Conduction and Convection Heat Transfer
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Lecture - 13
Two-dimensional Steady State Heat Conduction

So far, we have discussed about various cases of 1 dimensional steady state heat conduction. So, with us Prof. Som has discussed about those and today we will start with 2-dimensional steady state heat conduction. So, 1-dimensional problems have been discussed now we will discuss about 2-dimensional problems.

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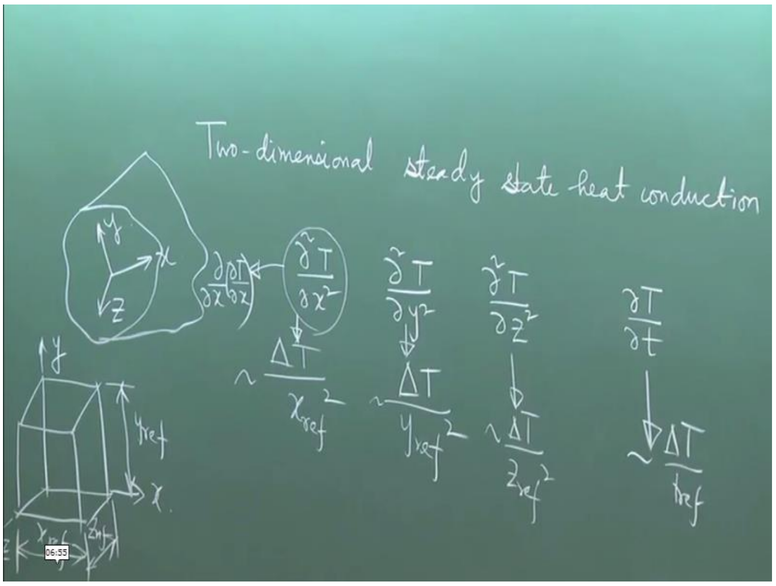

Now let us give a practical perspective to these type of problem. So, whenever you are solving a problem like in say Practical Engineering Industry or may be life science or whatever, nobody tells you that the problem is 1 dimensional, 2 dimensional or 3 dimensional. So, how do you assess whether a problem can be treated as a 1-dimensional problem, 2-dimensional problem, 3-dimensional problem.

Whether the problem is a steady state problem or an unsteady state problem. This must come from our own judgment and once that judgment is applied then we can apply the appropriate mathematical technique to solve the problem. So, let us think of that there is an arbitrary domain like this, some 3-dimensional domain. So, in this domain, you could have let us think of 3 octagonal directions x, y and z like this.

So, if you look at the heat conduction equation which we derive, you will figure out that in the heat conduction equation you will have terms like, these are special gradients and you also have unsteady term where the temperature variation with time govern by this type of a term. Of course, row c all these things are there I am not writing the full equation, I am just writing the important derivatives.

Now, let us say that you have a problem like this where there is a temperature difference imposed across the length of x reference. X reference is the length along x. What is x reference? So, let us may be draw cuboid like this. This is x reference, length along x. So, what is this basically? So, it is some characteristic delta T divided by some square of some characteristic length, that is the order of magnitude of these terms.

So, when we say order of magnitude by that we mean that what is like let us say maximum order. So, let us say that if x reference be the length then this is along x. Similarly, if the same delta T is imposed along y and let us say y reference be the length scale along y. Let us say this is x, this is y, this is z. So, this is of the order of remember this is not equal to this. It is just a rough estimate of its order of magnitude. So, this is delta T by x reference square.

This is delta T by y reference square. This is delta T by z reference square and this is T reference is the characteristic time scale of the problem that is the time over which you are studying the problem. It may be minutes, it may be hours depending on the time over which you are studying the problem. Now, the first keyword is steady state. I will discuss about this keyword very carefully.

So, when will a problem, a heat conduction problem be treatable like a steady state problem. When can we treat it as a steady state problem? So, of course from just a simple mathematical perspective we can treat it as a steady state problem when this term is not important right. So, when is this term not important? This term is not important when with respect to this T reference is large. That means what? That means the ratio of these two is small.

Now, how large, how small comparable to what and what we will come into that when we write the actual equation. But I am first trying to give you a very rough feel. As an engineer, it is very important that you get first an approximate rough feel for the problem and then you get into the mathematics rather than start with the bull work of mathematics from the beginning. So, if this is large and we are interested about characteristics of the large timescale. Then, it may not be important that we consider the unsteady problem.

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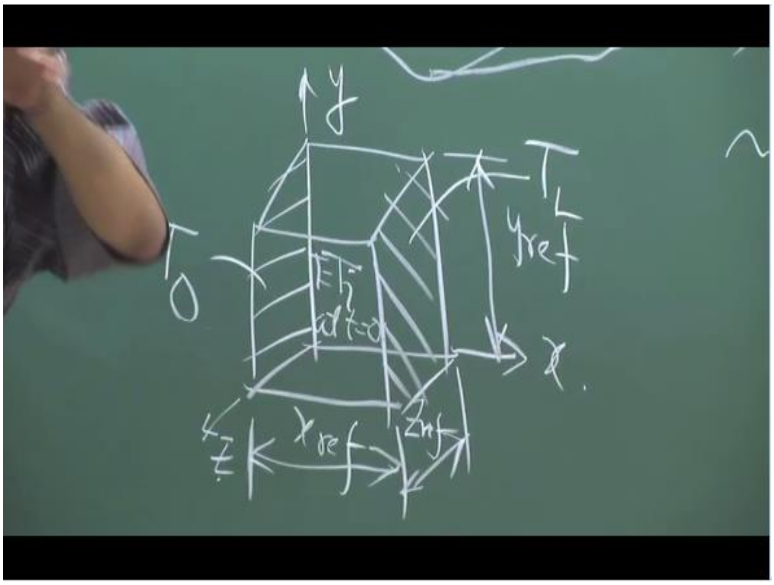

Remember in the world, all problems are unsteady in energy because let us say that you have these kind of a situation and initially there is a particular temperature within the domain. Now you suddenly subject this body to particular boundary conditions. So, let us say you subject these 2 temperature T 0 and you subject these 2 temperature T L. T is equal to T I at time equal to 0 for all x, y, z.

So, in this problem the temperature from its initial state will start changing. It will start changing because of the impose boundary condition. So, what does the boundary condition physically do? A boundary condition helps the disturbance to propagate from the boundary to the interior of the domain and how is it possible? It is possible because of some material properties.

Here, in heat conduction the relevant material properties are K, row and CP or when you are talking about a solid it does not matter whether its CP or CV simply C. So, K, row and C. Now, if you are thinking about the change with respect to time may be there is some initial change because of the imposition of the boundary condition, but after that the changes stop taking with respect to time. Then, we say that steady state has attained.

So, steady state has attained when there is no further change with respect to time. When there is no further change with respect to time then what happens then the unsteady term the gradient with respect to time is no more important and then you can treat it as a steady state problem. Possibly beyond a critical time when it achieves steady state then you can use steady state formulation, but below a critical time when there is a continuous change of temperature within the body as a function of time then you have to consider unsteady state.

So, no problem is perfectly steady or unsteady state. Depends on your interest over the act, what timescale you are interested. If you are interested over the timescale over which the problem has already attained steady state then you go for steady state analysis. This is the first point about steady state. The second point about steady state is that there is a tremendous misunderstanding between 2 terminology, steady state and thermodynamic equilibrium.

So, in many scenarios people say people use these terms interchangeably. People say that the system has attained steady state and the system has attained equilibrium simultaneously.

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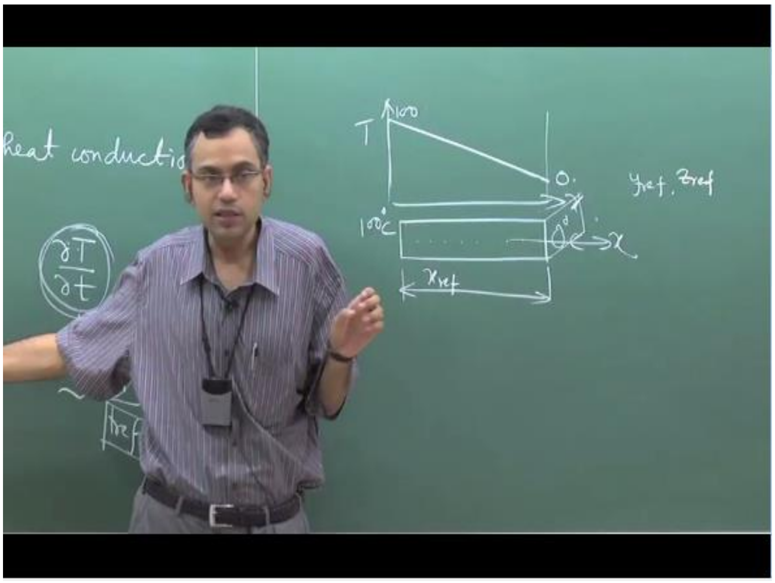

Now, let us take an example, let us say I give an example of a 1-dimensional problem just to illustrate the concept, but same concept can be applied to multidimensional problems. Let us say there is a rod, this end is immersing steam at 100 degree centigrade, this end is immersed in ice bath at 0 degree centigrade, ok. So, the question is, is this an equilibrium? can this be in steady state.

So, in terms of equilibrium, when do we say that the body is in equilibrium or thermodynamic equilibrium? It is in thermal equilibrium, mechanical equilibrium and phase and chemical equilibrium. When all these things are satisfied we say that a system is in thermodynamic equilibrium. So, one of the requirements is thermal equilibrium. What is thermal equilibrium?

Thermal equilibrium means the entire system is at a homogeneous temperature. Because if there is some gradient in temperature then that will trigger a local heat transfer to take place. So, if the system had to be in thermal equilibrium then throughout this system the temperature had to be something, some value, but here I mean from simple analysis you can say that if you plot temperature as a function of may be x, this is 100 and this is 0, ok.

So that means the temperature within the domain continuously varies with x and it is not a constant. That means this is an example which shows that steady state need not necessarily mean thermodynamic equilibrium. There is a difference between steady state and thermodynamic equilibrium. So, when we are talking about steady state, we are not caring whether it is in thermodynamic equilibrium or not. That is not of interest to the study of heat transfers.

That is of interest to the study of equilibrium thermodynamic, but not of heat transfer. So, now the question is dimensionality of the problem. So, we are clear that out of these 3 terms if any one of the terms dominates over the other terms. Let us take an example x reference much much less than y reference, z reference, ok. So, if that be the case then out of these 3 terms which term is most important?

The first term right and then it is as if like a 1-dimensional problem. Now, let me bring out a fallacious situation here. Look at this problem, ok. In this problem, this is x, this is x reference right. Let us say this is a bar which has some dimension along y and z, y reference, z reference all those are there, those are not drawn and 1 dimensional problem is solved. So, in these problem by the shape of a rod or a bar it is quite clear that x reference is much much larger actually the other way.

It is much much larger as compared to y reference or z reference, but still we are assuming this as a 1-dimensional problem along x, y. You understand my question? Here along x, the length is much larger. So, ideally if you go by this logic then it should be like the y reference and the z reference being small, it should be at least a 2-dimensional problem in the y, z plane, but we are considering a 1-dimensional problem along the x direction.

So, what is the inconsistency? If at all there is any inconsistency. See, here we have assumed that the same delta T is applied in all directions right. In this problem, the delta T is applied only along x direction. In other directions, no delta T is applied. So, it is not just the length. You also have to see what is the delta T that is imposed right. Here, the delta T is imposed along x, so the temperature gradient will be created along x, ok.

Now, it is quite clear that when we consider steady 1 dimensional, 2 dimensional, 3 dimensional, when we consider unsteady all these things must be very clear before we solve a problem. See when we solve a problem what we essentially try to do is we try to make a mathematical model. What is a mathematical model? A mathematical model is an abstraction of the physical reality.

It is not exactly what is the physical reality right. Otherwise every problem has to be solved 3 dimensional or unsteady, but mathematically when we simplify a problem we take into account some practical considerations and simplify the problem to an extent that it is mathematically simplified to the extent possible.

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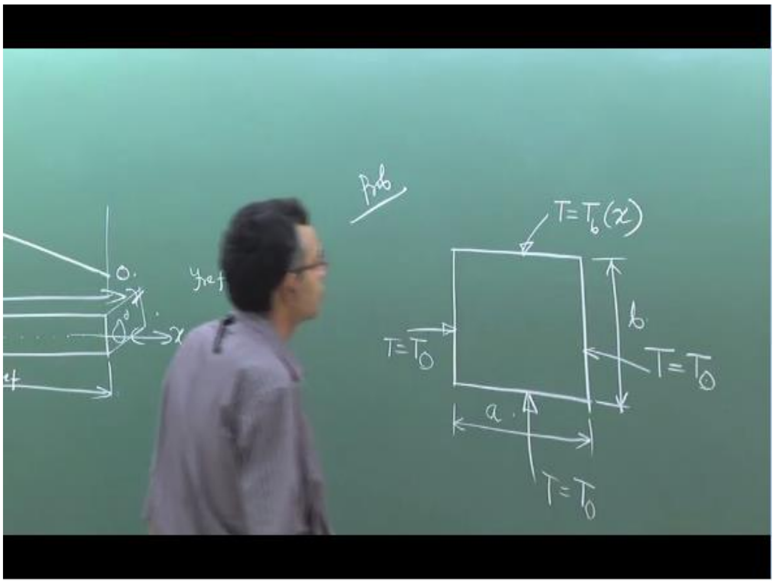

So, we will straight away get into a problem of a 2-dimensional scenario which will be our agenda for today's discussion. So, let us take a problem like this that you have a domain, rectangular domain with length a long x, length b along y, ok. Now let us assume that we are interested about a solution when steady state has already been attained. Later on, we will see that if you are interested for a solution when steady state has not yet been attained.

How to go for the solution, but to begin with let us assume that steady state has already been attained and we are interested for the solution. So, now can you tell whether it is a steady state problem and the largest dimensionality of this problem is 2 right. It is not a 3 dimensional problem because everything is planar and all the temperature gradients are imposed in the x, y plane.

Now, a similar problem can you think of when the problem looks like 2-D, but actually it's almost a 1-D problem. This is a 2-D problem. Why this is a 2-D problem? It is quite clear that you have a temperature gradient here. You have a temperature gradient here, you might argue that where is the temperature gradient here? This temperature and this temperature is the same, but that shows that not everywhere within the domain the temperature is same.

But there is a periodicity in the solution. Because there is a temperature gradient along y and the governing equation has some total of the effect of temperature gradient along y and temperature gradient along x. So, to make the sum total equal to 0, some temperature gradient along x should come right. So, it is a 2-dimensional steady state problem. We will solve this problem today.

But before that I want to capture your attention by looking into a scenario, when the scenario the schematic the drawing looks very similar to this, but the problem is actually a 1-dimensional problem, ok. Let us try to find that out.

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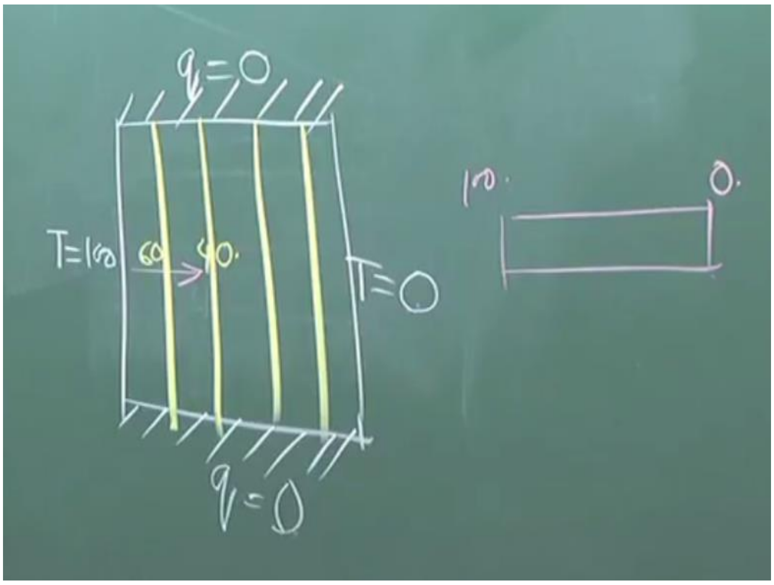

So, let us say that these 2 boundaries are insulated. In heat transfer, insulated boundaries are often drawn by this hatched lines, ok. You have seen in thermodynamics also that insulated boundaries are often schematically represented like this. So, these are insulated that means there is no heat flux. Let us say this is T equal to 100, this is T equal to 0, ok. So, can you sketch the isotherm lines here. What are isotherms?

Isotherms are lines of constant temperature. So, across the isotherm you have the temperature gradient, ok. So, let us try to draw it with a different color. So, you can see that when you look into a problem, always look into 2 things, one is geometry because of course geometry dictates the dimensionality. If you have already a 2-dimensional geometry there is no question that the problem will be 3 dimensional.

If you have a 2-dimensional geometry the maximum dimensionality of the problem is 2 dimensional, but you also have to look into the boundary conditions. So, here the boundary conditions are such that see no heat is able to get transferred along these directions. So, the primary direction of heat transfer is like this. Because the primary direction of heat transfer is like this, it is almost like the same problem where you have these 1-dimensional rod, here 100 and here 0, ok.

And the isotherms you will be getting will be like this. So, this is 100, let us say we draw it as some intervals so let us say this is 60, this is 40, I mean in this way it goes to 0 here. So that this line is the line where along which everywhere temperature is 60. So, you can see that this lines are vertical lines that means the gradient is along the horizontal. So, although the geometry is 2 dimensional, the boundary condition says that the problem is primarily a 1-dimensional problem.

Of course, again it is not perfectly a 1-dimensional problem. If you are a great perfectionist, it is a 2-dimensional problem. But for all practical purposes, the temperature gradient along y is not of great interest here. The great interest is temperature gradient along x. So now we will come back to the solution of this problem. So, let us make some assumptions. So, what are the assumptions? 2 dimensional steady state constant properties.

So, if we write the governing equation. Let us write the full governing equation.

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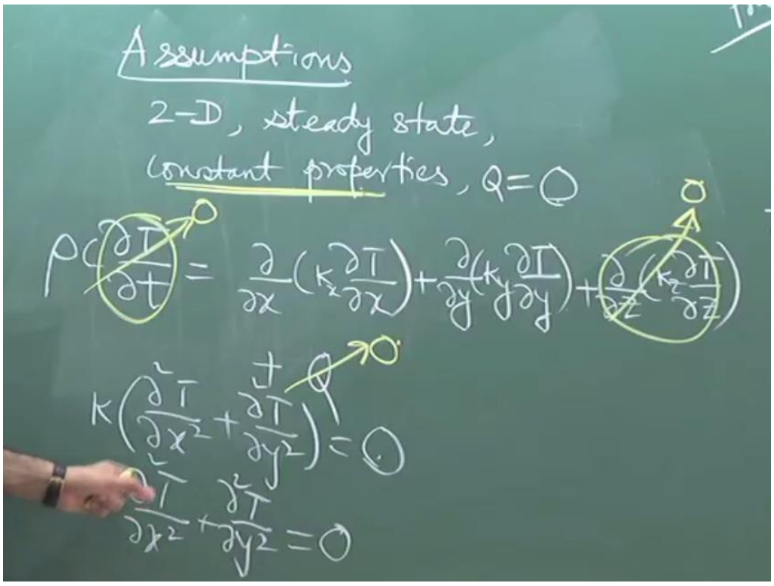

This is a solid so any term related to compressibility is not coming into the picture. Now, first of all it's a 2-dimensional problem, because it's a 2-dimensional problem - this term is not important. Then it's a steady state problem, so this term is 0 ok. Now once you say steady state problem, when you talk about constant properties, it is not necessary that all properties have to be constant just thermal conductivity constant is good enough.

Because once you say that it's a steady state problem row and C are no more important. So, whether they are constant or they are variable that will not make any sense. It is important that what K is and of course constant properties means that K is constant and then we will assume there is no heat generation. So, when K is a constant, K will come out of the derivatives, so you will have. Because K is not equal to 0, so you must have.

This prototype is not just important in heat transfer, but in any branch of applied physics and engineering and this is known as Laplace's equation. Of course, this is Laplace's equation in 2 dimensions. If you have another term like this for z that is Laplace's equation along 3 dimensions. Now we will try to solve this equation. Now there are many methods of solution and each method is very interesting by itself.

These bring the most elementary level course on conduction heat transfer. We will not start with very elaborate method of solution. We will start with one of the simple, but very insightful method of solution which is called method of separation of variable, but method of separation of variable will not always work and it works we will see I will derive the equations in detail and my teaching philosophy is very simple.

Whatever I cannot derive at your level, I will not teach. Whatever I can derive at your level and make you understand that only I will teach because that is what what you will learn. Not that you learn thousands of formula and reproduce without understanding where from it comes that is not the education that I want to impose on you. So, I will consider a simple case without any prejudice in mind we will start and we will solve the entire problem.

But I would like to let you know that method of separation of variable does not work always. So, when method of separation of variable does not work always. When does it work and when it does not work? We will look into it in detail but one of the important requirements like here how many boundaries are there? There are 4 boundaries right in this domain. So, in this kind of a problem, it is important that out of these 4 boundary conditions at least we can map 3 homogeneous boundary conditions.

So, what is a homogeneous boundary condition? Homogeneous means right hand side equal to 0. I mean of course this is not a mathematical definition of what is homogeneous, but this is a practical way of looking into it. Something right hand side equal to 0 that we call as homogenous. So, here T 0 may not be 0, right?

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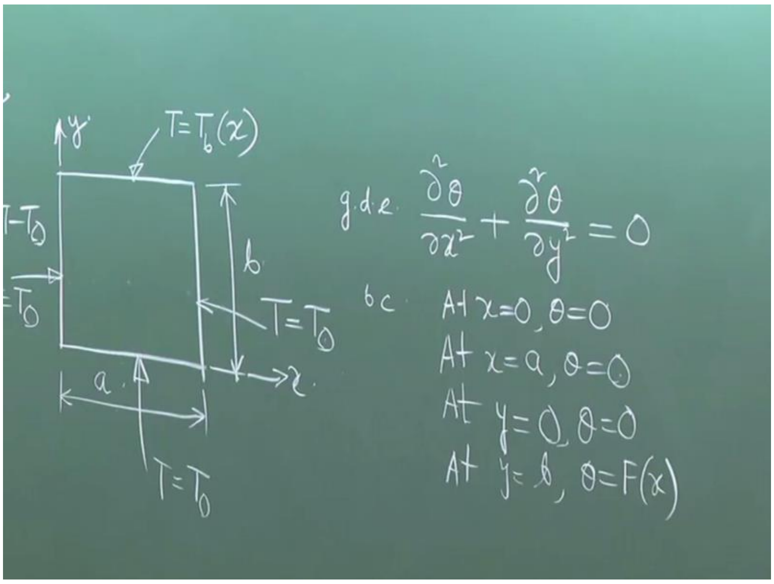

So, to convert this to 0 what we can do? We can just redefine; we can make theta is equal to T minus T 0 right. We can define in this way. See that is why deliberately these kinds of boundary conditions were given. I will show you that is 1 of this is relaxed how will you do? Ok so let us say this is T 0, this is not equal to T 0. How to solve that problem? I will give you some idea later on.

But, you can use a technique called as super position to solve that problem using the method of separation of variable itself. But, we will start with the simple one where we define theta is equal to T Minus T 0. The reason for defining this is to make all these 0 as boundary condition. So, the governing equation does not change. So, in separation of variable, what we assume? So, let us define the problem properly.

So, this is the governing differential equation. Boundary condition - so let us setup the x and y axis:

1. At x is equal to 0, theta is equal to 0.
2. At x equal to a, theta is equal to 0.
3. At y is equal to 0, theta is equal to 0.
4. And at y is equal to b, theta is some function of x.

So governing equation and boundary condition - it defines the physical problem.

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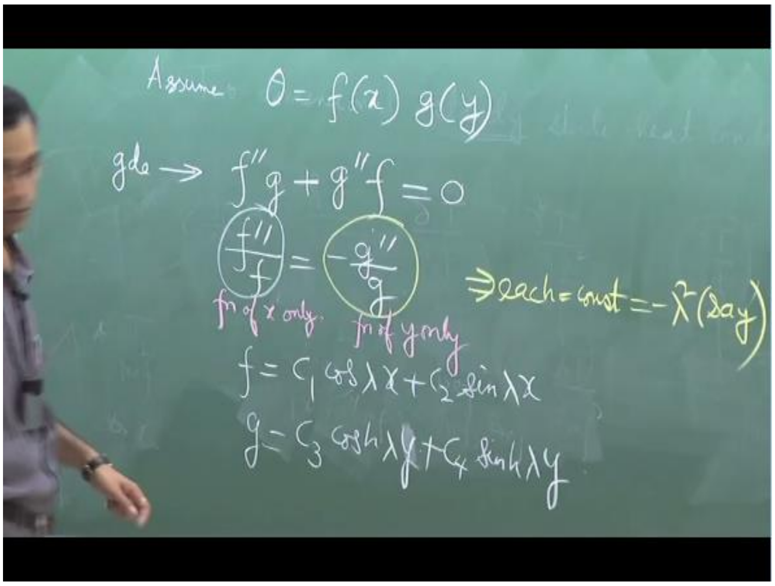

So, in separation of variables, what we do? We assume again I am repeating that in all cases you cannot assume this. It depends on the boundary conditions and we will see for what type of boundary conditions you do this. If you cannot assume this. What are the other techniques? There are other analytical techniques somewhat more cumbersome, but it is more straight forward to solve this problem numerically.

If you want to know how to solve this problem numerically for that you have to study "Computational Heat Transfer and Fluid Mechanics" or CFD. Now, I mean in all the books of "Heat Transfer" there is some elementary computational or numerical parts. If you are interested you can look into that but, I believe that is too elementary. If you are serious about learning how to solve this equation numerically get hold of any CFD book.

Or you can consider the online video lectures of CFD. There is one NPTEL video lecture of CFD by me. You can consider that if you are interested. That how to solve these equations numerically, but I am not getting into that here. In most part of this course, we will try to go for analytical technique. These are classical mathematical techniques which will give you a good grasp on the mathematics of the problem, which is necessary for many physical insights that you want to develop.

So, theta is equal to f x g y. So, from that equation what you (()) (35:45) So the second derivative with respect to x will mean f double dash g right. Dash is derivative with respect to x plus g double dash f is equal to 0. This is straight away from the governing differential equation. So, you can write f double dash by f is equal to minus g double dash by g right. See if somebody ask you that why do you go for method of separation of variables?

You should not give this answer because this was taught to me in the class that is why I am interested about method of separation of variables. So, method of separation of variable has one big issue that is why it is so important. It allows you to convert a PDE to a ODE. It allows you to convert, if it is applicable it converts a Partial Differential Equation to an Ordinary Differential Equation.

And there are much more elaborate and commonly available techniques for solving ODEs than PDEs. So, converting PDE to ODE there are several other techniques. We will see later on that there is a method called a similarity transformation that also does it. We will look into it when we solve unsteady problems, but the whole idea is that there are whole classes of problems or whole classes of mathematical techniques which allow you to convert the Partial Differential Equation to Ordinary Differential Equation.

So, this is one attempt to do that. How we do that? This is a function of what? x right. So, this is function of x only. This is a function of what? y only. A function of x only is equal to a function of y only for a general function. It is possible only when it is a constant right. Otherwise, it is not possible. So, this implies each is equal to constant is equal to minus lambda square say.

So, question is and a very big question is that why do we take this as minus lambda square and not plus lambda square, right. Always there are 2 ways of learning, 1 way of learning is I mean somehow you have got the idea. Another way of learning is learning through hard way and that is not very bad actually. Sometimes we undermine it, but it is not very bad. So instead of minus lambda square you may substitute plus lambda square.

And see that what kind of mess in which you come up while solving the problem, but that is the hard way of learning. But an insightful way of learning is also there. So, when you substitute it as minus lambda square. See what kind of solution you expect along x? Physical intuition. For this problem what kind of solution you expect along x? You expect a periodic solution along x because it is repeating again right.

So periodic means you have sine, cosine type of variation. So, if you put minus lambda square here, f double dash plus lambda square f equal to 0 will give sine and cos as the solution for it right. So, f will be C 1 cos lambda X plus C 2 sin lambda X right. This is straight forward homogeneous second order Ordinary Differential Equation in high school level you have done this. Then what about g? So, g will be exponential right.

E to the power x type of form, e to the power lambda X and e to the power minus lambda X. Sometimes for easy mathematical manipulation we instead of using e to the power x we use sin hyperbolic and cos hyperbolic x. Those are also linear combinations of e to the power x and e to the power minus x. Like you have solved fin problem. You have seen that in problems in fin you could write the solution in terms of e to the power mx and e to the power minus mx.

But usual it is written in terms of sin hyperbolic and cos hyperbolic just for mathematical manipulation. You can also write in terms of e to the power, there is no problem with that.
So, g is equal to C 3 cos h lambda Y plus C 4 sin h lambda Y. Now we will try to apply boundary conditions and solve for this constants to the extent possible. Let us give some numbers to the boundary condition. This is 1, this is 2, this is 3, this is 4.

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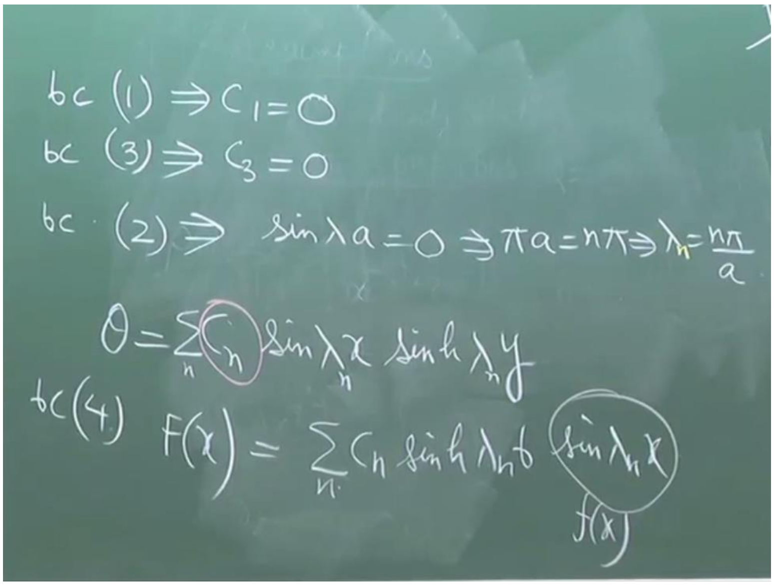

Boundary condition (1): At x equal to 0 theta equal to 0. Theta equal to 0 means f equal to 0. So, at x equal to 0 f equal to 0. So which term will be zero? C 1 will be 0 right. Now we apply the boundary condition number (3): At y equal to 0 theta equal to 0, so that means C 3 equal to 0, ok. So, the solution has got read of the Constance C 1, C 3. You have 2 more. Now you apply the second boundary condition (2):

At x equal to A, f equal to 0 means C 2 sin lambda A equal to 0 right. So, there are 2 possibilities, one is C 2 equal to 0 and other is sin lambda A equal to zero. What we do not take C 2 equal to 0 why? If we take C 2 equal to 0 the total f becomes 0 and theta becomes 0 that is a trivial solution. A trivial solution is theta equal to 0 everywhere. So, we are interested for a non-trivial solution and so for non-trivial solution sin lambda x equal to 0.

This is called as IN value problem in differential equation. Just like you have IN value problem in linear algebra, so for non-trivial solutions you have certain lambdas which give non-trivial solution to a system of algebraic equations. Here similarly you have a non-trivial solution not algebraic equation, but of a differential equation. So, these are called as IN value problems in differential equations.

So, you have sin lambda A equal to 0. So, lambda A is equal to n pi. So, lambda is equal to n pi by A. So how many possible values of lambda are there? Infinite number of possible values depending on infinite number of possible values of n. So, to mark that we sometimes give a subscript n to lambda to indicate that there are numerous infinite number of values of lambda.

So, theta when we write, so it is basically C 2 into C 4 let us call it C n sin lambda X sin h lambda Y. What is C n? C n is C 2 into C 4. Now for each value of lambda you get a solution like this. How many such cases are there? Infinite number of such values of lambda are there. So, you will get infinite number of such terms and the general solution is the sum of all those terms why? That is because of the linearity of the governing differential equation.

If theta is equal to theta 1 is the solution and theta is equal to theta 2 is the solution then theta is equal to C 1 theta 1 plus C 2 theta 2 is also a solution for this. So, in this way for lambda equal to lambda 1 you have a solution lambda 2, lambda 3 in this way all those terms sum together will also be a solution. That is because of the linearity of the governing differential equation ok. So, this is summation over n.

So now our task remains that how to find out this C n because once we find out this C n then that completes the solution of the problem. Only one constant means one set of constant because C n will be a function of n of course. So now we have only one boundary condition that we have yet not used. So, the boundary condition number (4): Theta is equal to f x at Y equal to b so summation of C n sin h lambda n B into sin lambda n X.

Remember this is what is f x. Now how to get this C n. So, we will see how to get this C n. Let us consider this equation, which I have marked by arrow.

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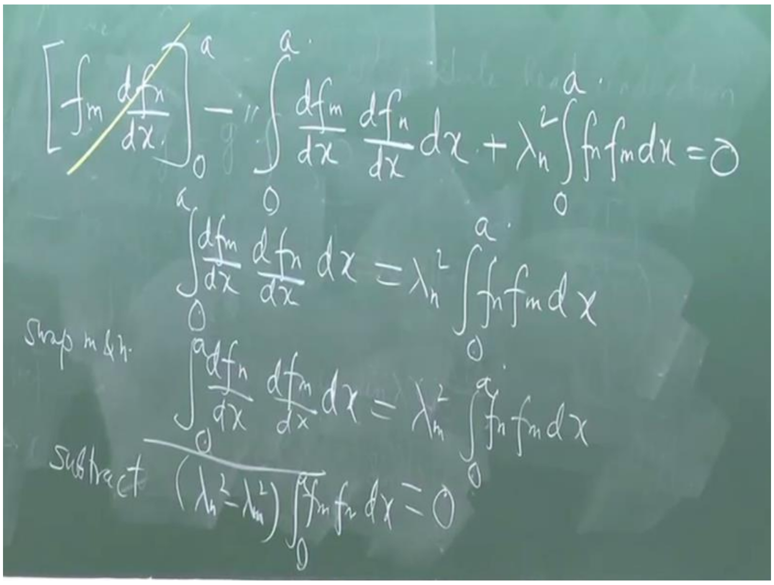

So, this equation just I have expanded it in full form and used a subscript n, this equation. Now what we do is we multiply this equation by f m, where m is not equal to n or a special case is there when m equal to n that is one value. In general m may not be equal to n, ok. So, we now integrate it with respect to x from x equal to 0 to x equal to a. So, when we integrate this we will use integration by parts and quite obvious because in integration by parts.

We have first function into integral of the second function. Second function we normally keep higher order derivative. Because when we integrate, the order of the derivative comes down. So, this is the first function. This is the second function. So, first function into integral of the second minus integral of derivative of first into integral of the second. Now I apply the boundary condition. So, this is the boundary term right.

What is the boundary condition? At x equal to 0 theta is 0 right. So, f equal to 0 and at x equal to L also theta is 0. So, you can see this boundary term goes out. Had it been that at one of the boundary instead of theta is 0 it is d theta d x 0 then also this term would have been 0 ok. So, this term has become 0 and you are left with integral d f m d x into d f n d x. Now you swap m and n.

So, if you do that you will get just swap m and n, ok and subtract that which you subtract you will get lambda n square minus lambda n square, ok.

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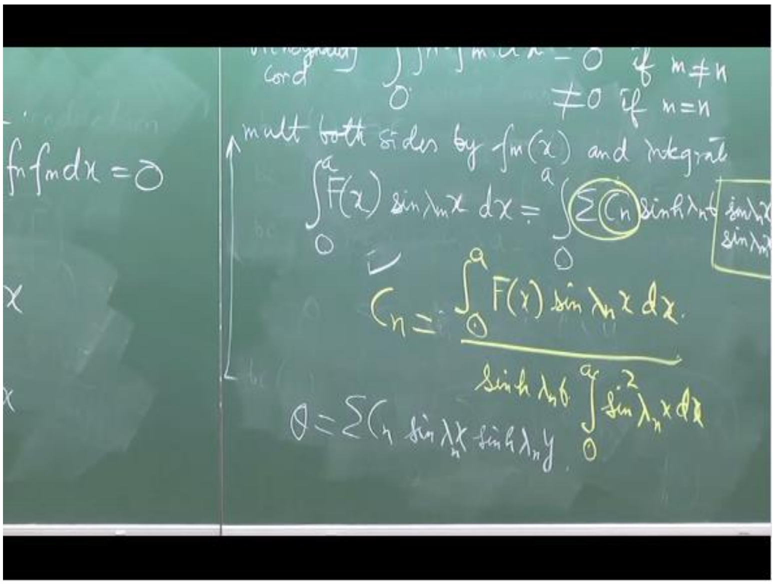

So, from here what can we conclude? We can conclude that integral f n f m dx from f equal to 0 to f equal to a. This is equal to 0 if m not equal to n right. In general, to satisfy this if m not equal to n this term is not 0. So, to make this 0, this is in general 0. But if m equal to n that is not necessary. So, this is not 0 if m is equal to n. This is called as orthogonality condition. So why such a name orthogonality condition out of Tom Dick Harry why orthogonality condition?

So, you are familiar with orthogonality of vectors right. When you have 2 vectors, when do we say that 2 vectors are orthogonal to each other? When their dot product is 0 right. They are perpendicular to each other. So, here in a functional space instead of dot product we talk about inner product and this is here instead of orthogonality of vectors it is orthogonality of 2 functions.

So, when you say dot product of a vector that is a special case of inner product. Basically, you are multiplying 1 component of 1 vector with the corresponding component of the other vector and adding it out. So here also you are multiplying the function with its corresponding n index with m index. So, when this is 0 then we say that it is an orthogonal function. So, we will use that orthogonality here. So how can we use the orthogonality here?

This is f and x right. We will multiply both sides by fm and x and integrate. So, what we will do multiply both sides by f m x and integrate ok. So now the charm of using the orthogonality condition is that this summation will go away and you will be able to isolate C n because the integral is essentially integral on this function and the integral will be 0 for all m not equal n only for 1 term when m equal n the integral will be non-zero.

So instead of the summation the C n itself can be isolated. So, you will have C n is equal to integral of 0 to a f x sin lambda n x d x divided by sin h lambda n b integral of sin square lambda n x d x, that completes the solution of the problem. So, this integral you can very easily evaluate you substitute lambda n a equal to n pi that was the value of lambda n right. So, lambda n is n pi by a that you can substitute here and here sin square theta you can write half into one minus cos 2 theta and integrate.

Very simple integrations are there ok. So, once you get the C n then basically you can write the solution in terms of an infinite series theta is equal to summation of C n sin lambda n x sin h lambda n y. I tried to do it from as fundamental as possible without referring to the Fourier series. But you can if you use your familiarity with the Fourier series you will see eventually the terms in the series that come, they quite clearly resemble with the terms in the Fourier series ok.

So, to summarize what we have studied today? We have discussed about what is the physical way of assessing whether a problem is 1 dimensional or 2 dimensional or 3 dimensional steady or unsteady and then we have invoked simple problem of 2-dimensional steady state heat conduction and we have solved that problem. So, we stop here today and we will continue in the next class. Thank you.