Conduction and Convection Heat Transfer
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Lecture - 16
One Dimensional Unsteady State Heat Conduction - I

In the previous class, we were discussing about the Lambda parameter approach of solving the unsteady state heat conduction problems. Now the Lambda approach, essentially neglects the special variation of temperature within the system. But there are certain problems where the special variation of temperature within the system is important and there you have to solve the temperature distribution as a combined function of position and time not just time but also a function of position and time.

So, we will discuss about such problems today. So, we will discuss about first, one dimensional unsteady state heat conduction.

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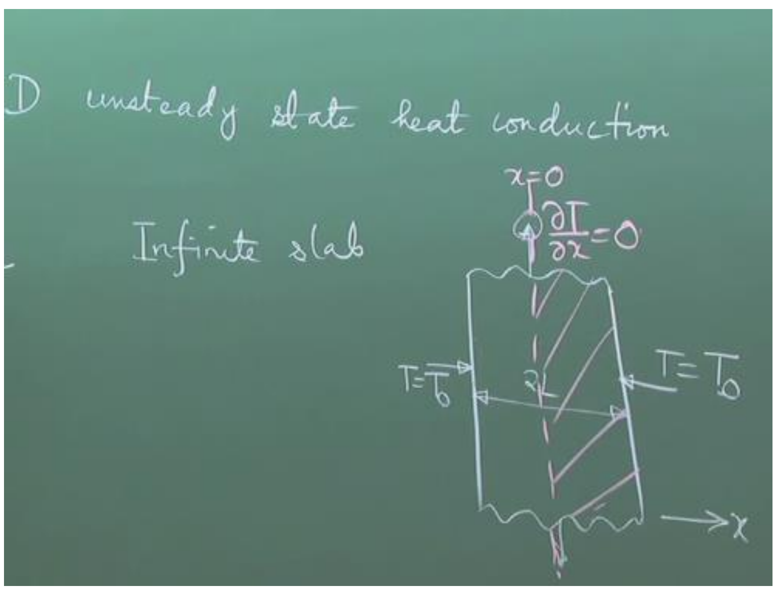

Now, it is a general broad topic and we will try to understand this particular issue with the help of some problems. So, we will first consider one problem of something called Infinite slab. So, what is this? Let us say we have slab which has a thickness of 2L and the direction in which it has a thickness of 2L is x direction. The other directions like these directions and the direction perpendicular to the plane of the figure are infinitely large.

Because they are infinitely large the gradients along those directions are very small and therefore the corresponding second order derivative terms are neglected in the corresponding heat conduction equation. So, if we do that we will come up with the simplified equation. But because we are working with a problem let us keep the boundary conditions and then we will solve the problem. So, let us say that the temperature here is T equal to T zero.

We will first work out, a problem when both the temperatures are same. If the temperatures are different, I will not work out that problem. But I will give you some idea that based on what consideration you can solve the problem even if the temperatures are not the same. But to begin with let us consider that these two temperatures are equal which is T equal to T zero. So, considering that all the thermophysical properties are constant.

So, let us first write the energy equation, plus let us say we put a term for heat generation Now, let us try to see that which term of these equations are negligible as compared to the other terms let us look into that.

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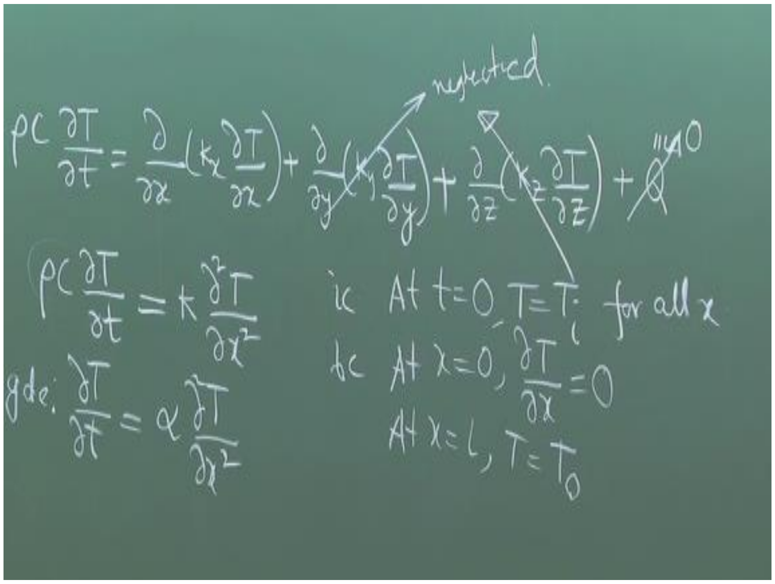

So, this slab is much wider and much longer in height as compared to the thickness. So, the gradients along y and z directions are neglected. This is also neglected. If these are neglected and also, we don't have any heat generation so this term is zero. Then how do we solve this problem? We solve this problem, by considering that Kx is a constant. If Kx is a variable it is not so easy to solve it analytically. This is our governing differential equation.

See, there is a very interesting thing when we solve this problem we say that we are assuming all thermophysical properties as constant. But you can see that actually taking K as constant is good enough it does not matter whether Rho CP are constant or whatever because anyway Rho CP come out of the derivative, right. So, Rho CP or for a sold CP become C Rho into C comes out the derivative not because Rho and C are constant.

But because of some other simplifications which we have already taken care of when we drive the energy equation. So, equation has the assumption that K is constant but it does not have an assumption that Rho C is constants, right. So, this is very illusive because Rho C out of the derivative may create an illusion that as if Rho and C are constants but these are not constants. So, this is the governing differential equation.

Now, this is a time dependent problem. So, this kind of problem in the theory of differential equation, are called as initial boundary value problems. So, you require initial conditions to specify that at time equal to zero what is the situation? So initial condition, at t equal to zero, temperature is equal to T i for all x. This is let us say initial temperature and boundary condition.

Now, you can play a little bit of trick by reducing the size of the domain you can see that this problem is symmetrical on this side on this end whatever is the temperature at this end also the same temperature is there. So, it is symmetrical with respect to the central axis. So, also when we say symmetry we have to keep in mind that we are taking about symmetry in geometry and symmetry in boundary condition.

So here we have both symmetry in geometry and symmetry in boundary condition. So now because of symmetry you can solve half of the domain. Let us put, the origin here as x equal to zero whatever is the solution that we get for half of the domain the remaining this half will be symmetrical solution.

So, we can reduce the size of the problem. For analytical solution, it does not matter that much but if you are solving the problem computationally or numerically I mean reducing the total domain size will reduce your computational cost that is you will require less number of grid points to solve the problem and so on.

Computational time and computational cost will be less. So now can you tell what will be the boundary condition here? It should be partial derivative of T with respect to x equal to zero. So, at x equal to zero you have this and at x equal to L, T equal to T zero. See you can make this problem very nicely attractable by method of separation of variables by transforming even both the boundary conditions to be homogeneous.

Right, if you define theta equal to T minus T zero then this will give you theta equal to zero at x equal to zero.

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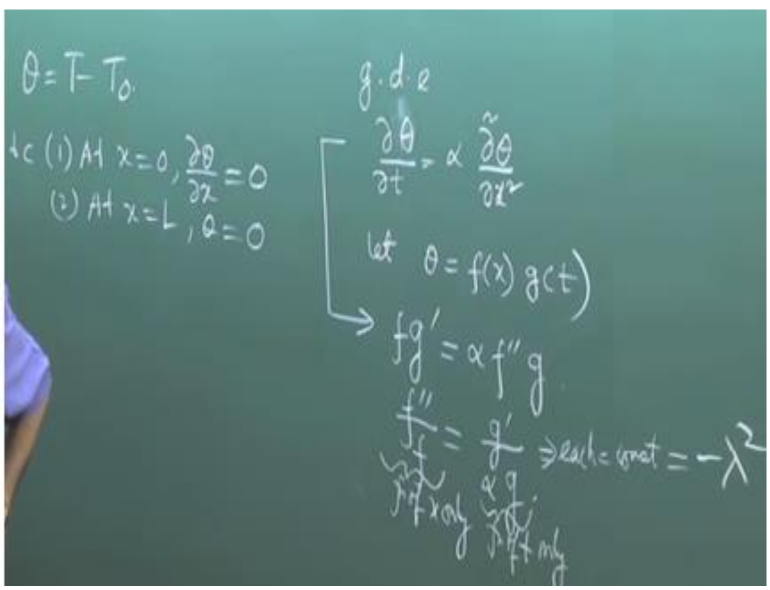

So, let us recast the problem by writing theta is equal T minus T zero. So, boundary condition at x equal to zero, At x equal to L, theta equal to zero. And the governing differential equation now, we will use the method of separation of variables. So, theta equal to let us say that theta is a product of a function of x and a function of time. So, you have g dash, fg dash equal to alpha f double dash g from this. So, f double dash by f equal to g dash by alpha g.

Now when you write this equation, this is a function of x only, right. This is a function of T only. So, function of x only is equal to a function of T only. That means each must be a constant. Now the question is whether the constant is a positive constant or a negative constant. That we have to carefully figure out. Again, if you make a mistake as I told you that there are two ways of figuring it out one way is the hard way.

That let us say, you make a mistake you will see that it will not eventually satisfy your initial or boundary conditions. But there is a simple physical way of figuring it out. See, this is dg dt, right. So, dg by g if you integrate it will be Ln of g. So, that will be equal to this alpha into the constant into T. So, what is g? G is the time dependence of the problem eventually as you allow the time as T tends to infinity what will happen?

The time dependent part of the solution will vanish because it will attain a steady state in the limit as time tends to infinity this problem will attain a steady state. Because this problem will attain a steady state what it will mean? It will mean that the unsteady part of the solution will decay with time, right. Because it will decay with time and g will an exponential function of T a negative constant will imply an exponential decay.

Otherwise it will be an exponential rise. So, therefore we can say that this is equal to minus lambda square. So, now let us apply the boundary conditions before applying the boundary conditions let us see the solution.

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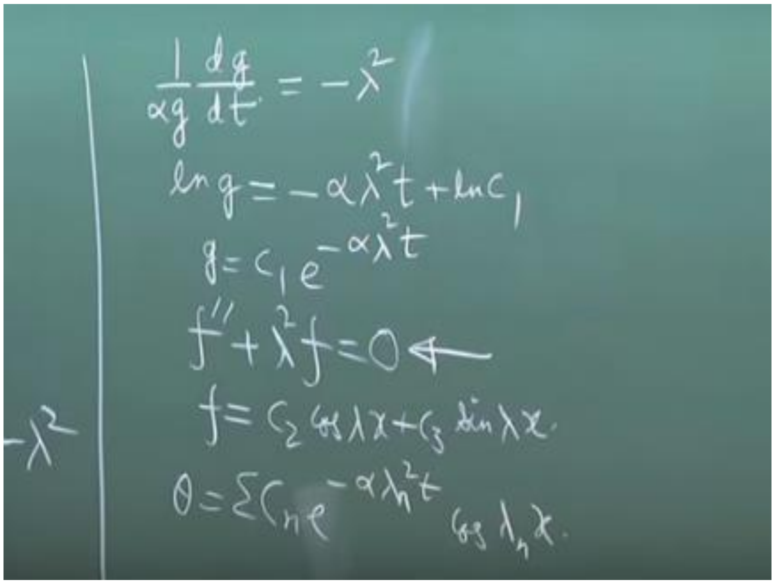

So, dg by dt, I y alpha g, dg by g is what? Ln of g is equal to minus alpha lambda square t plus ln of some constant C1. So, g is equal to C1 into the power minus alpha lambda square t. Now what is the solution for f? cos and sin, right. So, C two cos lambda x plus C three sin x. The solution is related to the product of f and g. Now let us apply the boundary conditions. Boundary condition number one at x equal to zero, delta theta dx equal to zero this boundary condition.

That means basically dx will be equal to zero. So, if you make df/ dx cos will become sin and sin will become cos, of course plus minus. I am not bothering. So, cos will become sin so at x equal to zero that term is automatically equal to zero and the derivative of this will be cos. So, at x equal to zero you have C three, x equal to zero if f has to be zero then C 3 must be equal to zero, right.

Because if C3 is not equal to zero then the entire solution is a trivial solution that is if f equal to zero solution itself is zero.

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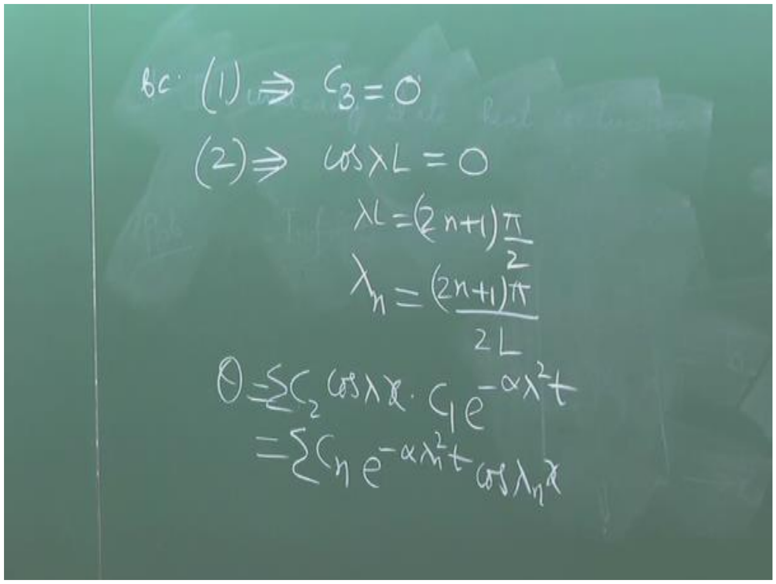

The next is at x equal to L, theta equal to zero means you have C two cos lambda L equal to zero again here C two cannot be zero because if both C one and C two are zero then what happens? Then it results in a trivial solution. So, only possibility is that cos lambda L equal to zero. That means lambda L is two n plus one into pi by two. Because there are infinite such possibly values of n you have infinite such possible values of lambda.

so, each lambda is denoted by lambda with subscript n. So, lambda n equal to two n plus one into pi by two L. This is the lambda. This is the so called IN value of that problem. Now, the solution theta is you have C three equal to zero C two cos lambda x into C one e to the power minus alpha lambda square t. This is the solution but we have to keep in mind that there are n such possible values of lambda for each value of lambda this is the solution.

So, for n such possible values you have summation of these for over n that should be the solution. That is because of the linearity of the governing differential equation. So, the total solution is a sum total of the solution for each possible value of lambda and lambda will have infinite number of possible values. So, now in place of C1 into C2 let us write this as Cn into the power minus alpha lambda n square t into cos.

So, the only part of the problems that remains is to calculate what is Cn? If we find out what is Cn then that completes the solution of the problem. So, how do we calculate Cn? To calculate Cn we will refer to this equation.

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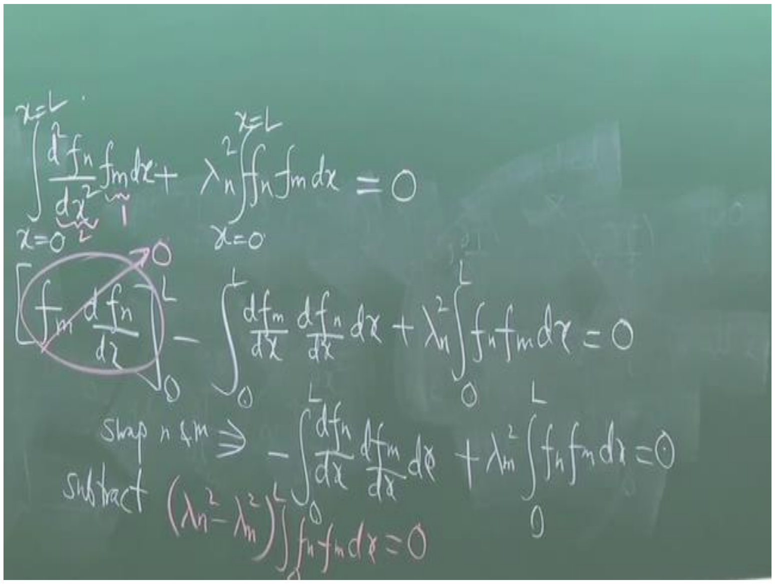

So, we write, now we basically need to derive an orthogonality condition in the context of two-dimensional steady state problem we discussed what is an orthogonality condition. And similar orthogonality condition will be derivable here so how do we proceed towards deriving the derivable remember the objective of deriving the orthogonality condition is we can isolate Cn and we can figure out that for one value of m equal to n only that is none zero.

And for other cases the coefficient of Cn becomes zero. So that out of the summation you can isolate the Cn from the summation. So, now what we did we multiplied it by fm and integrated by watts. The same thing we will do here. So dx then this is a higher order derivative so this we will put a second function and this we will put as first function. So, first function, this is second function.

So, first function into integral of the second minus integral of derivative of first into integral of the second. Now there are certain simplifications we can make see this boundary term at x equal to zero (()) (27:13) dfn dx equal to zero. Because delta theta delta x equal to zero and at x equal to L you have theta equal to zero that means f equal to zero. So that brings this entire boundary term equal to zero.

And that is why we actually had to use the homogenous boundary conditions. So that homogenous boundary condition clean up this boundary term either this equal to zero or this equal to zero either of this. These are homogenous boundary conditions. So that leads to products equal to zero and this term goes away. Now next what we do we write the same equation but swap n and m.

So, swap n and m and then subtract. If we subtract then what will happen these terms will be zero. I mean they will get cancelled and you will have lambda n square minus lambda m square integral of fn fm dx from zero to L equal to zero.

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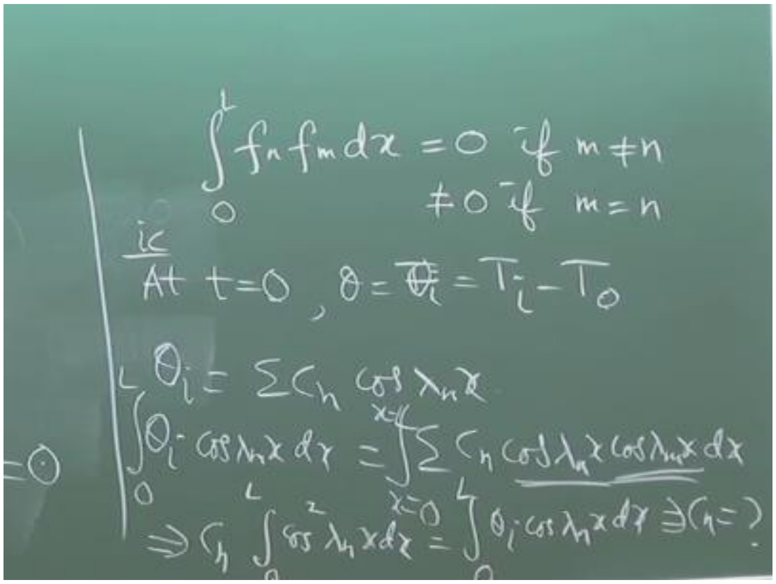

So, this leads to integral of fn fm dx zero to L equal to zero if m is not equal to n and is not equal to zero if m equal to n. which is the so-called orthogonality condition. Where fn is what? fn is cos lambda nx, right. And lambda n is given by this equation. So, we can now attempt to find out what is Cn by applying the initial condition so far as our solution is concerned we have used only the boundary condition. We have not yet used the initial condition.

So, at initial condition at time equal to zero, theta equal to say theta i which is Ti minus T zero. So, theta i is equal to summation of Cm cos lambda nx. So, what we will do we will multiply both sides by cos lambda mx, fn into fm and then integrate so theta i cos lambda mx dx. Now by the orthogonality condition this product integral of this product is zero if m is not equal to n.

Only in one case it is none zero when m is equal n.

So, this series will eventually be only one term, one none-zero term so that none-zero term. So that non-zero term if you isolate that means you can write Cn integral of cos square lambda n x dx, zero to L equal to integral of theta i cos lambda n x dx. So, this will tell you what is Cn that completes the solution of the problem because once you know Cn you can substitute that here to get theta and see this solution is very general.

Because if the initial temperature is a function of position then also you can solve this problem. If initial temperature is constant then you just need to take this out of the integral otherwise if this a function of x you just put it as a function of x I am not going into the integration of sin, cos, sin and these terms. These are all high school level stuff and we should not waste time here we should better use the time to develop physical insight now into the solution, right.

So, let us try to do that. Now let us try to see that if you want to plot the solution of this problem how will the solution look?

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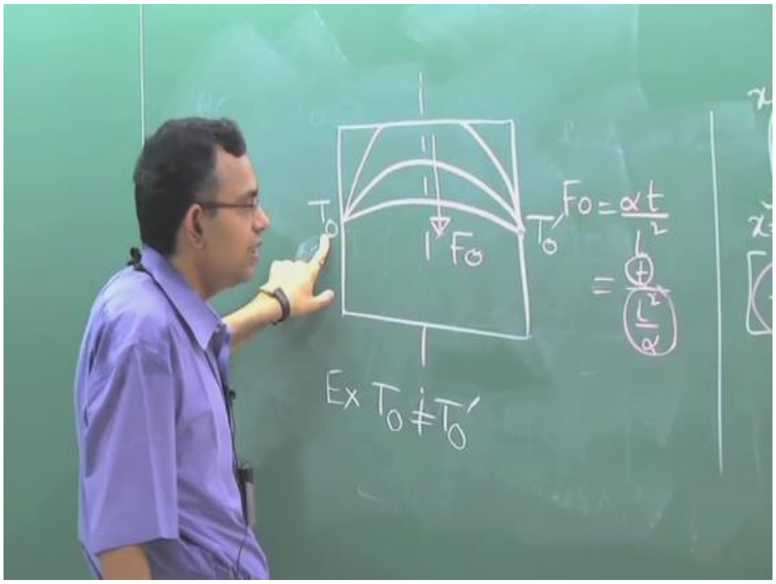

Let us say this is the domain. Of course, this is infinitely long but I have just truncated it and drawn it like this. So, if you consider the central line first of all it is the solution is symmetric with respect to the central line. So, at very early times the solution may be something like this, right. At very early times then the solution becomes this as you progress with time and eventually if you go for time tends to infinity what will be the solution?

It will be uniform throughout, right. Because there is no heat source so as time tends to infinity then the steady state solution will be what? A constant so this is progressing time. Initially there will be a difference between see the slab has an initial temperature and you are subjected into a boundary condition so what will happen? There will be a difference between these two and that difference will trigger some heat transfer but that difference will be slowly nullified as you are proceeding with time.

In the limit as time tends to infinity the time dependent solution goes away. So, when you have the time dependent solution gone then you will be basically having just the special dependence of temperature. Now this you can blot also in terms of none dimensional number we have discussed about this which is Fourier number. So sometimes to make a none dimensional representation instead of T you plot it as a function of alpha T by l square.

We discussed that alpha T by l square which is the Fourier number, this is t by L square by alpha. So, this is the time divided by diffusion time, heat diffusion time. In fact, the entire solution the time dependent solution you can write as a function of two none-dimensional numbers the Biot number and the Fourier number. You should try to make an attempt to write the solution in terms of the none dimensional numbers Biot number and Fourier number.

We have defined both of these numbers in our previous lecture. The next point is that what will be the change in solution if we change the boundary condition. For example, we have considered both the ends to be at the same temperature T zero but let us say this end is at a different temperature then this. Then you cannot directly use the method of superposition of variables. So, what we have to do is basically you divide the problem into two problems.

One problem is when both the ends are at the same temperature then you solve in this way and with that super imposed another problem when there is a differential of the temperature of the two ends and steady state solution of that. So, let us say that end is T zero and this end is T zero dash so one problem is that you solve our steady state problem when this T zero let us say an example with T zero not equal to T zero dash.

So, you can solve a steady state problem by considering this radiant and there you do not consider any unsteadiness the entire unsteady as you dump on other problem where you have both ends at the same temperature and then that total solution is a linear superposition of that two solutions where for the second problem where you show the unsteady problem you can use this method. For the steady problem, it is a simple one-dimensional steady state heat conduction.

So net solution is the sum total of the two solutions you have to be a little bit careful of how to give initial condition for the unsteady problem but that I leave on you as an exercise. Let us say that instead of having a given temperature boundary condition we have another problem where we use the given convective heat transfer coefficient boundary condition.

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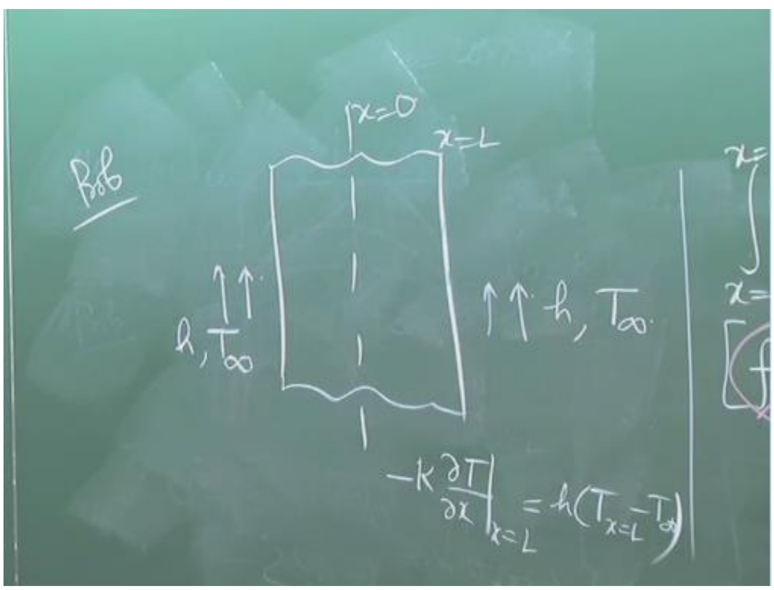

Let us say same infinite slab. Let us say that some fluid is flowing outside the slab with heat transfer coefficient h, and temperature T infinity. So, the only difference from the previous problem is that there is a different boundary condition. Previous problem what type of boundary condition it was? It was a Dirichlet type of boundary condition where the temperature was specified. What type of boundary condition is not there for this problem?

What is the boundary condition here? What is the boundary condition at this surface? So, physically what is the boundary condition? Whatever is the rate at which heat is transferred here at the same rate heat is transferred from here to the surroundings. Let us say that the system is at a greater temperature then the surroundings. So that means minus this is x equal to L, this is x equal to zero, minus K delta T delta x, at x equal to L is equal to h into T at x equal to L minus T infinity.

Now again the confusion always remains is that we are assuming that it is an unsteady state problem still we are assuming that the rate at which heat is transferred is the same rate at which heat get transferred to the surroundings. So is there any kind of inconsistency between this and the consideration of unsteady state. There is no inconsistency because a surface cannot store thermal energy does not matter whether it is steady or unsteady surface cannot store thermal energy.

So, at whatever rate heat is transferred to the surface at that instance at the same rate heat will be transferred from that surface. So, this boundary condition is the only difference between the previous problem and this problem. So, let us workout that let us try to follow the solution of the previous problem and then see what are the changes? So, at x equal zero delta theta delta x here we can also solve half of the domain and at x equal to L.

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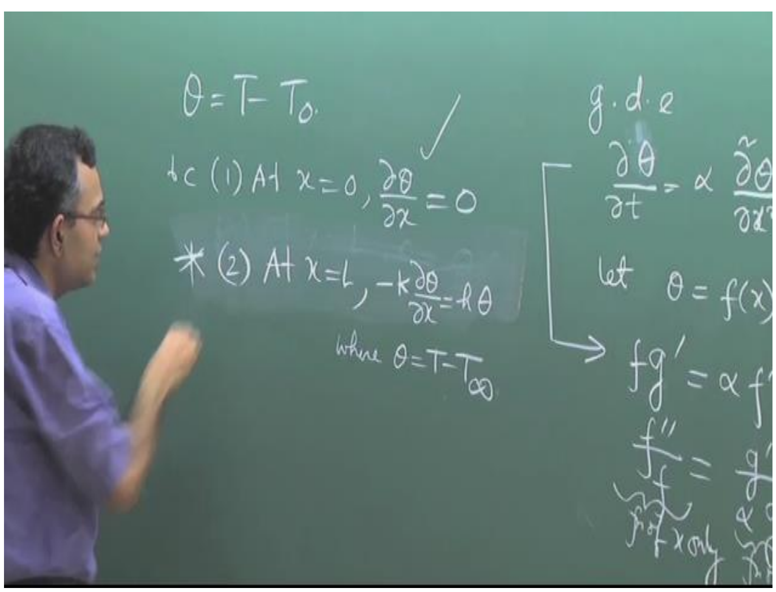

Here, we define as T minus T infinity, right. Minus k delta theta, delta x equal to h into theta, theta is T minus T infinity. This At x equal to l. So, this is the only change and let us quickly see how this change is reflected in the solution. So, this part of the solution remains the same because this is before applying the boundary conditions. So before applying the boundary conditions whatever was the solution general solution is the same.

So ultimately your general solution is when you apply the boundary condition so theta is summation of Cn e to the power minus alpha lambda n square t into cos lambda n x. How did we get this solution we get this solution by substituting this boundary condition, right and this boundary condition does not change from the previous problem to this problem so this form is fine but lambda n is different?

So how do you get lambda n? You have to use the second boundary condition.

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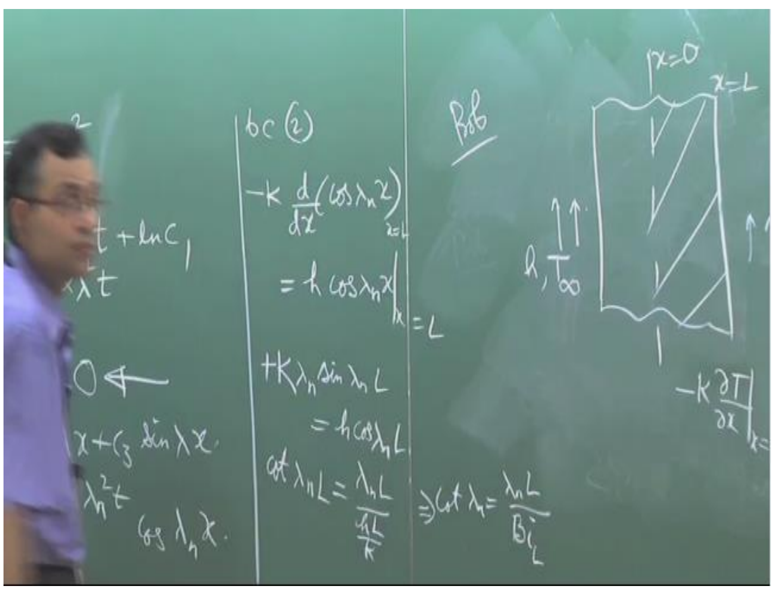

Minus k remember when you are differentiating with respect to x only f is important not g because g is not function of x, right. So minus k this will be basically df and dx, df and dx is d dx of cos lambda nx and this is h into theta, theta will eventually be f, g gets cancelled from both sides. So minus k minus will get cancelled so k lambda n sin lambda n L is equal to h cos lambda n L. So, you can write cot lambda n L equal to lambda n L by hL by K, right.

So, what we have done is we have multiplied by L because we want to use the none dimensional number the Biot number. Biot number is hL by k. So, cot lambda n L is equal to lambda nL by Biot number based on L. So, this is the iL value. See the iL value determination is not very straightforward because this is a transcendental equation. If you want to get a graphical feel of the solution then we can draw it here. So, this is lambda nL.

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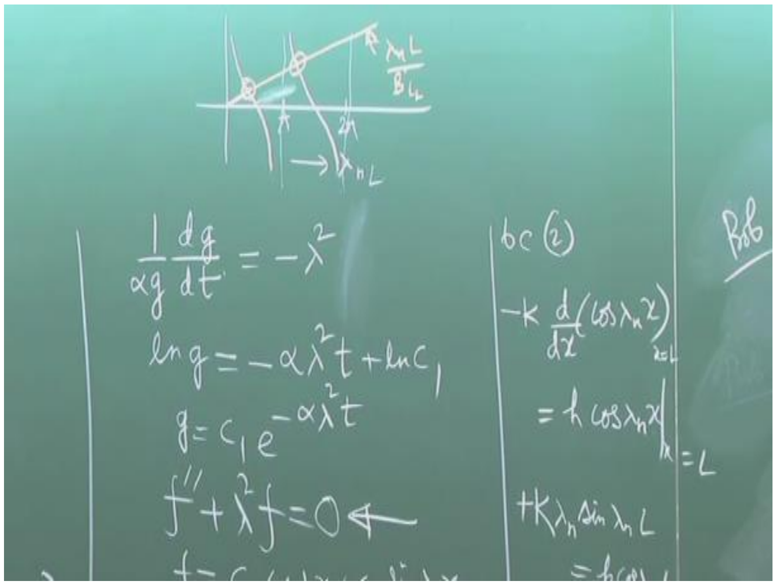

Pi two Pi like that we can sketch. So, cot lambda nL will be something like this, right. In this way, you will have infinite number of such segments and the other part of the solution is this is lambda nL by Biot number. This is y equal to mx, a straight line passing through the origin. So, this point of intersection are the iL values. So, if you are familiar with Met lab what you can do, in Met lab.

You can draw the graph of cot lambda L and you may or say cot x and you may draw the graph of x/ Biot number and find out the points of intersection. Those points of intersections are the Eigen values of this problem. For the previous problem this was like straight forward two in plus one into Pi by two. For this problem, it has to be obtained from this. Now what about the constant SC?

So, we will quickly see how to get constant Cn because we have got lambda n see all these problems are Eigen value problems just like in Linear algebra you require, in Eigen Value problems you require to find out the Eigen Value problems and vector. Similarly, differential equation you require to find out the Eigen Value and Eigen Value function. So, Eigen Value we have already determined now the Eigen functions are related by orthogonality condition.

And we have to derive the orthogonality condition here.

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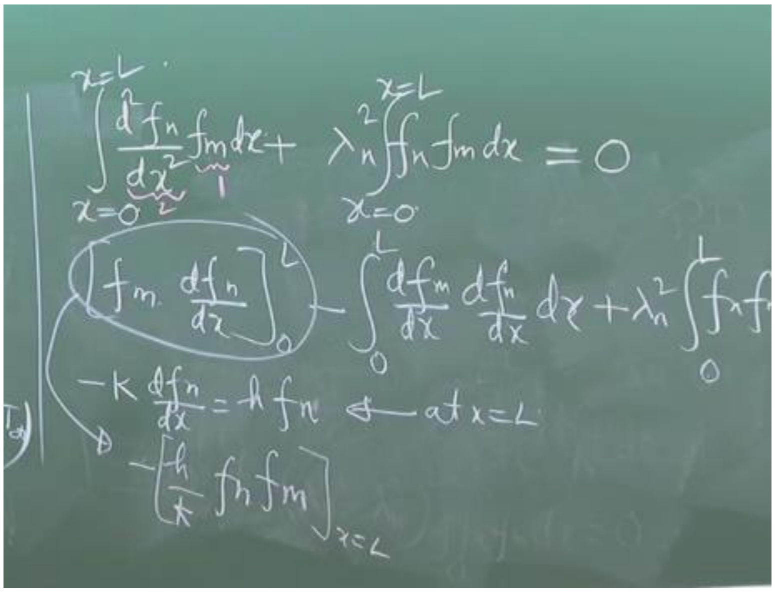

So, we will use this fm into dfn/dx. Why I am separately here? Because here the boundary condition is different so we have to make sure that we derive the orthogonality condition freshly from the boundary condition of this problem. So, what is the boundary condition of this problem at x equal to zero it is the same. So, at x equal to zero this term is zero. But x equal to L this term is not zero, right. At x equal to L what is this term? Minus k dfn/dx equal to age into fn.

This is minus k delta theta/ delta x equal to age into theta. This is at x equal to L. So, what will this become in place of d fn/dx we will have minus h/k fn fm, right. Because at x equal to zero any way df/dx is zero at x equal to L it is not zero but dfn/ dx is related to fn. So, we have written this now what we will do we will swap n and m and subtract and you can see because of the symmetry in fn fm these terms will be cancelled.

So, that is why see eventually sometimes the ignorance is a blessing if you do formula based study then you need not go through it you eventually know orthogonality condition will come some way or the other. So, you eventually write only the orthogonality condition but there may be a tricky boundary condition when orthogonality condition itself is not satisfied and then you cannot use the separation of variable method to solve that problem.

So, for every typical boundary condition you have to look for the orthogonality condition. Here fortunately the orthogonality condition can be obtained because when you swap fn and fm and subtract this term will be zero. So, eventually you will get this. So, although you get the same final answer for the orthogonality condition so far as the orthogonality condition is concerned but the root of getting that orthogonality condition is different for different problems.

Then the remaining solution is very similar to the previous one so you can get Cm from here just lambda will be different from the previous problem. So, only difference with the previous problem is Eigen value otherwise the structure of the problem is very, very similar. So, to summarize what we have discussed today some unsteady state problems where you have problem solved by the method of separation of variables.

So, we have addressed the method of separation of variables with the help of some examples of infinite slabs. We will later on consider some of examples which we call as semi-infinite problems and we will take it up in the next lecture. Thank you very much.