensional ratio. That means if you keep this ratio fixed then the non-dimensional solution is invariant. So, we will show that.

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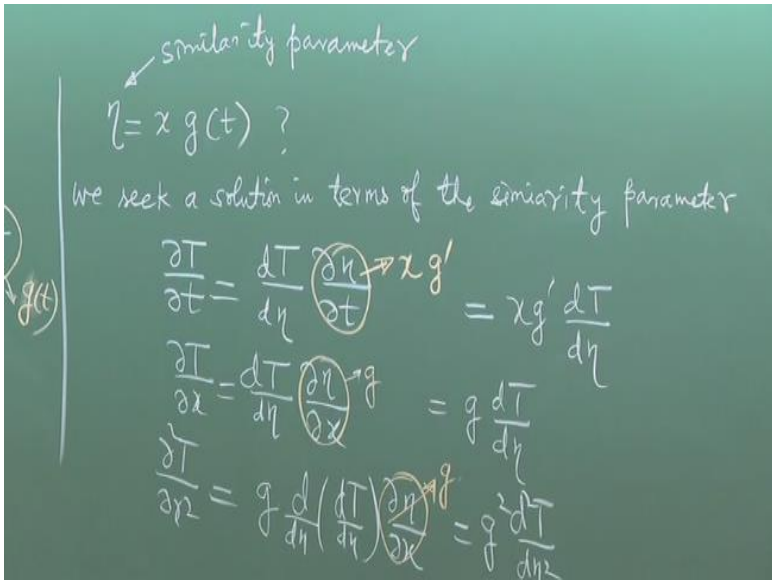

Now how do we show that for mathematical analysis we introduce the variable eta equal to x into a function of time. Question is where from our mind this is coming? This is coming from the fact that if this you call as or 1/this you call as g(t) then this is basically the solution depends on x into a function of time. So, see how physics guide mathematics, a mathematician will start from this form of solution but where from the form of the solution comes.

The form of the solution comes from the physical argument that the solution, if it is self-similar by itself then it should depend on x/x reference that is x by square root of alpha T and 1/square root of alpha t you can write as sum g(t). But the question is that this we already arrive from physical argument that g(t) is one by square root of alpha t or a multiple of one by square root of alpha t. But let us say that we do not keep that prejudice in mind.

So, can be show from mathematic also that gt will be of the same form as one by square root of alpha d. Let us try to do that so whether a similarity solution of this form this is called as a similarity parameter. So, we seek a solution in terms of the similarity parameter. See if you are by this time familiar with the method of separation of variables. It will appear to you that this is also a like a special case of method of separation of variables.

Separation of variable is like you can write it in terms of a variable which is a product of function of space into function of time. Here also it is like that but the function of space is the space variable itself and the entire solution is a function of this normalized variable. Question is can be express that or not that we have to figure out. You can always assume this to be a similarity parameter but for all problems similarity parameter does not exist.

So, we have to check mathematically whether the similarity parameter exists or not. So, let us see that how we can do that. So, if the similarity parameter exists then that entire solution is a single valued function of the similarity parameter that means it converts essentially the partial differential equation to ordinary differential equation. That means if this is a similarity parameter then the temperature will be a function of this parameter only.

So, will a function of beta only. So, delta T/delta T equal to dT/ d beta because T is the function of beta only and delta theta/ delta t. This is chain root. So, what is delta theta delta T this is x into g dash. g dash is bg dt. So xg dash dT/ d eta. We have to evaluate this term that on the right-hand side so delta T/ delta x. What is delta eta delta x, delta eta delta x is g. So, this is g dT/ d eta. The second derivative of this so this becomes g square.

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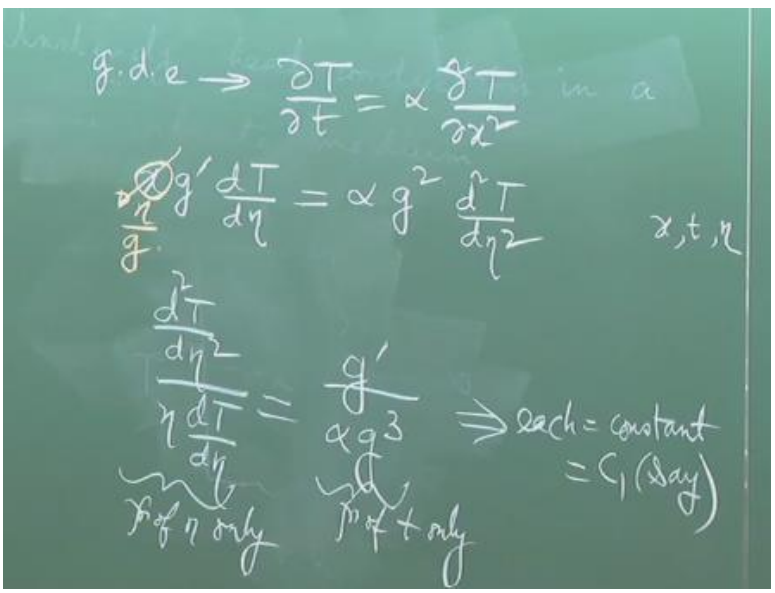

So, your governing differential equation to eliminate one of the variables. So, this equation involves three variables x, T and eta. Of course, T is dependent variable but I'm talking about the independent variables x, t, eta. So, to deduce the variables we will substitute x you can see x is nothing but eta/g. So, this you can write as eta/g. So, we have carefully written it this way because this is the function of eta only.

And this is the function of t only. So, function of eta only is same as a function of t only, only when it is a constant. So, if that we the case let us isolate this second equation and find out a form for g.

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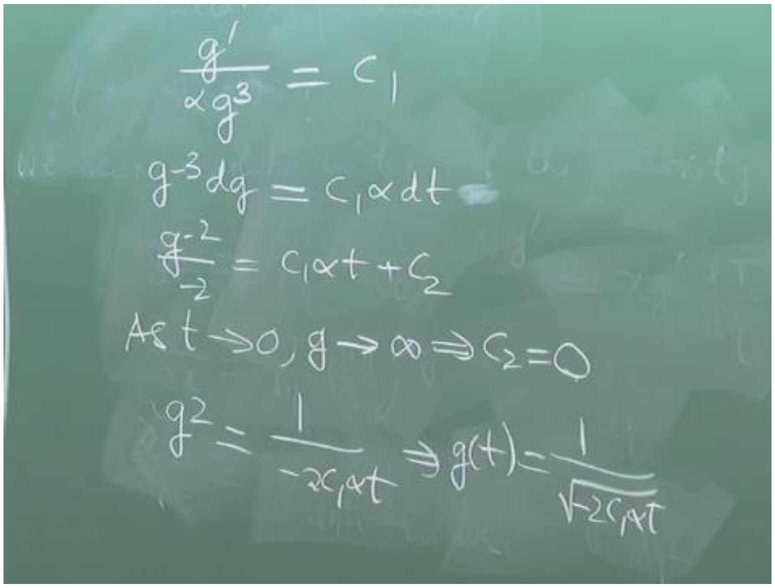

So, g dash by alpha g cube is equal to C1 so g to the power minus three dg is equal to C1 alpha dt and then you can integrate. So, g to the power minus two by minus two is equal to C1 alpha t plus C2 where C2 is a constant of integration. Now, how can be find out what is the value of C2? Clearly, even if we do not know the exact value the time dependent should be such that the characteristic length scale, so x/x reference equal to x into g(t) therefore x reference x we will scale with 1/g(t), right.

Even if we do not know what is g(t) at least we know that x reference should scale with 1/g(t). What is the x reference as time tends to infinity? Or as time tends to zero, as time tends to zero what is x reference? x reference is a distance from the wall up to which the thermal affect is filled? At the wall at which the boundary condition is given what is the distance from that wall up to which the thermal affect is filled.

So, at time equal to zero what is that it is only zero because only that wall knows that there is a heating just the boundary heating has started getting penetrated into the wall of the medium. So, the boundary heating at time equal to zero is confined still at the boundary. Therefore, x reference will be tending to zero as d tends to zero and gt is 1by x reference so gt will be 10 to infinity as x as T tends to zero.

So, we can say that as t tends to zero, g tends to infinity and this means from this equation you must have C2 equal to zero. So, from here you can write that g square is equal to 1/minus 2c1alpha t. So, this is something very remarkable because by mathematical analysis we are getting a form of the gt which is same as we did without any mathematical analysis but just by physical argument. So, the multiplying factor is not important because we say that it is of the order of.

So multiplying factor may be one, two, three, four whatever but the functional dependence gt, functional dependence is 1/ square root of alpha T and that without any deep mathematic just by physical arguments we could come up with. Now that means that we have come up with a situation when you could separate the function of eta with function of T and the function of T can be determined.

If we could not separate the function of eta with function of T we say that despite seeking a similarity parameter for the solution. There is no similarity parameter that exists for the solution. If we couldn't separate these two but because we could separate these two we can now say that yes, the similarity parameter exist which is of this from x into g(t). The next question is that when you have this xt into g t there is a parameter C1 here. So what value of C1 you should take?

See the only constraint is that C1 has to be negative because this is a real number and to make it real because it is negative and alpha is positive and T is positive C1 has to negative. So, choice of C1 must be negative. Within that what value of C1 you can take any value of C1 you can take it does not matter think about the situation see that a/ b equal to c/d this is what you have to satisfy in schools many problems of ratio and proportion.

We solve where we write a/b equal to c/d equal k, no matter whatever is the k. K is not a matter of our big concern. Big concern is some whatever may be the constant the equality of ab/cd has to be maintained and here also it is the same thing but the choice of constant only is to be constrained by negative constant. Now, we can take any value of the constant but for simplicity in algebraic manipulation if you take C1 equal to minus 2 then this two comes out of the square root.

So as an example, take C1 equal to minus two not a must. It is not that it has to be taken as minus two it is just one example it can be shown that whatever value of C1 you take within the negative paradigm you will finally get the same solution. So, it is not important that you have to take this but if you take this the only advantage is that the algebraic manipulation gets simpler because this two comes out of the square root.

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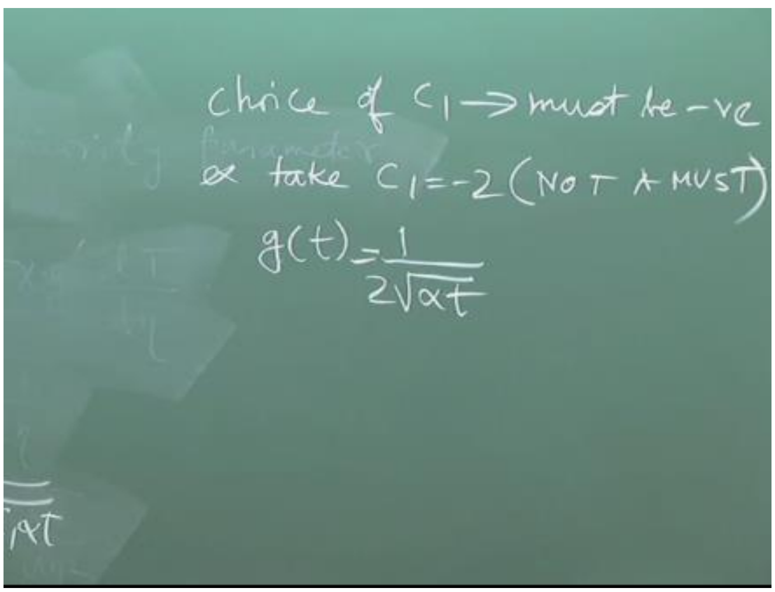

So, then g(t) if you take C1 equal to minus half then you get square root of alpha T so that is also possible but this is an example that we are taking. As a home work you should take C1 equal to minus half of another example and show that you converse to the same solution finally.

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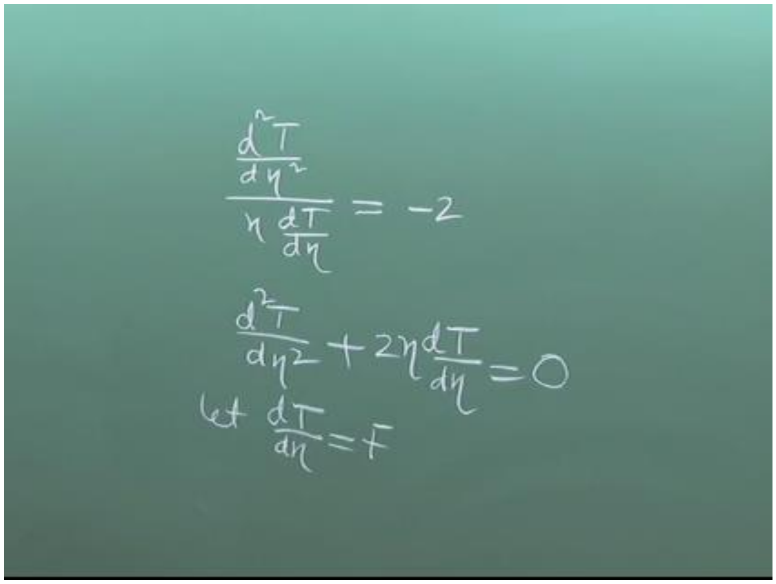

So, if you take C1 equal to minus two then you have d2T/d eta 2 by eta dT/d eta. This is equal to minus two from this equation. So basically, you have to solve this ordinary differential equation which is quite a simple task. So, let us assume dT/d eta equal to F.

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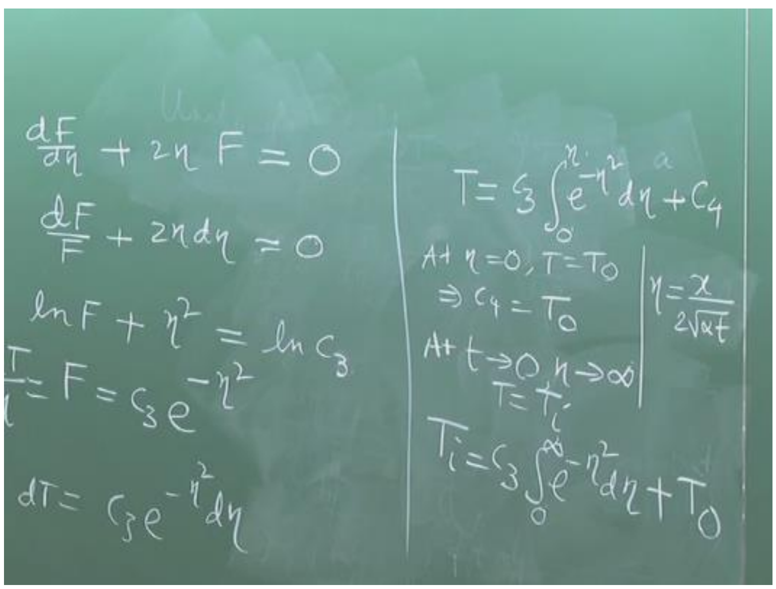

So that means you have dF/d eta plus 2 eta F is equal to zero. So, you can write dF/F plus 2 eta d eta equal to zero. So, ln F plus eta square equal to ln of C2 sorry C3, C2 we have already taken. Ln of C3 where ln C3 is a constant of integration so F equal to C3 into the power minus eta square. F is nothing but dt/ d eta. So, you have dT equal to C3 into the power minus eta square d eta. So, you have T equal to C3 integral of e to the power minus eta square d eta plus C4 that integral is zero to eta.

Now we will apply the initial condition and boundary condition to get the two constants this we have not yet applied. So, this is zero to eta. At x equal to zero what is T? T is T zero, At x equal to zero what is eta? Eta is also equal to zero. So, we can say that eta equal to zero, T equal to T zero that means C4 is equal to T zero. At eta tends to infinity. Remember what is eta? Eta is equal let us write eta here, eta is equal to x/ 2 square root of alpha t.

At eta tends to infinity sorry rather let us write at t tends to zero, you have eta tends to infinity that is what I wanted to write. As t tends to zero eta will tend to infinity. So, you can write T equal to ti at that t. So, Tl equal to C3 integral zero to infinity into the power minus eta square, d eta last T zero.

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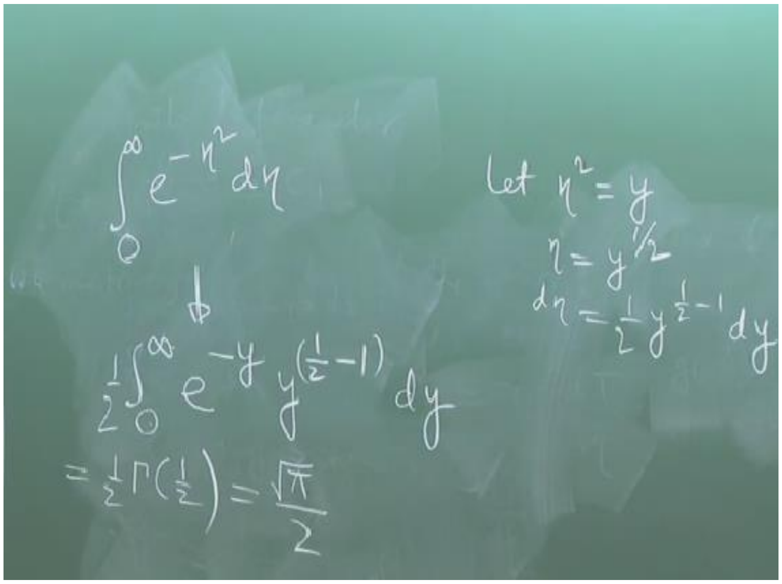

Now let us evaluate this integrals zero to infinity e to the power minus eta square d et. So, let us make a substitution, let eta square is equal to y, just new variable. So, eta equal to y to the power half, d eta so this will become integral zero to infinity e to power minus y into y to power ½ minus one. What is this integral zero to infinity, e to the power minus x into x to the power n minus one dx. It is gamma of half. Gamma function of half is root pi. So, this is root pi by two.

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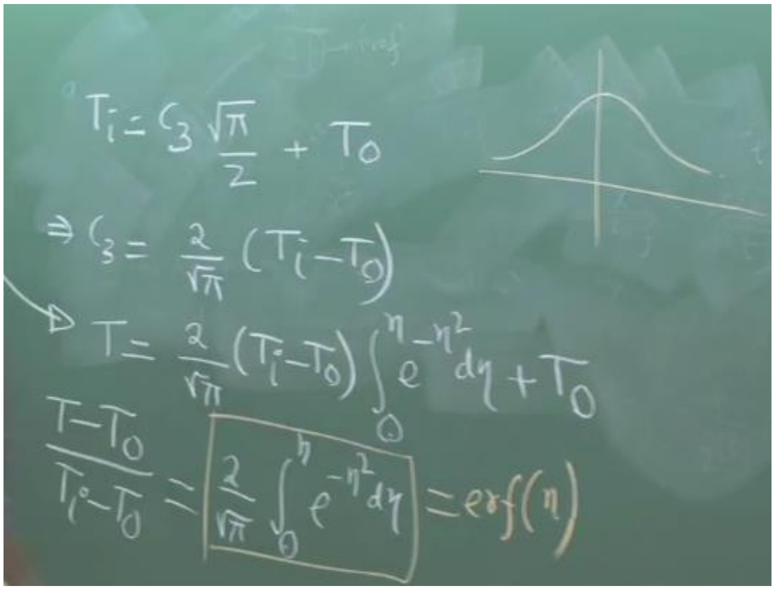

If that we the case then we can write Ti equal to C3 root pi by 2 plus To. So C3 is 2 by root pi into Ti minus To. So, then we can substitute in this equation and write T equal to 2/root pi Ti minus T zero. We have substituted C3 and C4 in this equation. So, we can in a compact form write T minus T zero by Ti minus T zero is equal to 2/ root pi integral zero two eta into the power minus eta square eta. By the definition this is what? This is the error function of eta.

So, the error function you must have studied in statistics where it is a function representing the normal distribution in a statistical sense like the probability distribution. The normal probability distribution which has a Gaussian type of distribution of probability of x as a function of x. That is that particular form is represented by this error function. So, you can also write it in a different way you can also write T is equal to T zero, let us subtract 1.

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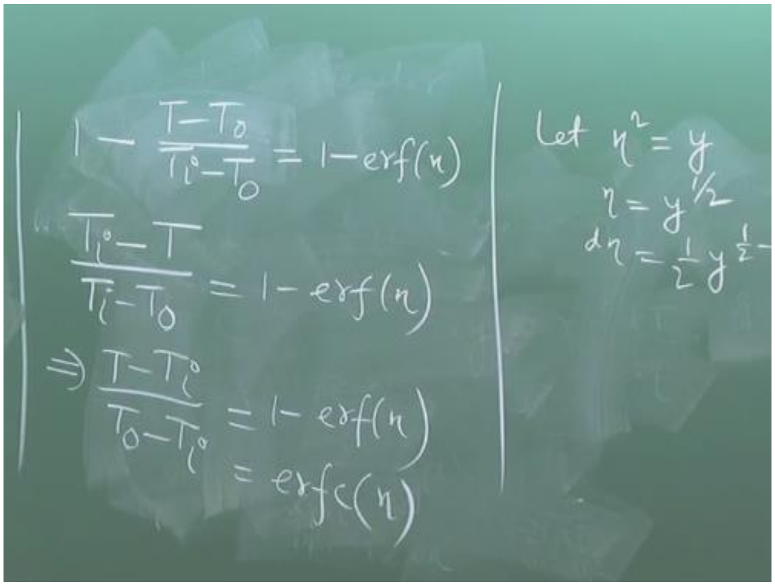

It is also called as complementary error function erfc. So, I have shown these two forms because you know different books will give different forms I do not know which specific book you are following. So, it is important to be familiar with the different forms but I mean this is just good enough to discuss about the solution. Now this is the mathematical part of the solution but again let us try to get some physical insight on the solution to this problem.

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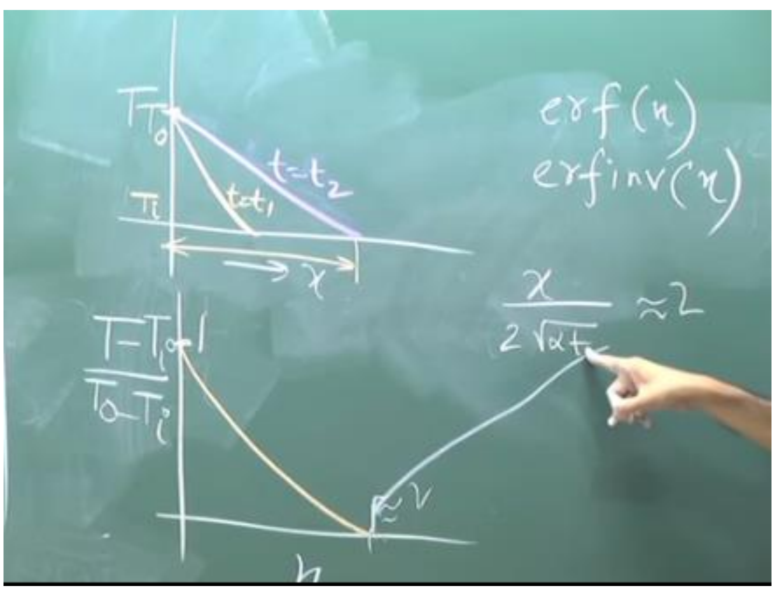

So, if you plot T as a function of x at x equal to zero. T is T zero then this these (()) (43:25). Let us consider another time. This is say time equal to T1 this is not a straight line. This is my problem whenever I want to draw a straight line I can't draw it and whenever I do not want to draw a straight line it becomes a straight line. This is t equal to t2, time. So, can you tell which time is more t1 or t2?

T2 is more because intuitively if you allow more time more and more length of the plate feels the thermal disturbance, right. So, it comes to the Ti. This is say Ti add some distance x that means beyond this critical distance x the wall does not feel the effect of the boundary temperature heating or cooling, whatever? So, when do you consider the wall to be a semi-infinite wall when the wall thickness is greater than then this thickness.

So that depends on what is the time at which you are considering. Now this is the distance, this is the thickness up to which the thermal disturbance of the boundary is failed. So, if the wall thickness is greater than this that it is as if infinity that is the notional meaning of infinity. It does not literally mean that it had to infinity it has to be greater than then the thermal penetration depth.

So that other end of the wall does not know that there is a boundary at which a temperature has been imposed. Now this is a dimensional plot but you can also make none dimensional plots. So, let us make this plot T minus Ti. This is a plot of one minus error function eta. So, if you want to make a plot in Met lab. Met lab has inbuilt function erf so you can use the inbuilt function erf in Met lab to plot the error function and to get the value of the error function.

Not only that let us say you know the error function but you have to find out what is eta? So that means you have to get the inverse of the error function. So, for that Metlab has inbuilt function erfinv. So, this problem is if you know eta what is error function? This problem is if you know error function eta what is eta? Because sometime you have to you know that this is the temperature. The question at what time and position that is the temperature.

So, then you have to find out error function inverse because you have to get the eta then you have to do this. See in old days you will find that still in books that is there. There is a table of error functions. So, for even eta what is the value of error function that is tabulated basically the numerical evaluation of the Gaussian distribution function. So, if you have some value of eta from the table you can find out the error function or if you know the error function from the table you know for eta that error function is there.

But I would say that in modern times that kind of approach is no more necessary because you already have Metlab you can sit with Metlab or any other numerical computational software. If you the value of beta just write the corresponding function type the corresponding function it will give you back error function. If you know the error function just write inverse of that, that will give you the value of the function you do not have to have a real physical look of table.

That is what you can do but that's just for implementation purpose now you can see can you tell what the value of this. What is the value of this point? At eta equal to zero what is T? T zero, so what is the value of this. One. Typically, if you calculate this I would request you to plot this using Metlab. You will see that this value is I can't remember exactly but it will be close to two. So, anything beyond this is like semi-infinite.

So, question is say x by two root alfa t that is roughly equal to two at this point. So which x is considered to be good enough for the plate to be considered to be semi-infinite it depends on what is the time. Larger and larger time you plate width also has to be larger and larger to make sure that this assumption is justified. But very short time even a small thickness can be infinite. So, let us say that you have a laser pulse by which you are treating a material.

This is very common in manufacturing process, very common in surgery that you have a laser pulse you are treating with the laser pulse treating the material. Now typically if you are doing the treatment for a very short period of time. Let us say (()) (49:58) for a medical treatment or milliseconds for a manufacturing process. Over that period of time the penetration depth may be a few microns to millimeter so in that respect even a plate with one centimeter is like infinity.

So, again I'm repeating the same concept over and again to make you understand that infinity is not a literal type, literal name that we are giving it here. It is a notional thing it is that if the thermal penetrating depth is shorter than the width then the width is like infinity. Now let us quickly work a problem and then we will call it a day today.

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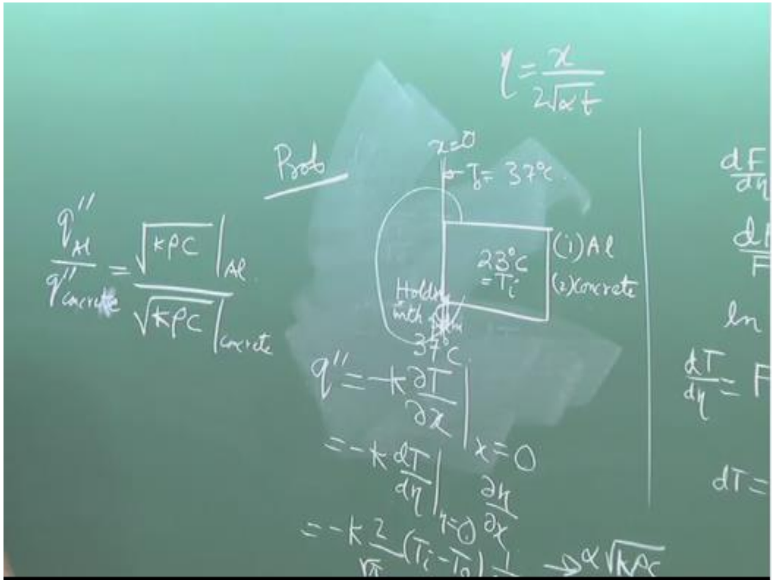

Let us say that this is a cold block at 23 degree centigrade, now you are holding with your palm that is at 37 degree centigrade. This may be of two material one material is aluminum the other material is concrete. Out of these two materials which will fill colder to your hand? That is the question, right. So out of these two materials this block can be of these two materials either aluminum or concrete. You hold it with your hand which is at 37 degree centigrade.

This is at 27 centigrade so colder then you hand but which will feel colder? Now I do want to get an answer from you what I want to get always is how to logically arrive at the answer not what is the answer because I mean once you logically arrive at the answer you can arrive at the answer but many times we arrive at the answer with a common sense without knowing that logic and that is actually very dangerous in science.

So, it is important that we know that or we figure out how to arrive at the solution. In fact, this is one of the assignment problems this chapter of unsteady heat conduction that we have given to you. So how will you know that which feels hotter or colder basically you have to find out in which case the influx is more so the heat flux, this is the boundary. This is basically Ti and at the boundary this is T zero equal to 37 degree centigrade, this is x is equal to zero.

So, minus K Delta T, delta x as x equal to zero. So, what is eta? Remember eta is equal to x/ two root alfa t. So, what is dt/ d eta, at eta equal to zero. Look at this expression dt d eta is equal to

2/root pi Ti minus T zero into e to the power minus eta square. At eta equal to zero that e to the power minus eta square part will be one. So, it will be 2/ root, Ti minus T zero. So minus k, 2 by root pi into Ti minus T zero into delta eta, delta x is 1/ 2 root alpha t.

That means what is alpha, alpha is k / rho into C, right. So, this will be proportional to square root of k rho c, right. This K divided by the square root of k becomes square root of k and 1/ square root of rho c becomes square root of rho c. Other, parameters remaining constant so for aluminum and concrete you have to substitute the corresponding values of k rho c. So, the ratio of heat flux with aluminum by heat flux by concrete is root over k rho c aluminum by root over k rho c concrete. So, let me write it down here.

So, heat flux aluminum by heat flux concrete so you can substitute these values get the values from this table and table which is there table of properties which is there in your book and figure out that what is the ratio of heat flux. Whichever heat flux is higher that will make you feel colder. Let us stop here today and in the next class we will discuss about the tutorial problems and the solutions.