**Introduction to Fluid Mechanics and Fluid Engineering**
**Prof. Suman Chakraborty**
**Department of Mechanical Engineering**
**Indian Institute of Technology - Kharagpur**

**Lecture - 32**
**Some Exact Solutions of Navier Stokes Equation (Contd.)**

We continue with the example of fully developed flow through a circular pipe and we in the last class arrived at the simplified forms of the governing equations.

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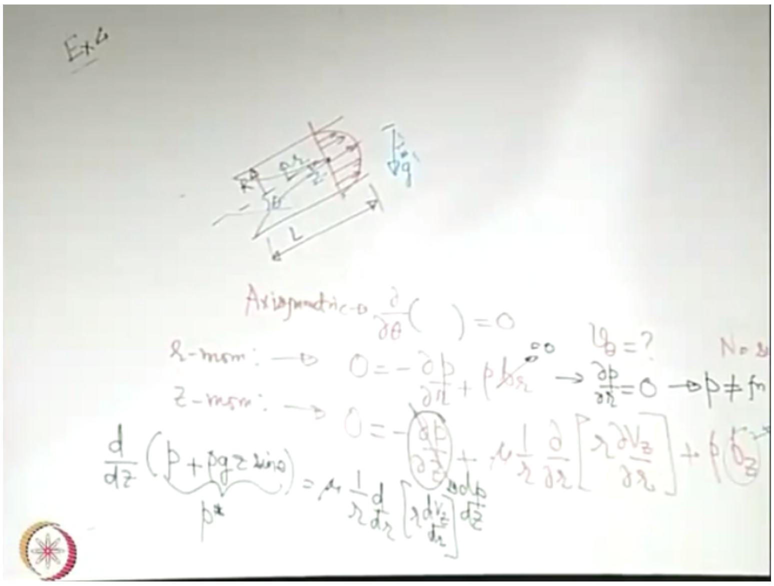

So let us continue with the governing equations which are written here. First let us consider the r momentum equation. So in the r momentum equation first of all let us see that what is this br? See br is the body force component in the radial direction. There is a body force component in the radial direction that is true but until and unless this extent of the pipe is very large.

It is like a very high radius pipe then that effect of that body force is not going to be important along the r direction. So this therefore maybe neglected, so you come up with the pressure gradient along r=0 or approximately=0 that means p is not a function of r that means p is a function of z only. So you can write when you come to the z momentum equation this as dp/dz.

And you can combine the term with the body force term to have d/dz of p+rho gz sin theta=mu*1/r d/dr of r dv z/dr because vz is the function of r only that we showed in the last class. Therefore, it boils down to an ordinary derivative. So this p+rho gz sin theta is like p+rho gh so this is like p star, the piezometric pressure.

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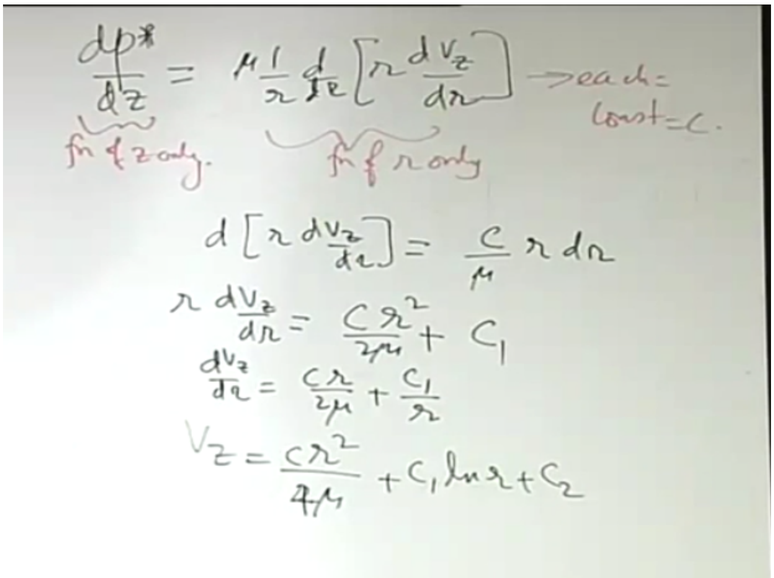


We again are having an equation where you have dp star/dz=mu*1/r d/dr of r dvz/dr where the left hand side is the function of z only. The right hand side is the function of r only, so this is function of z only, this is function of r only and again this may be valid if each is a constant. So each equal to constant say=c. So the first thing that we may do is to find out the velocity profile from this.

So we consider the right hand side and integrate it. So you have d of r dvz/dr=c/mu r dr right. So this you may integrate. So once you have integrated, you have r dvz/dr=c r square/2 mu+c1 which means dvz/dr=cr/2 mu+c1/r okay. So you may integrate it once more. So if you integrate it once more you will get vz as a function of r. So let us do that, so you have vz=cr square/2 mu+c1 ln r+c2, sorry 4 mu+c1 ln r+c2.

Now c1 and c2 are the constants of integration that you have to find out from the boundary conditions. So what are the boundary conditions here?

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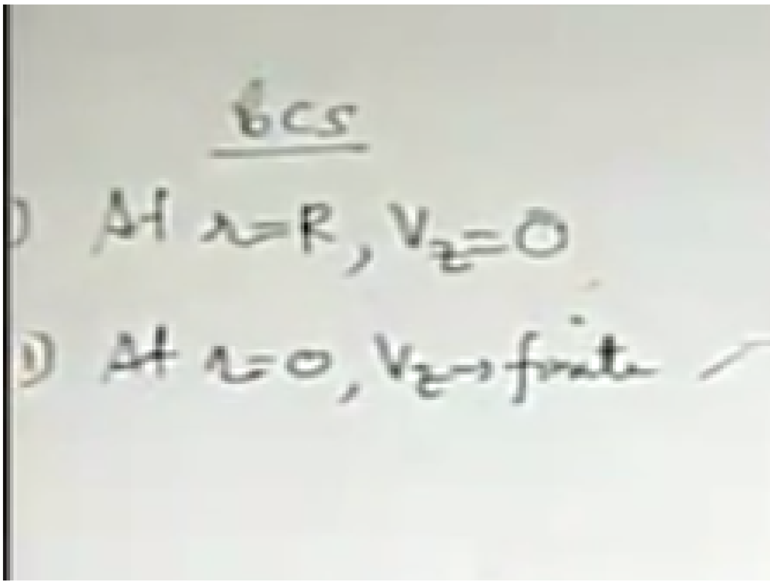


Boundary conditions, so one of the boundary conditions that is at the wall is straightforward at small r= capital R you have vz=0 no-slip boundary condition. Then what happens at small r=0 that is a point of singularity so to say but physical problem is that vz has to be defined at small r=0 I mean you cannot have a situation where there is vz undefined at small r=0 because it is a physical problem the pipe is entirely filled up with fluid with some velocity.

At r=0, it should be a well-defined velocity and that is possible so at r=0 vz must be finite. It cannot be infinite or it cannot be undefined in other words. Therefore, from the second condition you must have c1=0. This is from the boundary condition number 2 and from the boundary condition number 1. So this is boundary condition number 1 and this is boundary condition number 2.

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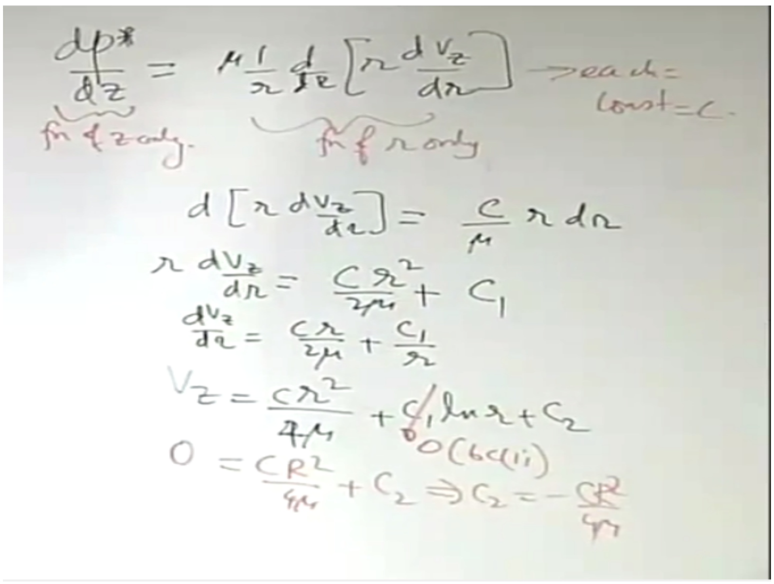

So from boundary condition number 1, you have 0=cr square/4 mu+c2 that means c2 is –cr square/4 mu.

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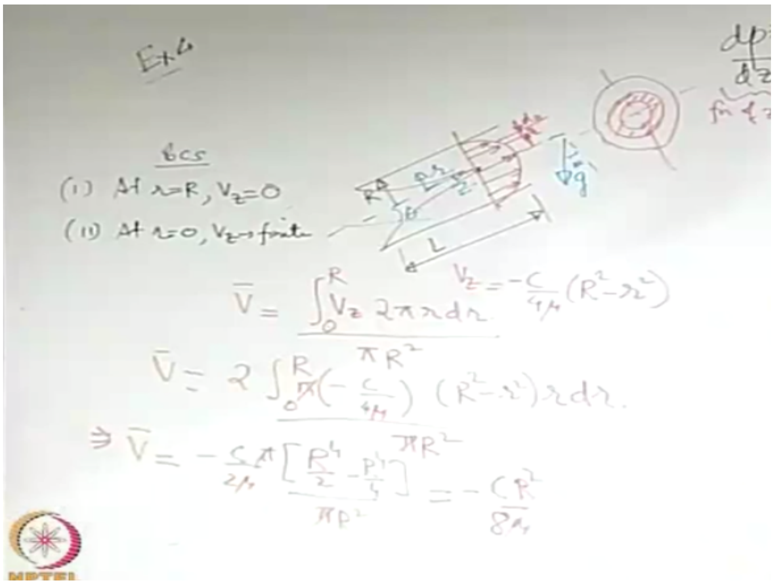

So the velocity profile becomes vz=c/4mu*or if you write –c/4 mu*capital R square-small r square right and as usual you see that the c which is like dp star/dz that has to be negative to drive the flow along the positive z direction, which means that vz has to be positive if c is negative. Now you may express c in terms of the average velocity what we did for the plane Poiseuille flow, for the Hagen-Poiseuille flow also the same thing is valid.

So what is the average velocity? v average, it is integral of vz over the area of cross section/the area of cross section. So vz what is the elemental area that you can choose? Say at a distance r you take a strip of width dr. So if you just draw its other view so you have taken at a radius r some strip. So you are talking about such a strip of width dr and this strip is 2 pi rdr. So vz*2 pi r dr from 0 to R/pi capital R square is the area.

So let us do this integration quickly, so 2*integral of 0 to R you have –c/4 mu. So what is v average in terms of c? So –c/2 mu then if you integrate it, so R four/2-R four/4/pi R square sorry pi is not there pi has got canceled. So this is=-cR square/8 mu. So this is one important result which we will use subsequently but at least let us write the expression for the velocity profile by writing c in terms of this.

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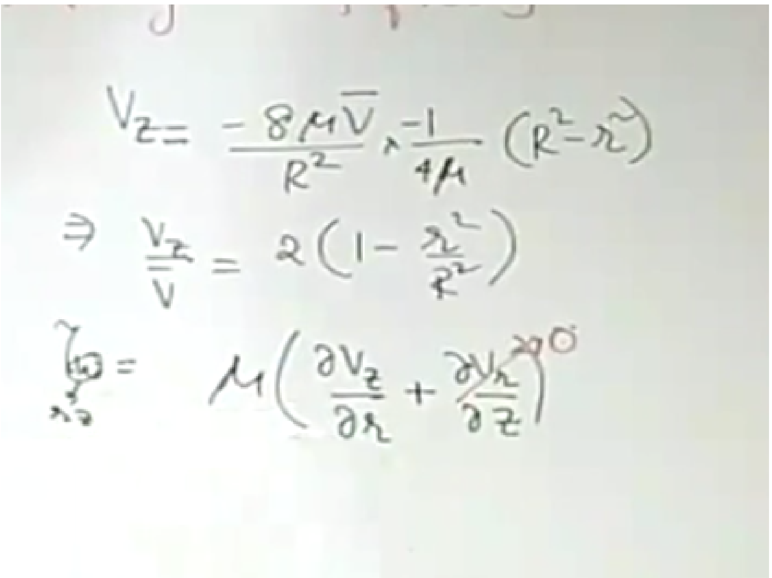

So vz becomes in place of c you write -8mu v average/R square that*1/4 mu. Again, there is a – sign*R square-small r square that means vz/v average is 2*1-small r square/capital R square. This is the velocity profile in terms of the average velocity. Recall the flow between 2 parallel plates, it was like 3/2*1-y square/H square, so it is quite similar. The coefficient is different just because of the different geometry.

And it is also a parabolic type of velocity profile because it is square with the local radius. Now you can find out what is the wall shear stress. So let us find out what is the wall shear stress. So it will be like mu, we will adjust the sign positive or negative later on, mu*rate of deformation. This is like tau rz, so the proper subscript with the index. Now one of the important things is see positive tau with like mu du/dy, so this is just like mu du/dy.

First of all, vr is 0 so this term is not there. So it is just like mu du/dy but the thing is see the r direction is towards the solid boundary. It is not away from the solid boundary, so that has to be compensated with the –sign here.

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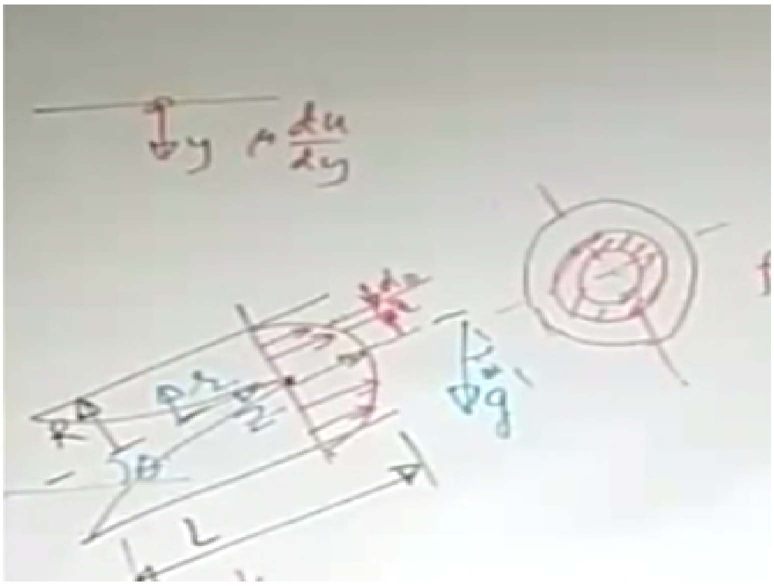

So if you have a solid boundary like this, if you have this as y then you have like mu du/dy. So y is away from the solid boundary towards the fluid. The reason is you want to write it as a positive shear stress, so you want u to be increasing in the y direction. So y=0 is like the no-slip and then u increases but here the r direction is opposite to that so you have to make it up with the –sign.

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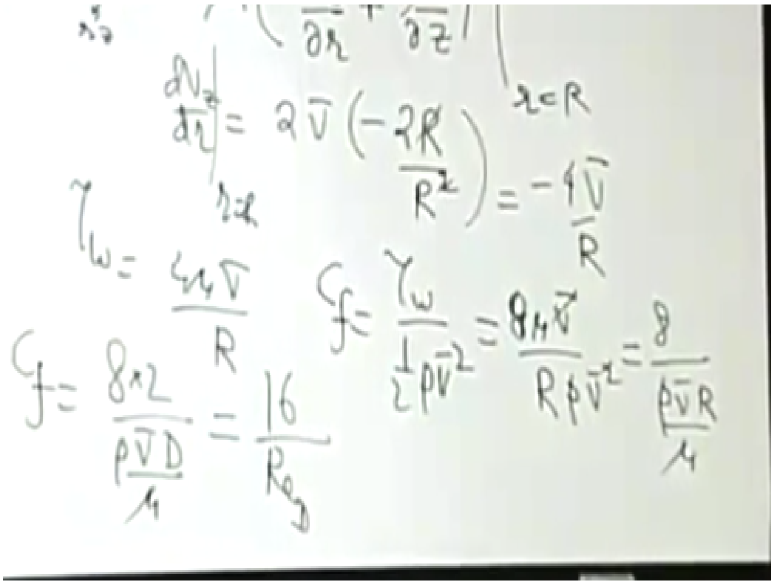

So the wall shear stress therefore if you just differentiate it with respect to r so dvz/dr is like 2 v average. So this entire thing evaluated as small r=capital R that is at the wall, wall is located at small r=capital R. So 2 v average*-2, so -4 v average/R is the dvz/dr right. Therefore, the wall shear stress is 4 mu v average/R. What is this coefficient of friction or the Fanning's friction coefficient?

That is tau wall/1/2 rho*v average square. This we introduced in one of our previous classes, this friction coefficient. It is a non-dimensional way of writing the shear stress. So it is 4 mu v average/R, basically it becomes 8 mu v average/R rho v average square. So it is 8/rho v average R/mu. The denominator is the Reynolds number based on radius R but usually for this type of geometry the length scale that is taken is 2R or the diameter of the pipe.

So if you just consider that then the cf becomes 8*2/rho v average*diameter/mu where diameter is 2*R. So this is 16/Reynolds number based on the diameter of the pipe. As engineers what we want to do with this friction coefficient? So what importance or implication it has to us? So if it is large, it is small, we may understand that if it is large maybe the frictional effect is large but how do we take care of that in a design?

And that maybe visualized more effectively if you consider now the dp star/dz term which is=c. So to do that let us come back to this equation. V average written in terms of c, so what is c? c is=dp star/dz.

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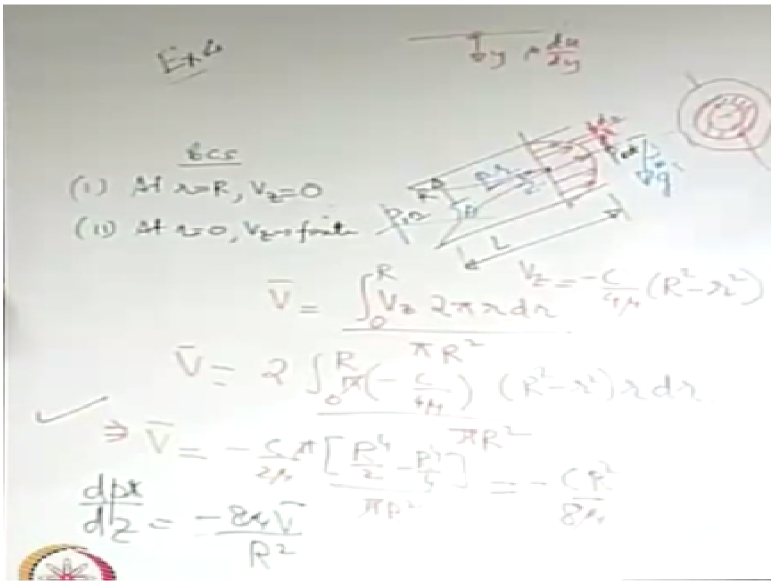

So let us write dp star/dz in terms of the other parameter so it becomes -8 mu v average/R square okay. So -8 mu v average/R square and what is this dp star/dz? This is if you have pressure at the inlet as say p in if you have pressure at the outlet as say p out and the distance over which this pressure difference is there is L, then first of all we could derive the dp star dz is the constant.

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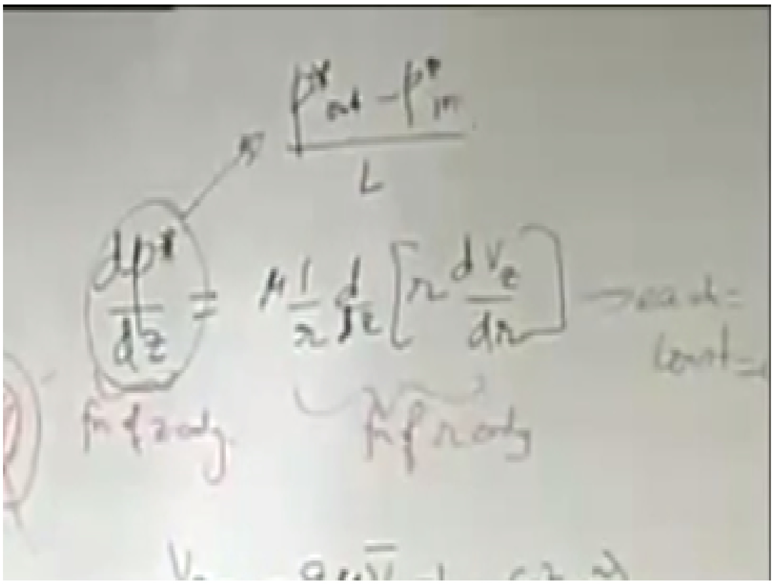

That means p star versus z is a linear profile. So this is as good as p star out-p star in/L okay.

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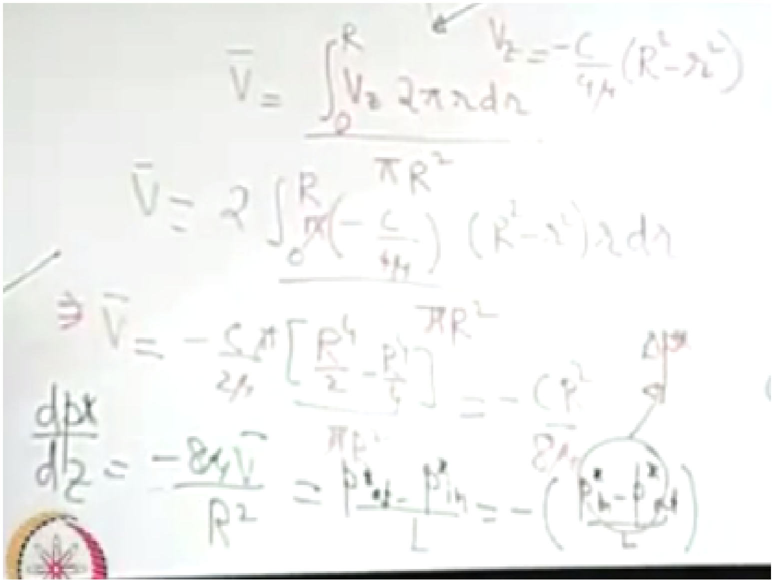

So this we just write as p star out-p star in/L. So in other words it is – of p star in-p star out/L. I mean why we are writing it in this way is see the p star in-p star out is the driving delta p star for the flow, the driving difference in piezometric pressure that should drive the flow. See why this driving pressure difference is necessary? It is necessary because you have viscous resistance in the flow.

So this is a manifestation of the consequence of the effect of viscous resistance that is being exerted in the flow. So you have to have a driving pressure gradient which is a favorable pressure gradient that is dp star/dz negative to make this flow occur. So when you have this one this delta p star, so what it in effect does? In effect it overcomes the viscous resistance. So the viscous resistance also maybe expressed in form of a pressure unit.

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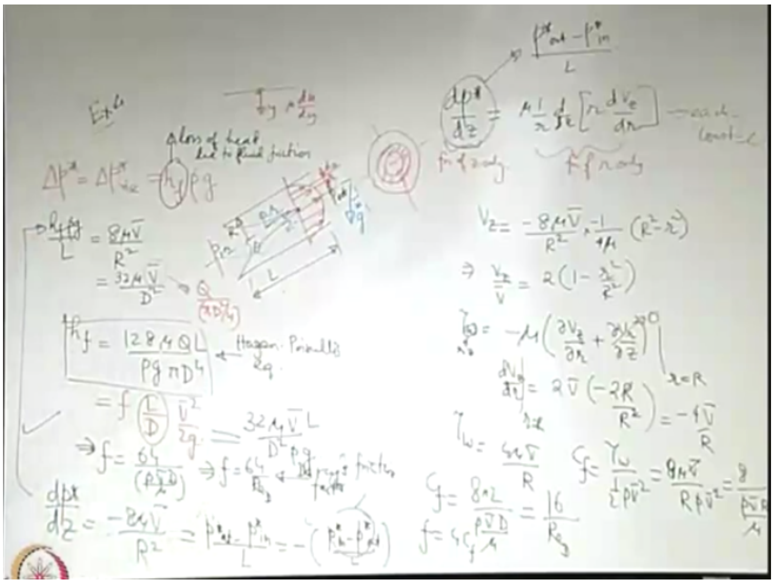

So the viscous resistance therefore you may say that delta p star is nothing but the delta p star because of the viscous resistance and we may write it in terms of a head or some unit of length so we remember that the pressure is nothing but some length unit*rho*g and the length unit is the equivalent unit of pressure expressed in terms of length with which we call as head so that is what we discussed also earlier.

So if you write it in terms of a length unit as head with a subscript hf for friction f for friction so we can write it hf rho g where this hf as engineers we understand that this is a loss of head due to fluid friction okay. Therefore, it is possible to write this expression dp star/dz if we come back here in terms of this delta p star as hf rho g so you have hf rho g/L=8 mu v average/R square okay.

Of course, it is possible to write it in terms of the diameter, so 8 becomes 32 here okay. So it is also possible to write the average velocity in terms of the flow rate. That is sometimes important because if the flow rate is Q, it is just Q/pi d square/4 because in the experiment you usually measure the flow rate. So it is we are trying to write the equation in terms of experimentally measurable parameters.

So from this what you get is hf=128 mu Q L/rho g pi d to the power 4 okay and this equation is known as Hagen-Poiseuille equation. So what it gives? It gives a direct indication of the loss of head due to viscous effects in the flow. If the flow had viscosity 0, there would have been no loss of head and we can see that the loss of head is directly proportional to the flow rate, it is proportional to the length over which the fluid is flowing which is understandable.

And inversely proportional to the fourth power of the diameter, so if you make the diameter very, very small the loss of head will be very large that means to drive flow through a very small tube say a tube of micron size you require a huge pumping power because there is a huge head loss because of the frictional effects that is one of the challenges in having a flow in a micro or a nano channel in a very small channel.

So the whole understanding is that like those are advance topics but you see that the basic of fluid mechanics gives us a clue that is what are the challenges as you make the sides smaller and smaller. What are the engineering challenges in terms of having the flow? Now it is also possible to express it in terms of a non-dimensional form by writing this in this way by writing this as some factor f*L/D*v average square/2g.

So this is just a non-dimensional way of writing it. See L/D is a sort of aspect ratio of the pipe, it is a non-dimensional parameter and v square/2g is a kinetic energy head. So the head loss expressed as a fraction of the kinetic energy head. This f is that sort of fraction. So what is that? So if you just equate these two, so you have hf as rho g sorry f L/D v square/2g and that is=whatever expression of hf what we had.

So 32 mu v bar/D square L/rho g right. So from here what we can conclude what is f? 64/rho v bar*D/mu okay just by equating these two, so that means this f becomes 64/Reynolds number. So this sort of non dimensionalization was first introduced by an engineer known as Darcy. So this friction factor is also known as Darcy’s friction factor. So it is just a different way of writing the friction or frictional coefficient.

You can see clearly that f and cf are related by f=4*cf, just is the same thing expressed in terms of in different ways. So one looks into it in the view point of a pressure drop, another looks into it in the form of a wall shear stress and here they are related because you have to overcome the wall shear stress and therefore you have a pressure drop. So that is how they are related and this 4 coefficient is just because of the difference in which they are defined.

But physically they imply the same. So if it is Fanning's friction factor that is cf and if it is Darcy's friction factor it is f. So now we have a clear idea that if we have flow through a pipe, there is a head loss. The head loss is because of the viscous effects in the flow and you require a driving pressure gradient. The driving pressure gradient is to overcome an energy loss.

The energy loss maybe expressed in terms of a head and that head is given by this expression. So that is what we learn and as engineer it is important because say you are given a length of the pipe, see let us think about how this helps in design. Say you are given a length of a pipe, say 10 centimeter length of a pipe. You are given a flow rate that you expect out of it, say some meter cube per second.

You know what is the viscosity of the fluid and density of the fluid like that, so your problem is that what should be the power of that pump that is necessary to drive the flow. This is one of the very basic elementary problem. So to know that you have to know that what is the pressure drop that is taking place and first of all if you find out what is the hf? What is the head loss over that length?

Then that*rho*g is the pressure drop and the pump should have enough power to overcome that.

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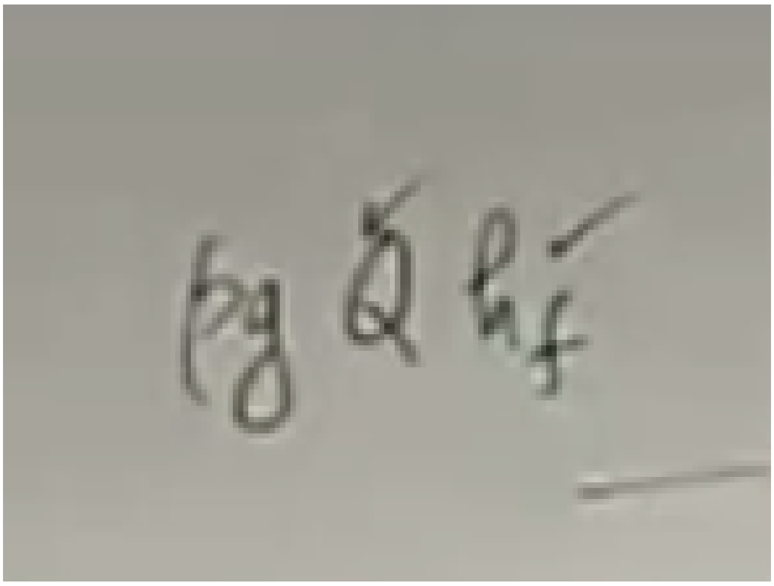

So the pump should have a power that is developed that is given by rho g*Q*head loss, it has to overcome that loss. So this is like a unit of power. So if you know that what is the flow rate, if you know what is the head that the pump has to supply. Then you know that what is the power of the pump that is necessary and so a very basic elemental mathematical work also is necessary even for a very crude design work, which is there in the day today life of fluid mechanics like flow through pipes okay.

Now we come to another example, example 5 and that will be like the final example before we will be moving to the next chapter. So the final example is about flow between 2 long rotating cylinders this is a problem that we are going to revisit because this problem we introduced when we were discussing about viscosity and its measurement.

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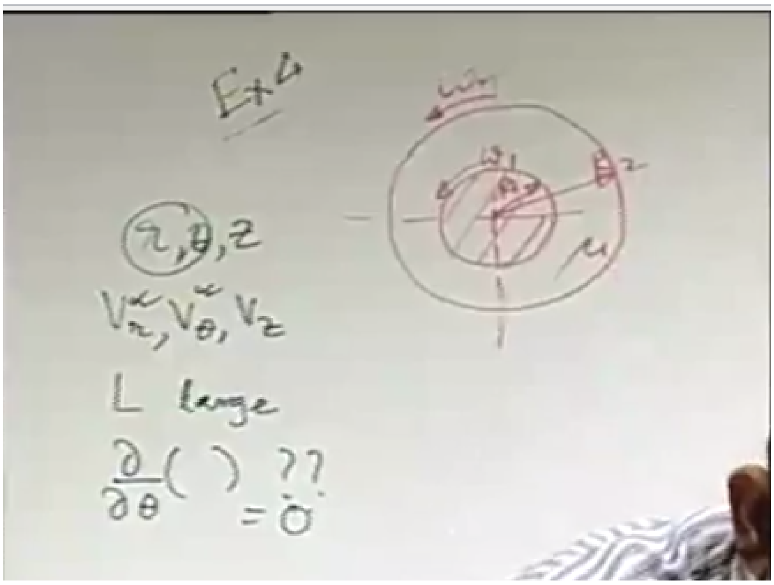


So you have 2 concentric cylinders of radii R1 and R2. Let us say that the inner cylinder is rotating with an angular velocity omega 1 and the outer cylinder is rotating with an angular velocity omega 2 and these angular velocities are like these are not functions of anything, these are constant angular velocity then we are considering a steady flow of a Newtonian fluid of constant properties in which the fluid is kept in the gap between the 2 cylinders.

This type of problem is important because we have discussed that it gives a principle of the measurement of viscosity of an unknown fluid. So principle of a device known as viscometer. So we have earlier qualitatively use the 1-dimensional form of the Newton's law of viscosity to like get an estimation of what happens here but now what we will do we will try to do it in a more careful way by using the proper equations of motion.

So first of all we have to understand that what are the important variables, which are involved here. So again by the nature of the problem, the geometry of the problem, the cylindrical polar coordinate system seems to be a proper choice, so the r theta z coordinate system okay. Now the corresponding velocity components are there vr, v theta, vz. The first assumption that we will make is that the length of the cylinder, which is perpendicular to the plane of the figure is very large.

What is the consequence of making the length large? Large means large in comparison to the radius. So if L is large then what is the consequence in terms of your analysis? It effectively boils down to a 2-dimensional problem in the r theta plane. So the z gradient is not important. So when you come to the r theta component you have vr and v theta, these things are there. Next is what about the partial derivatives with respect to theta?

See when you are solving a problem, it is not that the assumptions are given to you. It is important to come up with a assumptions based on the physical description of the problem. See the inner cylinder rotates with an angular velocity, which does not have any preference over theta right. It is not that at different theta it is different. The outer cylinder also rotates with an angular velocity that does not have any preference over any theta.

And therefore in between we expect that whatever is the behavior should not have any preference over theta, so that means the derivative with respect to theta for whatever the flow parameters that should be 0. So that should automatically follow from the physical description of the problem. Next, what we will do, we will now go to the basic equations, again we will go to the cylindrical coordinate system equations, Navier stokes equations.

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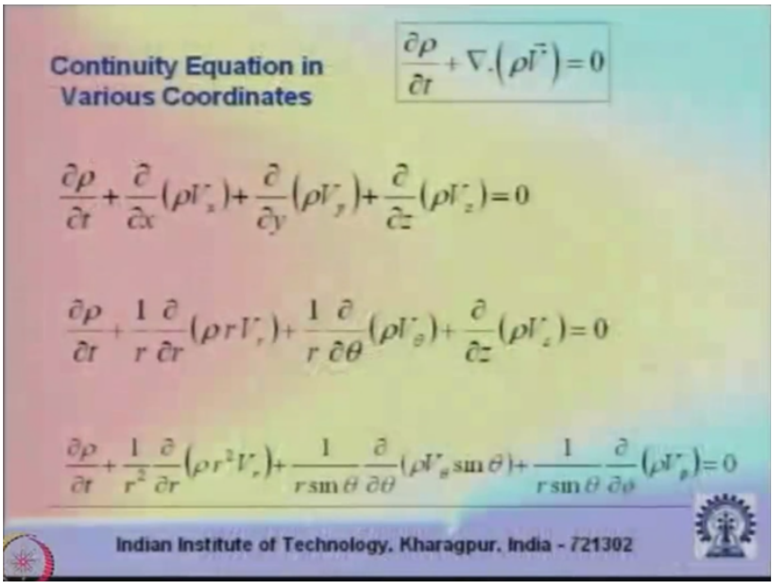


But first we will go to the continuity equation, so if you go to the continuity equation you see we are talking about the continuity equation in the cylindrical polar coordinate system first term because we are considering the steady flow, the density gradient derivative that is 0 with respect to time. The second term is like if you want you should keep it. The third term and the fourth term see because there is no derivative with respect to theta the second term is 0.

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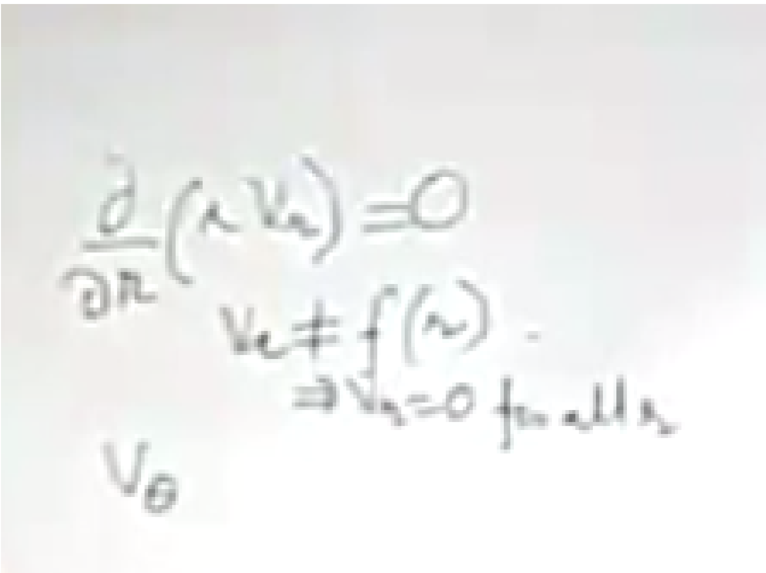


No gradient with respect to z, so that term is 0. So only one term is R*vr the derivative of that that means d/dr of rvr that is=0, just like what we had for Hagen-Poiseuille flow and what will be the obvious conclusion that vr is not r*vr or vr is not a function of r. So we know that at 2 different radii vr is 0, R=R1 and R=R2 and that is good enough for us to say that vr=0 for all r because vr is not a function of r.

So the problem has boil down to only 1 velocity component that is v theta okay. So with this understanding let us now go back to the momentum equations.

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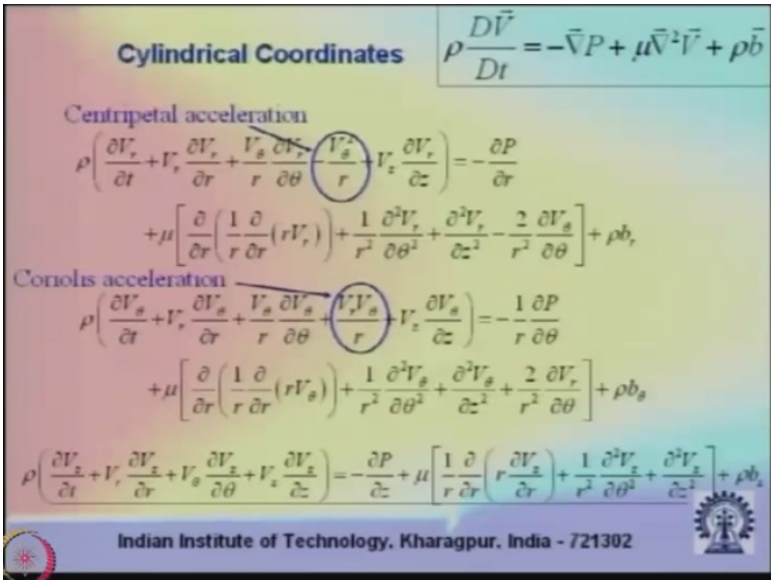


So let us look into the momentum equations, first the r momentum equation that is the first equation written in the slide. So you have vr=0, so all the terms in the left hand side are 0 except the term involving v theta. So in the left hand side you have –rho v theta square/r okay and it is the centripetal effect. So we may understand physically what it is, so when it is a rotating thing, it is a centripetal acceleration because of the tangential component of the velocity.

Right hand side you have first term-dp/dr the term is there. The remaining terms first term has vr so that is 0, second term has again vr that is 0, third term again has vr and fourth term has the derivative with respect to theta. So that is 0, there is no body force along r okay.

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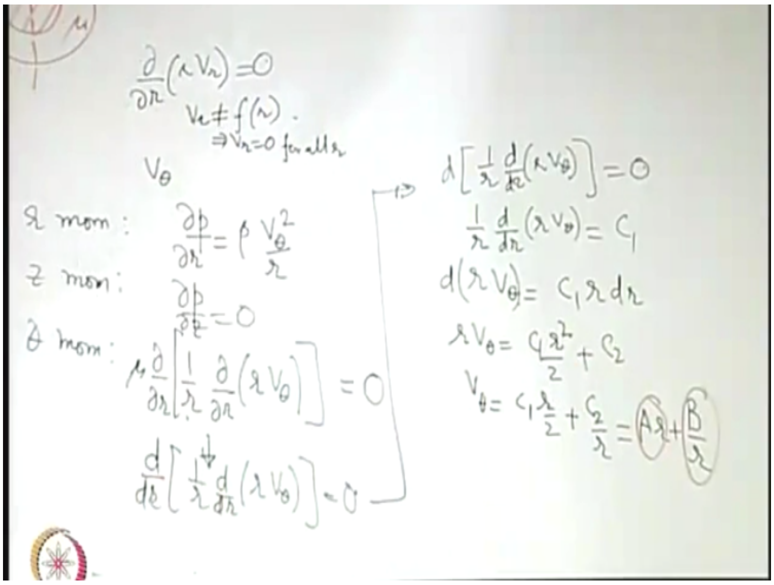

So you have the r momentum equation what it gives? dp/dr is rho v theta square/r. **"Professor - student conversation starts."** No, no that is what I said that is always a special case. So you also have to satisfy the boundary conditions and consider it for all r for all possible cases, vr proportional to 1/r is only a special case but you have to consider that it should be true for all possible velocity fields, it is not just for one particular velocity field. **"Professor - student conversation ends."**

Not only that you have to satisfy vr say 0 at r=R1 and r=R2. So if you have say r*vr as some constant c, so that is vr=c/r then at r=R1 how do you ensure that that vr will be 0? You have to ensure that at the boundaries you have normal component of velocity 0. That you have to keep in mind. So the r momentum equation you have dp/dr as rho v theta square/r. Let us look into the other momentum equation.

See the z momentum equation is useless here because it involves all the terms, which have vz and there is no body force along z and only it gives that dp/dz=0 that is the z momentum equation only termed at remains important is dp/dz=0 okay. The most important momentum equation will be the theta momentum equation. So theta momentum equation if you see that is the second equation written in the slide.

First you go to the left hand side you have, first the flow is steady so the first term is 0, next you have vr is there so vr*that is 0, third term derivative with respect to theta, so that term is 0, fourth term because vr is 0 that is=0, fifth term vz is 0, even if you do not consider that the gradient with respect to z is 0 because z is large. So in either way the left hand side totally becomes 0.

Right hand side first term dp/d theta is 0 because there is no theta variation. Second term is very much there because you have v theta as a function of r which is what we are going to solve. Next term it is a gradient with respect to theta, so that is 0. The term after that you have derivative with respect to z that is 0. The next term you have vr that is 0 and you have no body force along theta.

Therefore, it boils down to mu sorry theta momentum mu* let us write the terms 1/r d/dr sorry mu*d/dr 1/r d/dr of r v theta that is=0 right. So see now v theta could be function of what? v theta could be function of r theta and z. It is not a function of z because there is no gradient with respect to z, the length is large, v theta is not a function of theta because there is no variation with respect to theta for anything.

So v theta is the function of r only and therefore we may write it as an ordinary differential equation. So we may just write it equivalently as d/dr of 1/r d/dr r v theta=0. So if you integrate it, you will get d of 1/r d/dr of r v theta. That means 1/r d/dr of r v theta=say some c1. So r v theta=c1 or d of rv theta is c1 r dr right. So if we integrate it, r v theta=c1 r square/2+c2 and v theta therefore becomes c1 r/2+c2/r.

So it is like of the form Ar+B/r right. So if you look into this equation can you recall that the corresponding components of these velocities are related to the vortex flows that we have studied earlier. So what is the first component that is like a forced vortex and this is free vortex or irrotational vortex. So it is like a combination of free and forced vortex that is going to be the resultant velocity field.

Now if you want to get the pressure distribution, you have to go back to the r momentum equation to get the radial pressure distribution. So you know now v theta is the function of r, you may substitute and integrate it with respect to r to get the pressure as a function of r. We are not going into that but what we will do is we will find out what is the expression for the wall shear stress.

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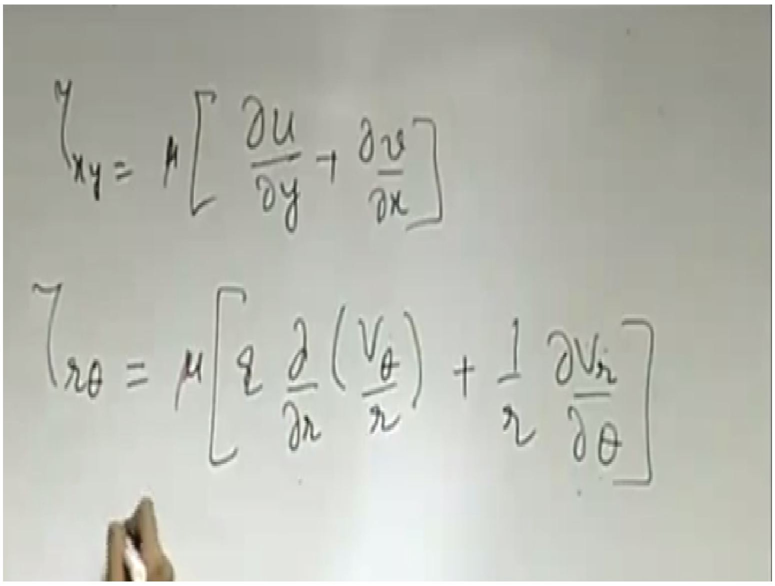


So we are interested in tau r theta or just shear stress at different locations. So if you want to write so there is an expression for tau r theta and I am just writing it down here and I will tell you that what is the origin or rational behind this expression. So this is for a Newtonian fluid just written just expressed in terms of the r theta coordinate system. So it is just like just think of 2 coordinates like if it was tau xy that was what mu du/dy+du/dx right.

So basically you are having a cross gradient x component of velocity with y gradient and y component of velocity with x gradient. So similarly you have a theta component of velocity with r gradient and r component of velocity with theta gradient and this 1/r these adjustments are there because of the use of the cylindrical polar coordinate system. So these adjustments are also quite obvious.

That if you have say this term, so you have vr with respect to theta, see you want a gradient that means it should be the velocity divided by a length. So this is d theta so r d theta is like an elemental length in a cylindrical polar coordinate system. So that is why 1/r term has to come outside. So in this way you may relate this term with the general understanding of the curvilinear systems.

So again it is not very important to go into the derivation of this term but we will just utilize this to calculate the stress.

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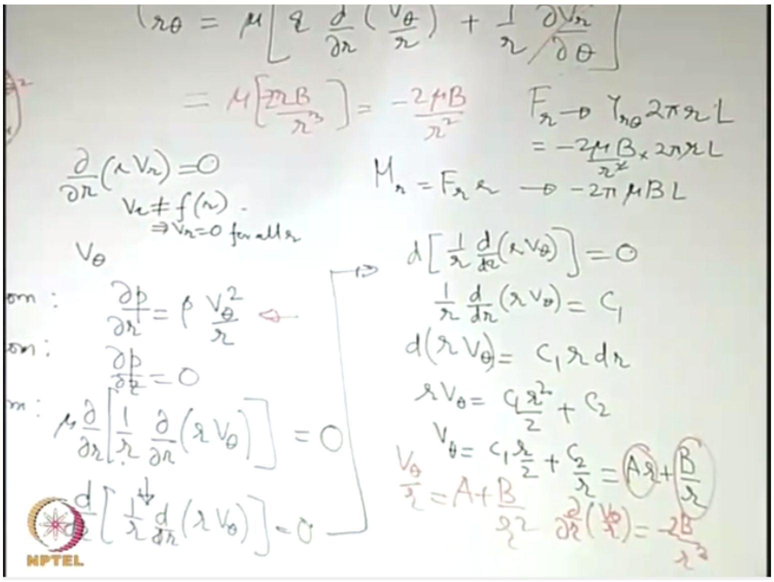


So now write this is mu r d/dr of v theta/r, so v theta/r is we have not yet completed the expression for v theta because A and B are 2 constants of integrations to be evaluated but that we can straightforward do from the boundary condition but at least let us look into the form first. So v theta/r is A+B/r square okay. So v theta/r is A+B/r square and therefore d/dr of v theta/r is –2B/r cube.

So you have this as -2B/r cube and theta derivative is not there. So -2 mu B/r square. So one important thing is independent of theta but dependent on r. The next thing is that what is the elemental shear force because of this? So if you consider an element at a radius r you have a line element like this because it is independent of theta otherwise you could have taken a small line element with r d theta but because it is independent of theta we are taking a total peripheral line element.

So what is the elemental shear force on that? So the shear force is f at a distance r is tau r theta*2 pi r*L. So 2 pi r is the perimeter of this times the length is the outer surface over which this tau r theta is tangentially acting. So that becomes -2 mu B/r square*2 pi r L and what is the moment of this force with respect to the axis of the cylinders? At a radius r that is Fr*r and you can clearly see that is=-2 pi mu BL which is independent of r.

So if you remember that while dealing with the understanding of the principle of viscometer we say it that same moment or torque is transmitted at all radius and this is what it shows that at any arbitrary radius, the torque due to the shear force is independent of the local radius.

This is one of the very important understandings. Now to find out the constants A and B you have to use the boundary conditions. So what are the boundary conditions?

Let us just write the boundary conditions and try to find out A and B.

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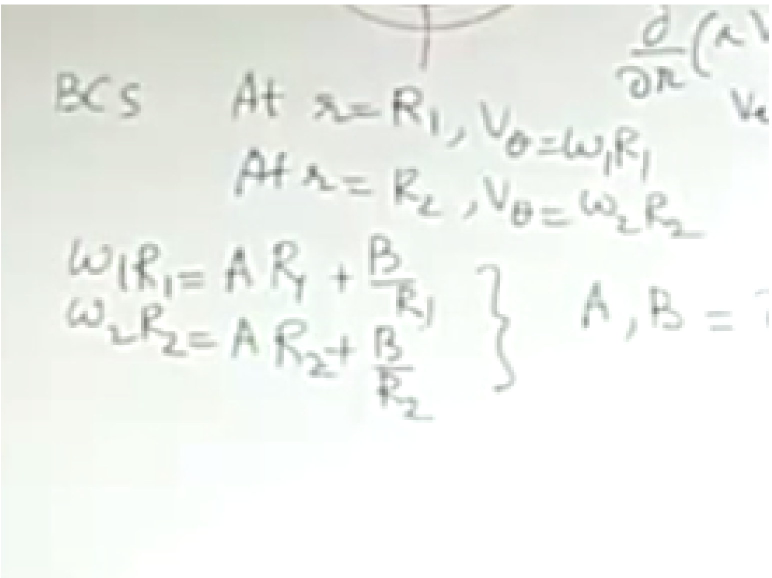


So the boundary conditions are at r=R1, v theta is omega 1*R1 no-slip boundary condition so velocity of the fluid same as velocity of the solid and at r=R2, v theta is omega 2*R2. So you substitute that so omega 1 R1=Ar1+B/r1 and omega 2 R2=Ar2+B/r2 so it is possible to find out A and B from these two right. These are like capital as per our notations, so AR1+B/R1 AR2+B/R2.

Now let us say that we want to solve a problem which is a bit of a modified version of this one. What is that modified problem?

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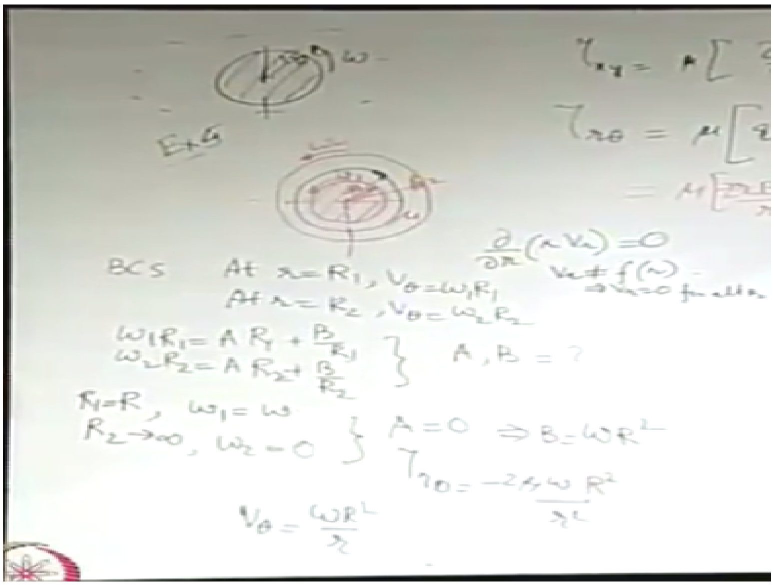

So the modified problem is you have one single cylinder, solid cylinder rotating with an angular velocity omega in a fluid of large extent. So one cylinder of radius r and of large length rotating with omega, so there is no inner or outer concept. So how can you solve this problem by considering the solution of this one? So omega 1 so R1=capital R, omega 1=omega, R2 tends to infinity and omega 2=0.

So if you do that let us see that what forms the constants of integration step for that particular case. So you have R2 tends to infinity and omega 2=0. So when R2 tends to infinity this term is 0, omega 2 is 0 that means A=0. So this will straightaway give you A=0 that means B=omega*R square. So tau r theta in that case becomes -2 mu omega*R square/small r square.

And what is the v theta there? v theta is just B/r because A is 0, so omega R square/r. So it has become like a free vortex. We have seen earlier that a free vortex flow is an irrotational flow right. So that much we know that means the vorticity vector if you calculate that will be a null vector that we showed by calculating the circulation and the vorticity. Now let us try to find out what is the shear force on the fluid.

And that is something which is very, very interesting. So we have calculated the shear force locally at a radius and maybe they are talked due to that but over the volume what is the total shear force that is acting. To do that what we will do is we will write the momentum equation just in a vector form.

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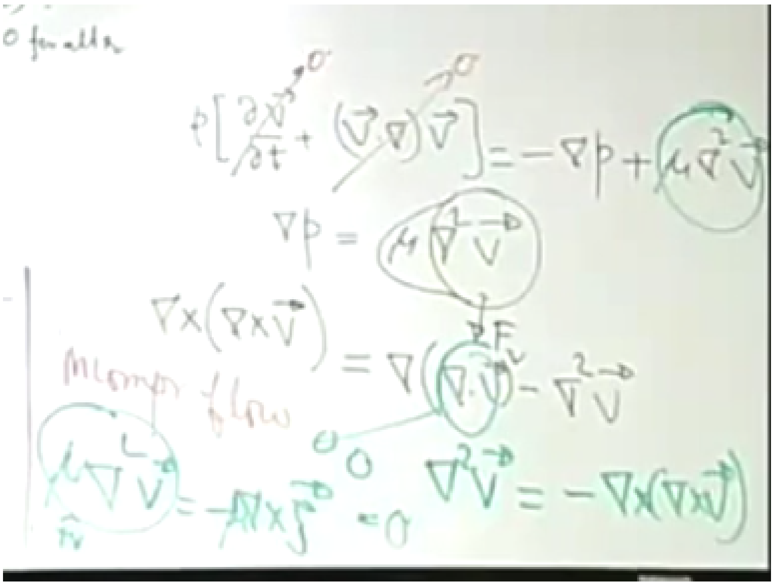

So the momentum equation is rho okay. So this is the vector form, we have derived this vector form and the advantage with the vector form is whatever vector operation that we will be doing on it so longer as it is in a vector operation form we do not explicitly mention which coordinate system it is, so it remains valid for all types of coordinate systems.

So the problem or the situation that we were looking for in that particular situation the left hand side was 0 always by the consideration of fully developed flow and steady flow. So this term was 0 and this term was 0 because of the sort of either fully developed or maybe 0 gradients in certain directions say theta or z direction or for whatever is the left hand side always came to be 0 for the example that we are talking about.

So in the vector form also it is like that. So you have the gradient of p=mu*this. Now let us try to use this particular term and see this is what? This is basically a force per unit volume right. So if you want to find out that what is the force per unit volume because of viscous effects, this is the term that is coming into the picture okay. So this you can say that is the F per unit volume mu*this.

Now what we will do is let us say we want to take the curl on both sides or just by looking into another way let us try to find out this term I mean we may take a curl of this side this and do but let us just look into this. So we want to take the curl of the velocity vector. So that is given by the vector identity right. So in this vector identify let us now try to use this vector identity.

So what we will do is we will consider an incompressible flow. So incompressible flow when you are considering an incompressible flow, what is the consequence of that you have the divergence of the velocity vector=0 okay. So when you have divergence of the velocity vector 0, then you have the del square v= - of curl of v right and curl of v is the vorticity vector right.

Therefore, you can write that del square v=-curl of the vorticity vector that is zeta is the vorticity okay. So this force that is mu del square v therefore we can write this as –mu*curl of the vorticity vector. So this is what? This is the viscous force per unit volume right. So this is shear force per unit volume. It does not matter whether the equation is simplified to this form or not.

Even if the left hand side all the terms are there, still this happens to be the viscous force per unit volume, so this particular term. So only these terms were simplified to this extent for the problems that we discussed but for all problems it will not be like that but for all problems for a Newtonian fluid with a constant viscosity this is going to be the viscous force per unit volume. Now you look into the special case.

For this free vortex, we know that it is an irrotational flow. So that means curl of the velocity vector is 0 that means the vorticity vector is a null vector. So that means this term is 0 okay. So you see an example, if you recall one of our very introductory lectures in fluid mechanics we mentioned one thing that fluid mechanics is a beautiful subject because it gives something which is non-intuitive and this is one example.

You have shear stress !=0 but shear force=0. If you recall we mentioned this as a non-intuitive example that you have shear stress != to 0 right. This is B is !=0 it is varying with the radius but you would have find that because of the irrotationality the shear force is 0 (()) (54:53) with this example one of the very non-intuitive things we learn out of fluid mechanics that there may be cases where the shear stress is not 0 but the shear force is 0.

And that we can get from very elementary mathematical consideration of this type of flow. So to summarize we have looked into examples of exact solutions of the Navier-Stokes equations mostly for steady flows, we have not done it for unsteady flows that is not there in the purview of this elementary course but at least for steady flows and fully developed types of flows we have worked out the solutions.

And we have come to conclusion that simple closed form solutions of Navier-Stokes equations are possible for such types of flows and this give rise to certain important insights from engineering and scientific principles. So we will conclude this chapter here and from the next lecture, we will start with another important facet of the equations of motion for viscous flows and that is the turbulent flows. So we will start with the introduction of turbulence from the next lecture. Thank you.