Introduction to Fluid Mechanics and Fluid Engineering
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Lecture - 31
Some Exact Solutions of Navier Stokes Equation (Contd.)

We were discussing about the exact solutions of the Navier-Stokes equation.

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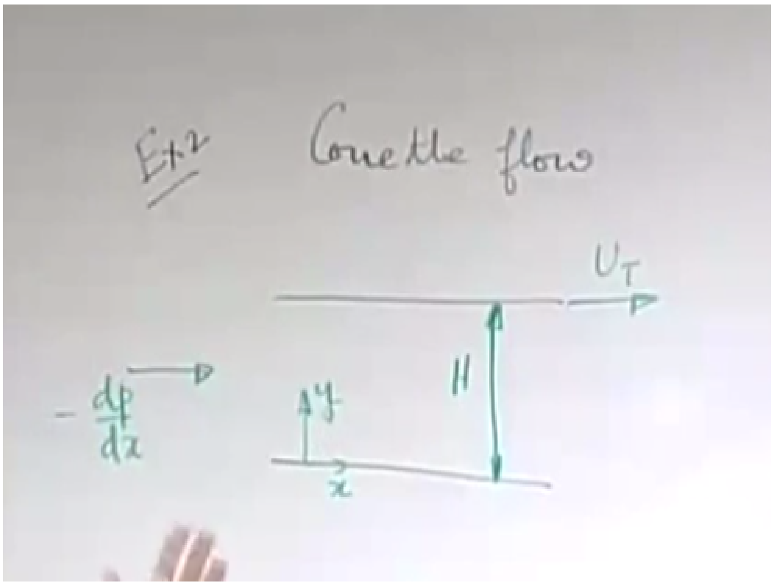

And we will consider now our second example, which is known as a Couette flow. So what is the difference between this and the plane Poiseuille flow that we looked in the yesterday's class? If you have say 2 plates, which are parallel to each other and if you have a pressure gradient which is acting on the fluid then the flow which is being created the fully developed part of that that we call as plain Poiseuille flow.

Now in the Couette flow, there is a modification to that. What is the modification? Now the plates are not stationary but one plate is moving relative to the other. That means you now have say as an example the top plate is moving with a velocity uT relative to the bottom plate. So it is not always necessary that the top plate will move with the velocity or bottom plate will move with the velocity.

The important thing is there should be a relative velocity between the 2 plates in whatever sense. So that relative velocity is being created by creating say a design motion of the top plate. There are cases when this design motion maybe oscillatory or time dependent in nature and that type of problem is not within the scope of this particular course but in an advance fluid mechanics course which is there in the subsequent years that is discussed.

So here we are talking about there is a uniform time independent velocity with which the plate is being moved. So with that understanding we have to see that what extra effect does it create beyond the plane Poiseuille flow that we have discussed? See if you forget about the effect of the plane Poiseuille flow; just think that this is there. This relative velocity is there. What it creates in a fluid is a velocity gradient.

So it is an automatic inducer of velocity gradient because the bottom plate has 0 velocity, top plate has velocity uT and let us say that the gap between the 2 plates is H. Then, roughly of the order of uT/H is the rate of change of velocity or the velocity gradient. So it creates a rate of deformation by itself by the boundary condition. So this is the way in which you create a shear by the boundary condition itself.

So this is sometimes known as the shear driven flow. So this is a mechanism of inducing shear or in other ways if there is a shear in the fluid, it is a one way by which you may simulate it artificially by having a velocity gradient, which is roughly like uT/H. So we want to simulate a rate of deformation. This is by giving a velocity to the plate and keeping a gap it is possible to impart the order of magnitude of the rate of deformation that you want.

On the top of that you might be having a pressure driven flow because of by virtue of a pressure gradient, some dp/dx is there, some negative dp/dx is there. We have seen that that is what that can write the flow through pressure gradient. So you have a pressure gradient driven flow, you have a shear driven flow and the resultant is the combination, what kind of combination that we will like to see here.

So let us set up coordinate system say this is y axis and this is x axis and let us write the equations of motion here. So we will keep all the assumptions which were there in the plane Poiseuille flow valid for this problem also. So steady flow, constant properties and fully developed flow, incompressible of course all these things. So when you say constant property by default it is incompressible that means density is constant.

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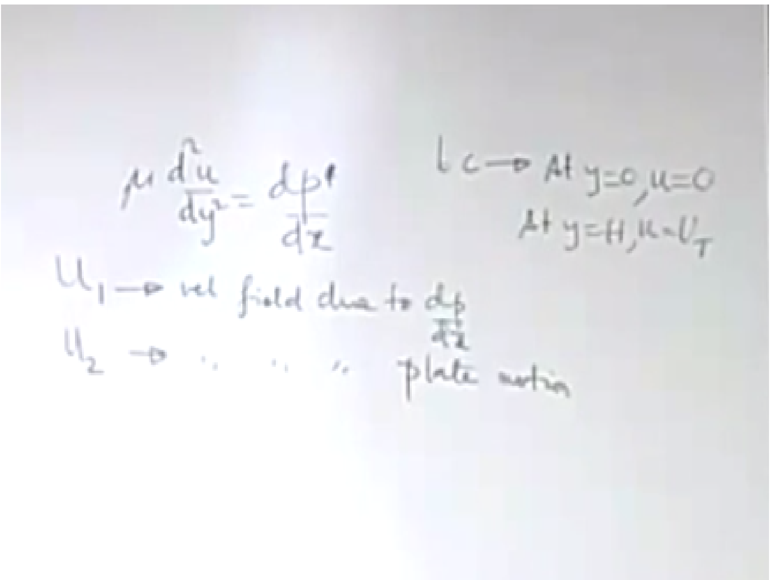

So with all these considerations into account, we have seen that you can write the equation of motion as mu d2u/dy2=dp star/dx right. See that was there for the plane Poiseuille flow problem, for this problem it does not change because what has changed is the boundary condition but in terms of like whatever is there within the domain of the problem, there is no special effect that has come into the picture or that has gone away.

So the governing equation remains the same. Now of course it is very easy and trivial to integrate it twice to get the velocity by using these boundary conditions but what we will do is we just do it in a different way, why the different way, we want to isolate the effect of this additional thing which has come into the picture. So we will consider that say u1 be the velocity field due to the pressure gradient.

And let us say u2 is the velocity field due to this plate motion. That is induced shear, so let us write a governing equation and boundary condition for u1 and u2 or u if you want to write the boundary condition, original boundary conditions at y=0, u=0 by no-slip at y=H, u=uT right. That is also by no-slip boundary condition.

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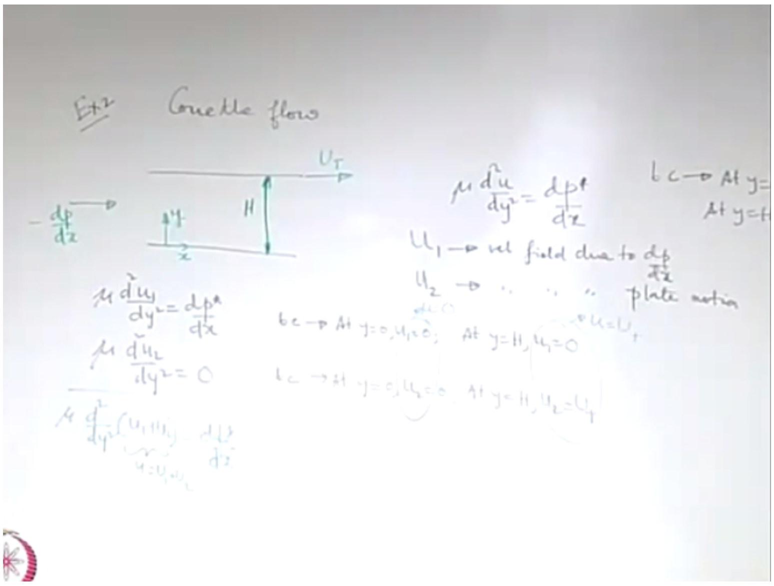

Now let us say that we are writing this problem in this way mu d2 u1/dy2=dp star/dx with a boundary condition at y=0, u1=0 and at y=H, u1=0 and mu d2u2/dy2=0 boundary condition at y=0, u2=0 and at y=H, u2=uT. If you just add these 2 equations, you will get mu d2/dy2 u1+u2=dp star/dx and if you add these 2 fx, it makes u=0. If you add these 2 fx, it makes u=uT.

So what u? u is nothing but u1+u2. Why we are able to do it so easily because of the linearity of the governing differential equation. Since this is the linear differential equation that is the governing equation if u=u1 is the solution and u=u2 is the solution, then u1+u2 is also solution. So that is why you may use this linearity or exploit this linearity to decouple it into 2 different problems.

See the first problem with u1 is what we actually solved yesterday that is the Poiseuille flow problem. Only the coordinate axis we have shifted, from the center line we have taken the y=0 as the bottom plate. So that is just a cosmetic change, in principle it changes nothing. So that problem is there as one part and the second part with u2 it totally isolates the effect of the motion of the plate.

So you are now in a position to adjust or pinpoint the effect of the plate from the solution of the u2 problem. So let us quickly solve the u1 and u2 problems.

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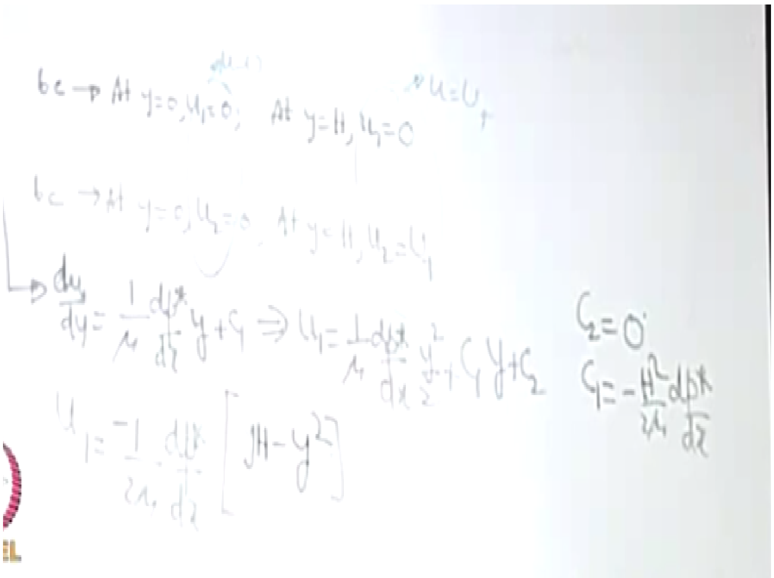

So the u1 problem if you want to solve so basically you have to integrate it 1 so mu or let us write du1/dy is 1/mu dp star/dx*y+c1 and u1=okay. So use the boundary conditions for u1, so at y=0, u1=0 that means c2=0. So the boundary conditions are c2=0 and at y=H, u1=0 so c1 is –H square/2 mu dp star/dx. So the u1 profile is 1/2 mu dp star/dx may be we put a –sign outside*H square-y H right. So this is for u1.

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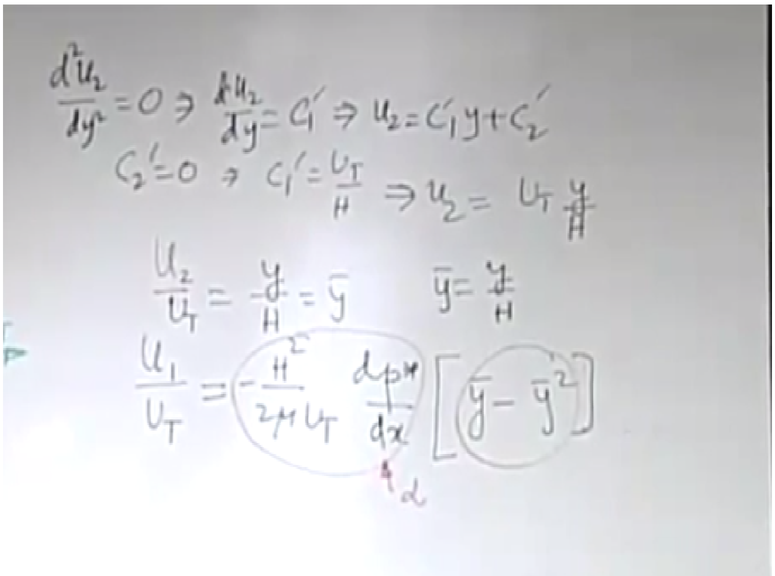

For u2, it is even simpler so d2u2/dy2=0 which means du2/dy is c1 and u2 is c1y+c2 may be another c1 say c1 prime+c2 prime. So use the boundary conditions at y=0, u2=0 that means c2 prime=0 at y=H, u2=uT that means c1 prime is uT/H. So u2 becomes uT*y/H. It is possible to write the expressions the velocity profiles for u1 and u2 in a non-dimensional form.

For example, you can write u2/uT=y/H, this is a non-dimensional way of writing it and let us give a new variable name y bar=y/H just for writing convenience. **"Professor - student conversation starts."** Which one? Last line, yes y H-y square because – sign is observed already right okay. **"Professor - student conversation ends."** Now if you non-dimensionalize u1, let us also non-dimensionalize these with respect to uT.

So we may write this -1/2 mu uT dp star/dx may be multiply both numerator and denominator by H square. So it will become y bar –y bar square okay and see if you have a careful look into these terms, this term in the bracket is non-dimensional, the left hand side is non-dimensional so whatever is here is also non-dimensional okay. So let us just give a name to this.

Let us say that this is alpha, which is a non-dimensional parameter. It depends on see this alpha is a physically a measure of what? Physically a measure of the relative strength of the pressure gradient in driving the flow with respect to the plate velocity in driving the flow right. So although it is a mathematically non-dimensional parameter but physically it represents the relative driving effects of the pressure gradient and the shear imposed.

Now with this understanding let us try to write the resultant velocity which is just simply u1+u2.

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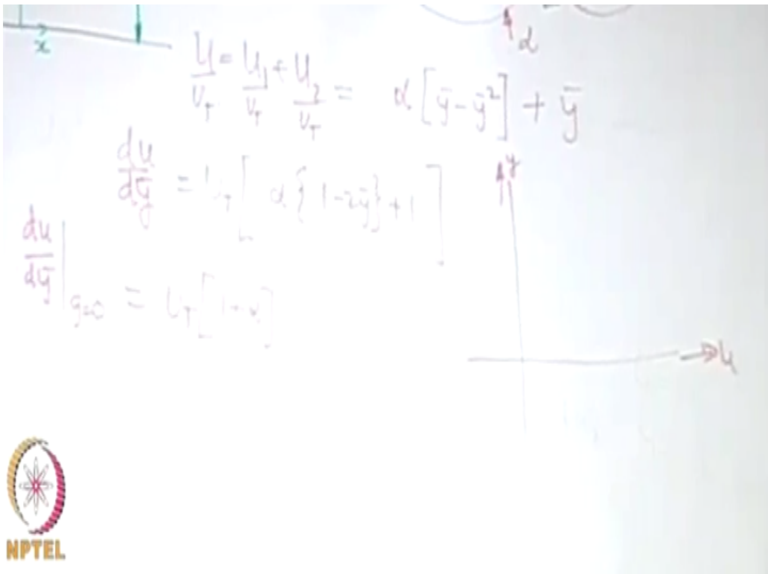

So you have u=u1+u2 which is=or maybe u/uT as u1/uT+u2/uT that is=alpha*y bar-y bar square and +y bar okay. So one important thing is that this alpha may be positive or negative.

See think of a case when uT=0, if uT=0 and if you want to drive a flow along the positive x direction only way is you must have dp/dx as negative. That we have seen or dp star/dx as negative if you consider the piezometric head as the total driving head.

Now if you come to a situation when uT is positive then there might be a case when dp/dx is positive but still the flow is maintained along the positive direction because positive dp/dx will try to have a sort of resistive effect on the flow along x but uT is trying to drive the flow along the x direction. The resultant maybe such that uT is successful and then the flow maybe along the positive direction.

So let us try to see that what are the typical velocity gradients. So for different values of alpha let us first find out what is du/dy and we will get important picture. So du/dy bar say=uT*alpha*1-2 y bar+1 so let us try to find out what is du/dy bar at y bar=0. So that is=uT*1+alpha. So if you want to draw the velocity profile say this is the sketch where we want to draw the velocity profile.

So along the x axis we will plot velocity, along the y axis we will plot the y coordinate. So if you see let us consider a value of alpha, which is negative. So value of alpha negative means what? Value of alpha negative means what happens to this? So alpha is negative means dp/dx is positive. So dp/dx positive is adverse pressure gradient. So we have discussed it earlier.

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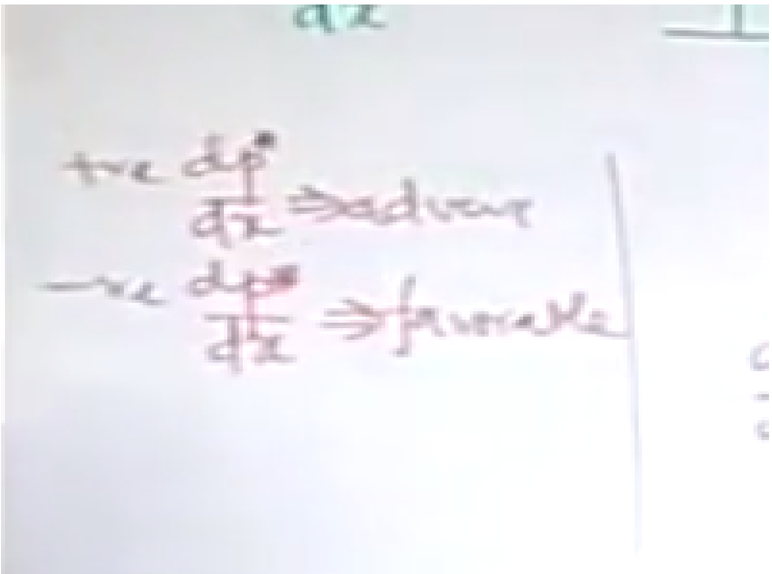

So always remember that positive dp/dx or dp star/dx implies adverse because we have seen that negative dp star/dx is something which helps in maintaining the flow along the positive x direction. So this is called as favorable pressure gradient. So adverse means it tries to have an opposition to the flow. So when you have positive dp/dx that is if you have say if you have a positive dp/dx that means you have negative alpha.

Now the situation is the du/dy at y=0 depends on the extent of negativity of alpha. So there maybe one range of alpha where alpha is<-1 and another range is alpha is>-1. If alpha is>-1 although it is negative but sum of this will be positive. So you will have a positive slope of the velocity profile but if you have alpha<-1, let us take an example say alpha=-2 so if you take an example of alpha=-2, this is –uT in a non-dimensional form of course.

So what it means is that if you have a negative slope of the velocity profile that means the velocity see at the wall it is 0 because of no slip.

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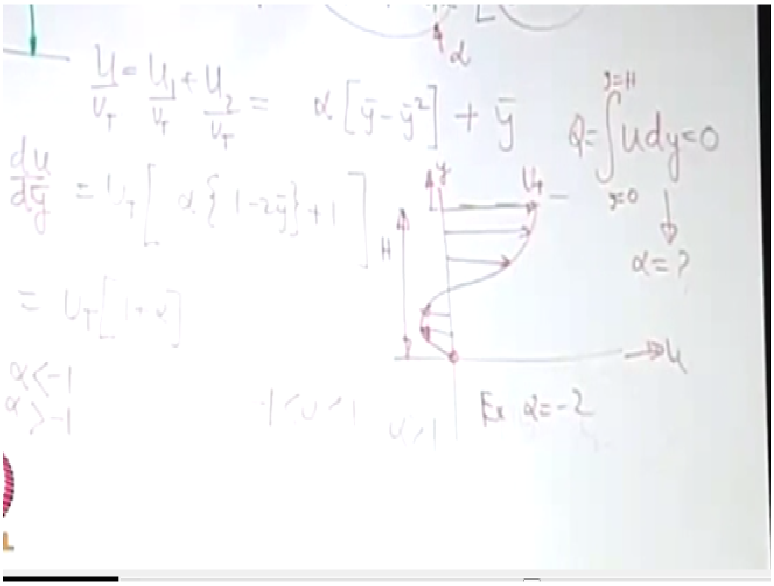

Then the velocity actually becomes negative because the slope is negative and if you want to consider the upper plate see let us say here is the upper plate here you have to have the velocity as uT, which is like this that is why the boundary condition. So let us say that is=uT. So somehow this started with negativity but it has to match with here, so it must somehow cross this axis.

So in the process it will have a sort of minima at a point which you can easily find out by differentiating u with respect to y and setting it to 0. So you are having a case where you may have a sort of back flow say this is an example with alpha=-2 as an example. So the important understanding is that you may get different sorts of velocity profiles depending on the value of alpha.

Say you have alpha=1 as an example. So if you have alpha=1, you see that the situation is different. For alpha=1 you have du/dy as something which is positive at y=0 and when you want to sketch the velocity profile what are the things you should look for and this you should practice at home, I am not going to do it for all possible ranges of values of alpha but that you should do yourself.

But important characteristics of the plot you should look for what is du/dy at y=0 that is how it starts off and where is the location of du/dy=0 that is a sort of location of maxima or minima in the velocity profile. If it starts off with the negative slope, then it will be minima. If it starts up with a positive slope it will be a maxima somewhere. So that you have to find out whether it is a maxima or a minima.

Where is it located is it because it has to eventually located between 0 to H and why we have chosen this example for demonstration say alpha is a negative number, which is in magnitude >1 because it gives a sort of interesting competition between the pressure gradient and the driving shear. If you take alpha as something positive, then this dp star/dx negative is the trivial case.

Then what happens the pressure gradient also drives the flow in a positive direction and the motion of the top plate also drives the flow along the positive direction. So that is like they just aid each other but here is a case where the pressure gradient is adverse, it tries to oppose the flow whereas the motion of the top plate tries to aid the flow. Where the opposition effect is strong?

Opposition effect is strong close to the bottom plate, which is the farthest away from the location where the external induction of the motion along the positive x is there. So here you see the locally back flow and as you go further and further away towards the top the effect of the top plate will dominate.

Now there may be a situation when let us say that whatever is the effect of the integrated effect of this velocity same as the integrated effect of this negative velocity then no matter whether you have some velocity distribution the flow rate of each section maybe 0. So that means you may come up with a situation where integral of udy from y=0 to y=H that is=0 that is Q. So you find out what is that condition? What should be the value of alpha for that? That you may easily do by integrating this and finding the value of alpha. You will see that it will come for a value of alpha corresponding to dp/dx positive.

Because if dp/dx is negative that will never occur, both effects will help each other. This is the case when it will occur when both effects will oppose each other. So that the resultant flow rate is 0, so although you have velocities locally but the integrated effect of the velocities is 0. So the combination of different pressure gradient and shear that you should try to look into by trying for different values of alpha.

What values typically you should try say alpha between -1 to 1 and alpha>1 example say alpha=2 like that and alpha < -1 and that example that we have taken as 1 alpha=-2. So for these ranges you try to assess the nature of the characteristics and try to make a sketch of the velocity profile. This type of thing is very important. So you make a sketch of the velocity profile does not mean that you actually plot it.

You assess the nature of the variation by the slope and so on and try to make an assessment of qualitatively how the velocity will vary from bottom plate yo the top plate.

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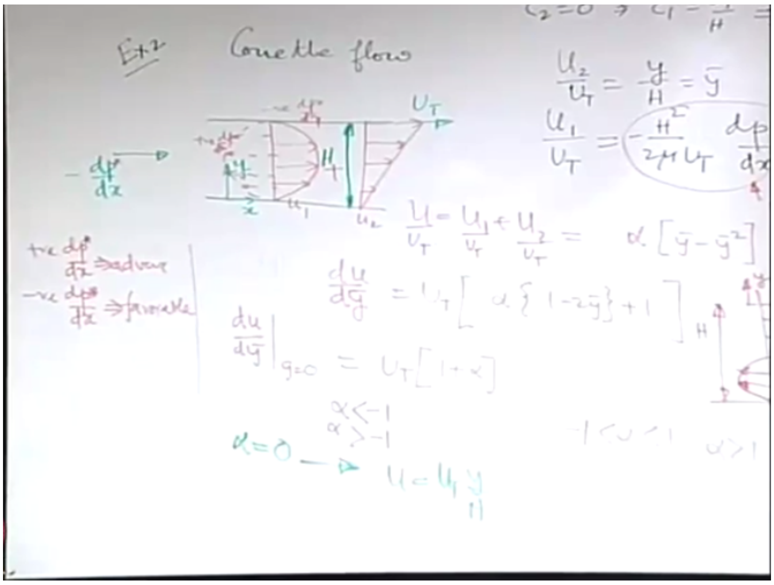

Now a very simple situation corresponding to a special case of this is alpha=0. So when you have alpha=0 that means there is no pressure gradient, so neither favorable nor adverse and then just the solution is very simple u=uT*y/H. This is known as the simple Couette flow that is the Couette flow without any pressure gradient. So this is the case where only the shear is driving the flow and the velocity profile for this is linear.

So you can see that if you want to superimposed to velocity profiles to get the resultant velocity profile, so you have the velocity profile as the sum of the velocity profile due to Poiseuille flow which is like this for dp/dx which is negative, this is for negative dp/dx+a linear velocity profile because of the motion of the plate. So this is u1, this is u2. So if negative dp/dx, the resultant is the algebraic sum of these two just.

That is what we got from the differential equation u1+u2 but if the dp/dx is in the other way you might have this is say positive dp/dx, you might have the pressure driven velocity profile directly in the opposite sense and then the resultant of these 2 maybe either positive or negative. That is very obvious but that is one of the important observations from the individual velocity profiles.

So alpha=0 is a special case of just a simple Couette flow. Let us take a third example.

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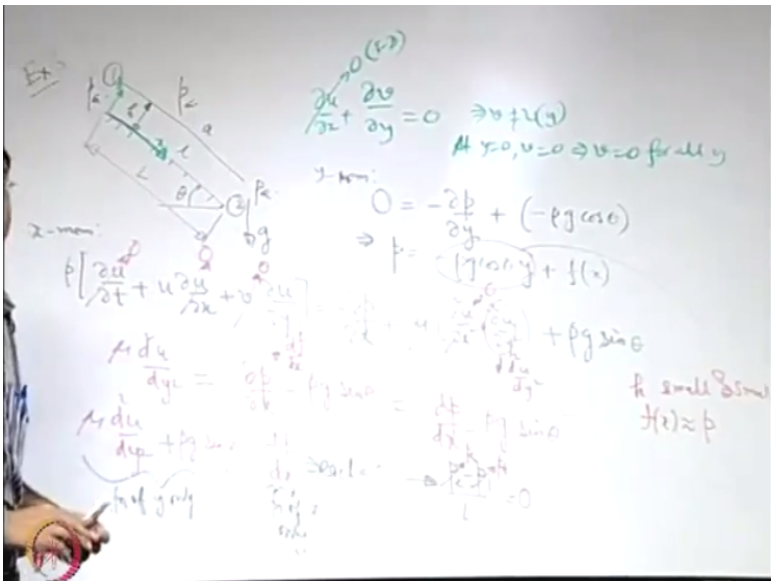

Let us say that you have a inclined plane over which you have a thin film of liquid of uniform film thickness h. So here there is some liquid and here there is air. So this line which is at the top is the interface between the liquid and the air. Gravity is acting like this and the inclined plane makes an angle theta with the horizontal and pressure at air at all locations is p atmosphere.

So you have even like the left of the film p atmosphere the right of the film p atmosphere like that and this is just a liquid film. So if h is very thin, it is known as a thin film flow and we will like to see that what kind of velocity profile in the film you get in this case and again let us assume steady constant property and fully developed flow. These are the 3 assumptions that we are considering here also valid.

So let us write the momentum equations. What is the consequence of the continuity equation? So let us set up the coordinates, most of the things are very similar to what the 2 cases that we have already done. So we will summarize that. So let us say that this is like x axis and this is like y axis. If you try to use the continuity equation, then what is the consequence of the continuity equation? V is identically=0 for fully developed flow.

So the situation does not change here why? You have the continuity equation for incompressible flow=this. For fully developed flow, u is not a function of x, so that means v is not a function of y. So here you are not maybe sure about the interface between the liquid and the air but the bottom wall at least you know that at y=0 v must be=0, no penetration boundary condition.

So at y=0, v=0 this means v=0 for all y. So you are having to deal with a situation where you have only u as the velocity component. Again, we are modeling it as a 2-dimensional flow so the third dimension perpendicular to the plane of the board is quite large in comparison to the x and y dimensions. So if you write the y momentum equation, so again it will look like the same as what we did for the Poiseuille flow.

So we are just summarizing it left hand side you have all the terms involving the gradients of v so that those are 0 because v is 0 then –this+again all the terms involving v+the body force along y. So what is the body force along y? So –rho g cos theta. So you have p=-rho g cos theta*y+some function of x. Then our more important concern is the x momentum equation. So let us write the x momentum equation.

So if you write the x momentum equation for steady flow, so you have rho, for the steady flow, the unsteady term goes away so anyway let us write all the terms and see which term goes away. So this is y momentum, now we are looking for x momentum. Then what is the body force here? rho g sin theta, it is along positive x, g sin theta is along positive x. So now let us see which terms are there and which terms are not.

First of all, because of steady flow this term is 0, then fully developed flow means this term is 0 and v is identically=0. So left hand side again has become=0. Since fully developed flow you have the second order derivative of u also with respect to x=0 and u is the function of y only. So this becomes the ordinary derivative d2u/dy2. So you have mu d2u/dy2=now what is dp/dx? Dp/dx is df/dx basically.

Now one important assumption is that you have this gap h is very small so that we call as a thin film. So if h is small what is an important consequence? The effect of the body force along the y direction is not important because of the very thin gap, the effect of gravity within the thin gap is not important. So because h is small, this particular effect is small. That means fx is approximately=p or for a thin film it is as good as p.

There is no difference so you have basically this is=dp/dx-rho g sin theta. So you can write it bit differently, you can write mu d2u/dy2+rho g sin theta=dp/dx right. Again you see the left hand side you can write as function of y only of course this is like a constant but constant is the special case of function of y, so this is the function of y only. This is the function of x only.

And this implies each must be a constant, so when each is a constant then dp/dx is the constant means what? Let us say that L is the total length of this, dp/dx is the constant means the variation of pressure with x is linear. So if you say that the left end is 1 and the right end is 2 then basically you have dp/dx is what? p2-p1/L. Now from the boundary condition you see that both p1 and p2 are p atmosphere right.

Because it is exposed to uniform atmosphere so this is p atmosphere and this is p atmosphere therefore c=0 okay. So this is the case where c=0 but the way in which the flow is revealed is because here the gravity acts like a driving pressure gradient. So we have seen there is a piezometric head that is important and the gravity part of the head is giving the piezometric head.

So it is from a higher elevation to lower elevation and that is what is driving the flow. That is what is like equivalent negative dp star/dx. If you include this in the p star although p change is not there but the gravity effect is there and then like it is very easy and straightforward to integrate it twice to get the velocity profile. We are not repeating anymore but what we will now try to pinpoint is what will be the boundary conditions.

See many problems have same governing equation but the solution of one problem differs from the other through the boundary condition. So let us try to see that what are the boundary conditions here? So let us write the boundary conditions.

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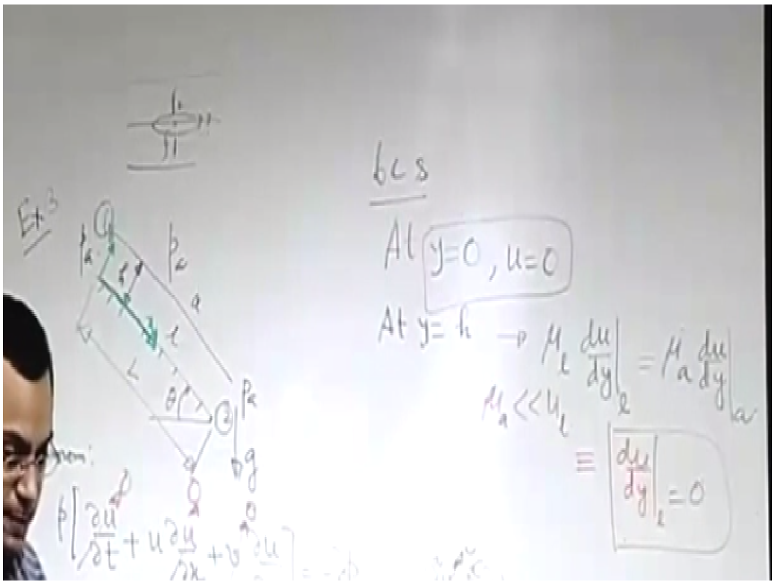

So boundary conditions, what are the boundary conditions at y=0? What is the boundary condition? Yes, what is the boundary condition at y=0? **“Professor - student conversation starts.”** U=0 that is fine what is the boundary condition at y=H? Why du/dy=0 who says that there will be no shear stress here. **“Professor - student conversation ends.”** God has said that there will be no shear stress here, where from you get that principle, which fundamental principle tells that there will be no shear stress here?

Which fundamental principle you tell? There is no such fundamental principle. The fundamental principle is like this. Here you have 2 fluids, here is a fluid and this liquid is a fluid. So fundamental it is like a problem where you have say fluid 1 and fluid 2, you are thinking about the interface condition. At the interface what should be the condition? You have continuity in certain things?

What are the things which are continuous? First of all, velocity is continuous because at a point you cannot have different velocities when you consider 2 different fluids. So velocity at fluid 1=velocity at fluid 2 at the interface but that is something what is like what we cannot use here because we do not know what is the velocity. The other thing is that there should be continuity in shear stress.

Because whatever is the shear stress at the interface in the fluid one side same shear stress should be there in the fluid 2 side. So it is a continuity in a shear stress. Because at a particular interface there cannot be different momentum transported the 2 sides. So the shear stress must be continuous. So the continuity in shear stress means here it is just a 1-dimensional type of flow that means mu du/dy must be continuous.

So at y=H, you must have mu of the liquid*du/dy measured from the liquid side at the interface=mu of the air*du/dy at the air okay because the viscosity of air is much, much less than that of the liquid, mu a is much, much<mu l. This term is much, much less that is mu a/mu l if you write this term is so small that it multiplies with whatever that smallness remains.

So this becomes as good as du/dy=0. So whatever you have said du/dy=0 here as the boundary condition or equivalent to 0 shear is correct but in an approximate sense. If I now replace air with another liquid, then you will be in trouble because then if you do not know what is the fundamental principle from which comes out still you will write a wrong thing. So important is the fundamental principle is continuity of the shear stress.

Here the special case is just because air it is a coincidence that air has much, much less viscosity than the liquid water say this is water. So roughly like 1/1000 so then in that case this right hand side is so small that we can say that it is as good as 0. So now the 2 boundary conditions are at y=0, u=0 and at y=H du/dy=0 and then based on this boundary conditions you can integrate this to get the velocity profile that I am not going to do, very simple exercise okay.

Now what we will do is we will see the consequence of having 2 fluids, which are such that viscosity of one is not negligible in comparison to the other. So we have seen one such movie earlier and what you will do is we will look into the one of those movies again.

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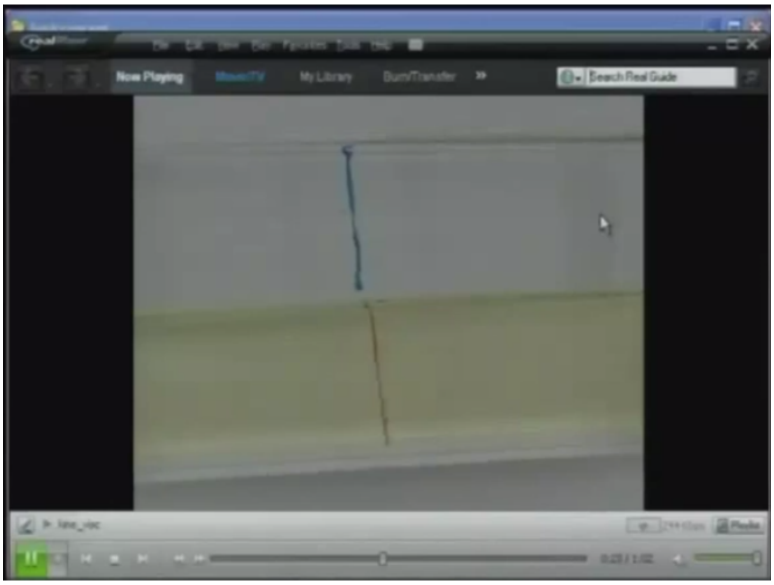

So let us look into one of those movies where we were basically looking into effect of viscosities of a 2 layered fluid. So just try to see that you have a 2 layered fluid, you can just perceive from the difference in color that you have 2 different layers and just colored dyes are there, 2 different color dyes are there in the 2 fluids to indicate the velocity profile and you see that the velocity profiles are such that at the interface there is no continuity in the slope.

That is du/dy is discontinuous at the interface that you can clearly see. The reason is it is not the continuity of du/dy that is important, continuity of mu*du/dy because mu of the 2 fluids are different, du/dy at the interface has to be adjusted. So when you have such 2 different fluids with 2 different viscosities you should not look for a continuity in du/dy, you should look for a continuity in mu*du/dy, that is what is the important thing okay.

Now let us come back to what we were discussing and we will now try to look into one example problem where we will use a different coordinate system. So till now we have used the Cartesian coordinate system to solve certain problems. Now we will come to an example 4 and then subsequently 1 example 5 where we will be using a different coordinate system.

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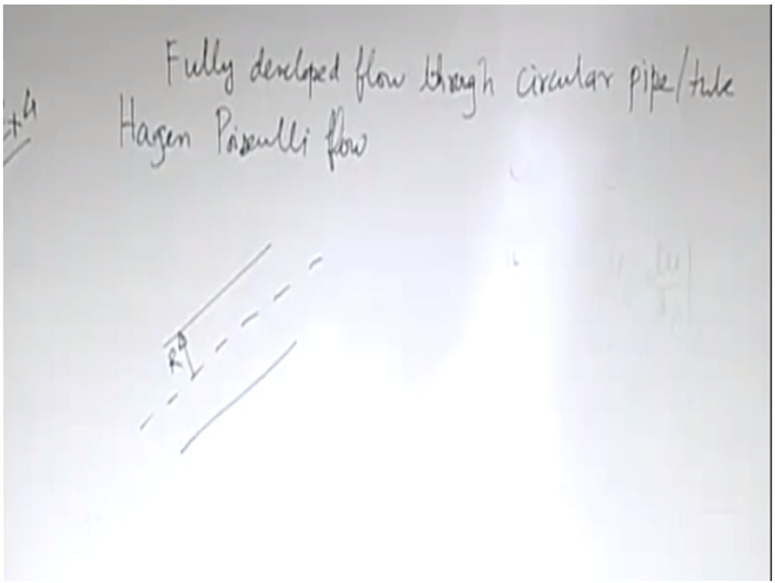

And the choice of the coordinate system is often driven by the geometry of the flow. So here what we are going to do is we are going to discuss something which is important from engineering applications, fully developed flow through circular pipes, so that is known as Hagen-Poiseuille flow. That is basically fully developed flow through circular pipes or tubes or whatever.

So let us say that we have a circular tube or pipe like this. When we draw it looks like a parallel plate channel because of course we are drawing the projection of it in the plane of the board but keep in mind that we are talking now about a problem with cylindrical symmetry where because the pipe surface here we are talking about is a circular cross section.

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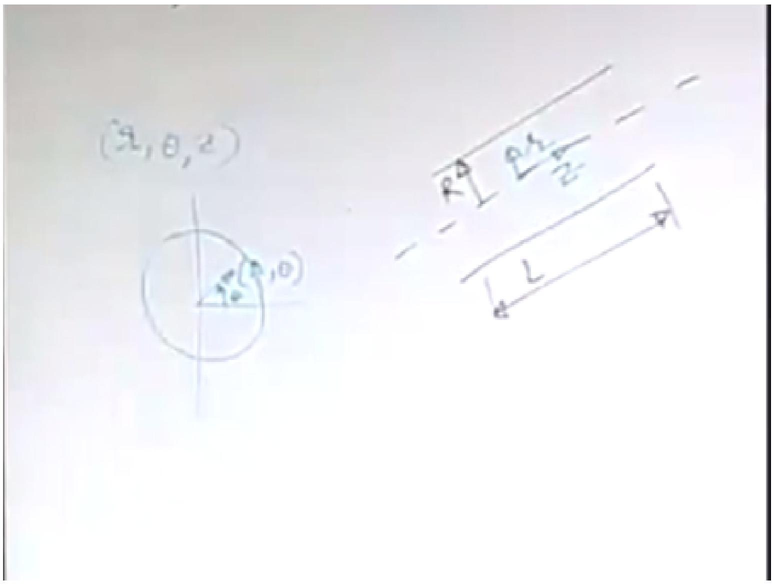

So you have a pipe of say maybe we are talking about an axial length of L and fully developed flow. So the coordinate systems what we use for this case are the cylindrical polar coordinate systems, which are more common. So you have small r as the radial coordinate, z as the axial coordinate and if you take the cross section of this, so if you draw separately the sectional view, it is a circle.

So the cross section wise the coordinate system is a polar coordinate system. So at a point if you consider it is r, theta. So this is r and this angle is theta. So the coordinate system requires r theta z and these are again mutually orthogonal coordinate system because theta direction is perpendicular to r and r is perpendicular to z and that means you have now r theta z all these are basically mutually orthogonal to each other.

Now we will be using therefore now the Navier stokes equation in the r theta z or the cylindrical polar coordinate system. So already we have communicated the corresponding form from the equations through the course website. You must have those equations with you, important thing is it is not necessary that these equations you have to remember. We will provide all these equations to you I mean during exams and so on because our test is not whether you can remember these equations.

So whatever is there with you is something what you will have if there is a question related to that it comes in the exam. Our important objective will be how to simplify those equations and come to the solution of this problem. So let us look into some of those equations. I will project those equations in the screen so that which are already there and it will be possible for you to follow it easily.

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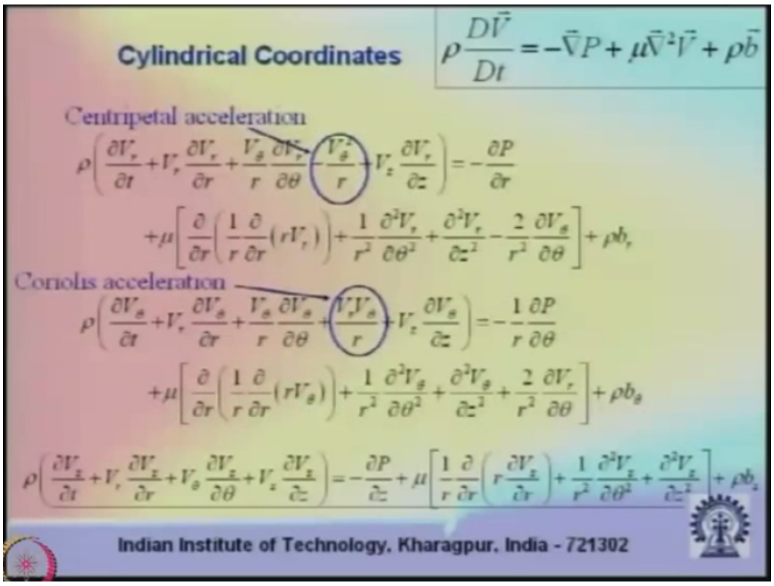

So let us look into the cylindrical coordinate form of the Navier-Stokes equation and we will try to solve the Navier-Stokes equation with the cylindrical coordinate form. To do that we will always start first with the continuity equation and see that what the continuity equation gives to us.

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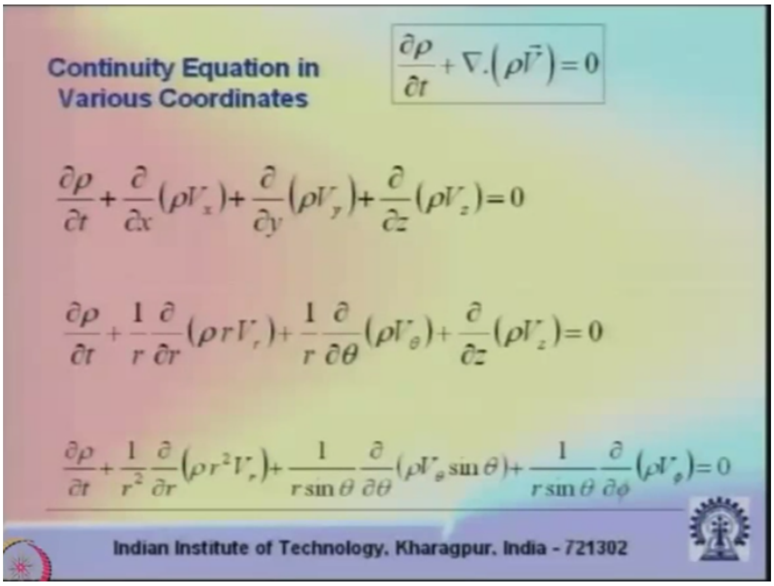

So let us look into the continuity equation form in the cylindrical coordinate. So if you look into the continuity equation, if you look into say the slide the second equation, so the first term is that the density variation with respect to time, so we are considering a constant density fluid. So that term goes away, so let us see what the continuity equation gives to us okay.

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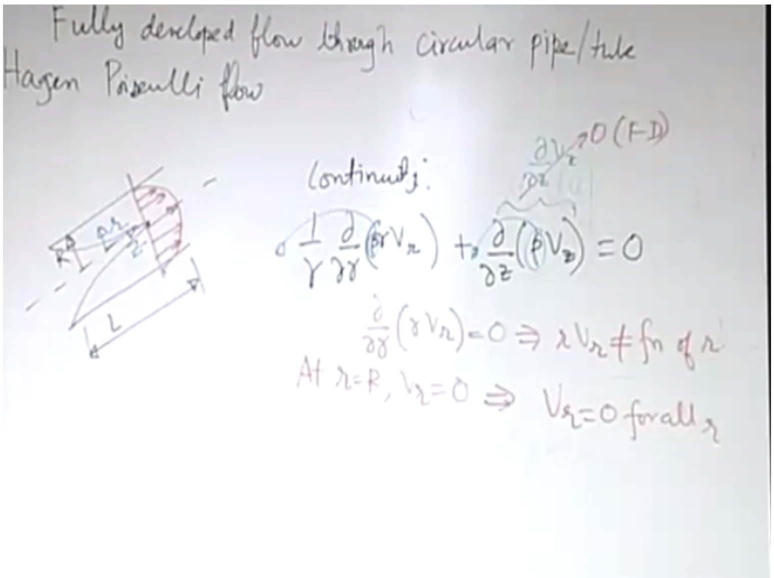

Next, let us come back to the slide again. See the second term in the continuity equation. The second term it remains because we are not very sure that what is going to be there. So 1/r *d/dr*rho*rho r vr so that is the term and what about the third term, so you have a third term. What about this term? First of all, rho is the constant so rho is the constant and hence it comes out of the derivatives. So here also rho comes out of the derivatives.

Then this term boils down to what? It boils down to the derivative of the gradient of vz with respect to z okay. What is the fully developed flow? See here also the concept of parallel plate channel does not change. That is, you have the boundary layer, which are merging and beyond that the velocity profile whatever is going to be there that does not change further with the axial coordinate.

The axial coordinate is the z coordinate. So that means vz is the velocity component along the axial direction that does not change with the z coordinate any further. So fully developed flow means this is=0 just like du/dx was 0 for parallel plates here just replace u with vz and x with z axial coordinate. So then you are left with d/dr of r vr that is=0 right. We are not considering that singular case.

These are the singularity at r=0, so we are not considering that singular case but for a general case. So that means r*vr is not a function of r right. Only where it is possible, there are 2 ways, so one way is if you do not look into the boundary condition say you do not look into the boundary condition but you just want to look into these so then if you have vr=c/r of that form but then that will be a conclusion made without looking into the boundary condition.

Because this has to be satisfied within the constraints of the boundary condition not just as a general case. So what is the constraint of the boundary condition that at the wall at small r=R you must have vr=0 that is a no penetration boundary condition, that is the fluid cannot penetrate through the wall in the radial direction. So which means that you have since at r=R vr=0 which means vr=0 for all r from this one except r=0.

That is the singularity, we are not talking about that. So whenever we are coming up with the solution we are cleverly avoiding the singularity point and then for the remaining that from this since this is not a function of r basically you forget about the r=0 case that means it is as good as vr not a function of r so if you have found out that vr=0 at r=R it should be 0 for all r. So from the continuity equation or conclusion is that vr=0 just like v=0 for parallel plate channels.

Next, we will go into the momentum equation. Let us say so we have how many versions of the momentum equation.

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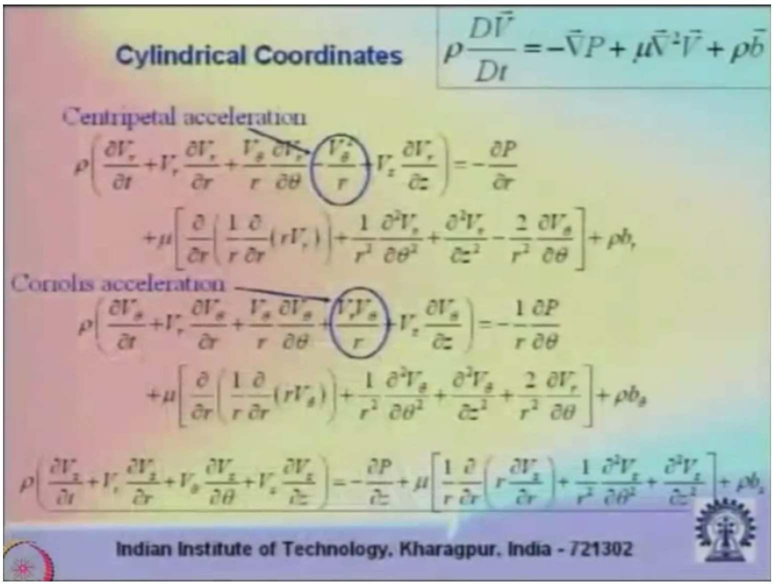

So if you see the cylindrical coordinates you have the r theta and z. So let us look into the case with theta. See first of all what about the derivatives with respect to theta? So if you look into these equations, you will see that there are lots of terms which are having derivatives with respect to theta. So if you want to figure out that what happens for the derivatives with respect to theta, you look into the equations very carefully.

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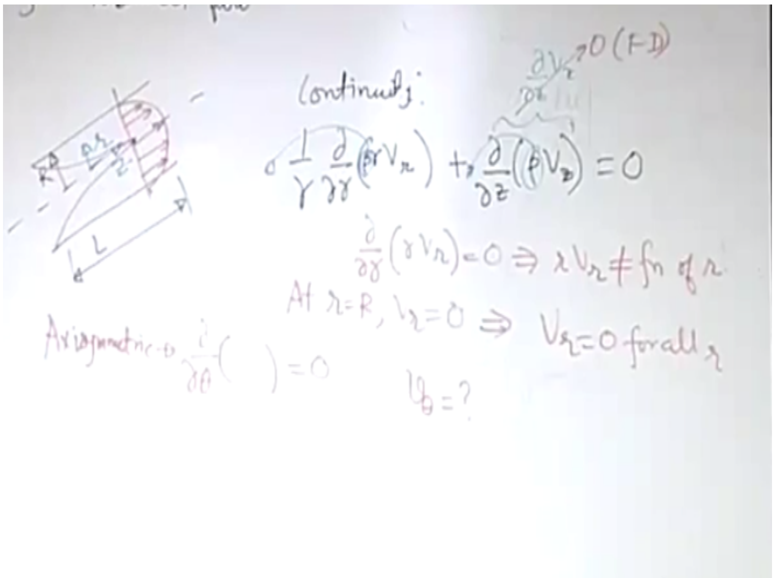

Now when you have derivatives with respect to theta in all the momentum equations such terms are there, so terms of the form derivatives with respect to theta. Question is that when will it be 0. See theta is what? Theta is the polar coordinate, so if you have symmetry with respect to the axis that means that is called an axisymmetric flow. Then the symmetry with respect to axis means that there will be no variation with respect to theta.

It is only symmetric with respect to axis at which theta it is located is immaterial. So axisymmetric problem will always mean this=0. When you have axisymmetric problem which means that this=0 you have to keep another thing in mind that axisymmetric problem does not ensure that v theta=0, you could still have v theta but that not a function of theta. So if you have a v theta component, it is something called as a swirl component.

Because it tries to have a rotation of swirl in the flow but here we are considering that there is no v theta. So if there is no v theta I mean if you look into the continuity equations see the continuity equation is satisfied.

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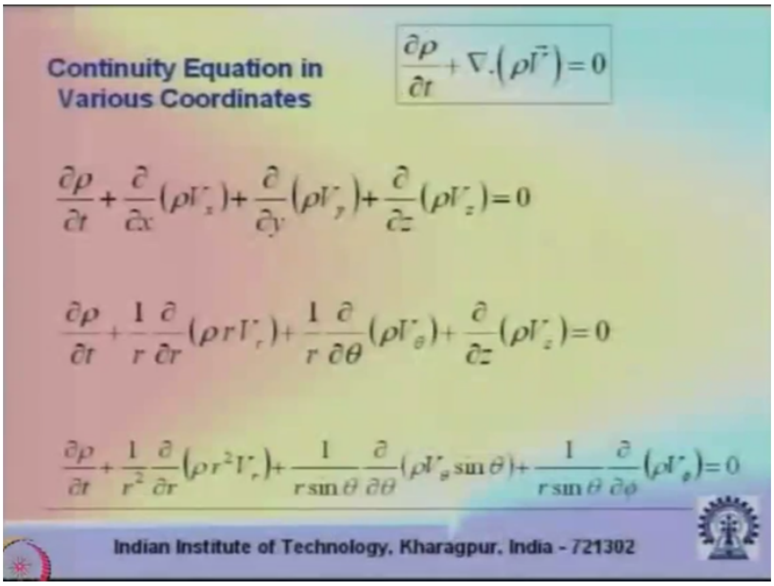

Let us go back to the continuity equation, you see that in the continuity equation there is a term d/d theta of rho v theta so that term we omitted because of the axisymmetric condition. So in the continuity equation, we did not consider v theta that does not mean that v theta is not there. Simply, the theta gradient of v theta was 0 because of the axisymmetric condition. So till now we have never utilized the condition that v theta is 0.

But now we will utilize it by noting that it is not a swirling type of flow. There may be swirling type of flow in a pipe where there is a rotationality that is imparted to the flow but here we are considering that such rotationality is not imparted to the flow.

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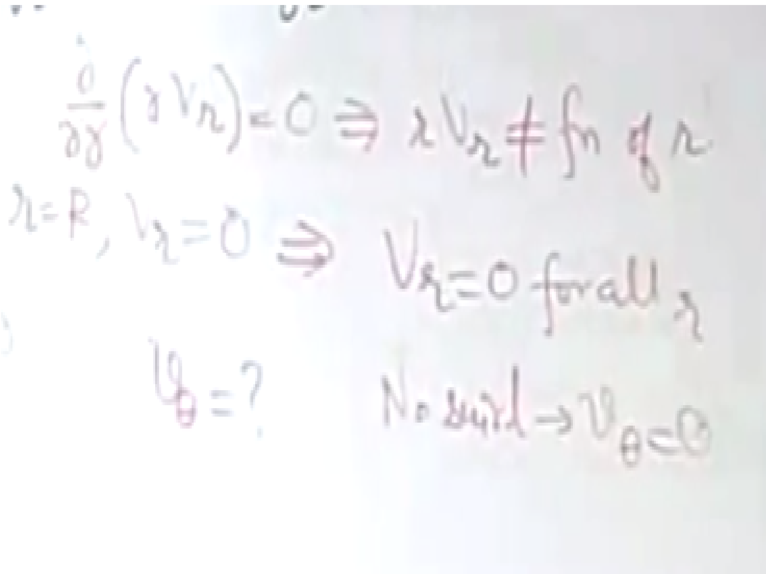

And that means you have for this problem so no swirl is considered that means you have v theta=0. So you see certain conclusions are there. Conclusions at the end are important but where from you arrive at the conclusions I feel is even more important. So now we go back to the momentum equations again.

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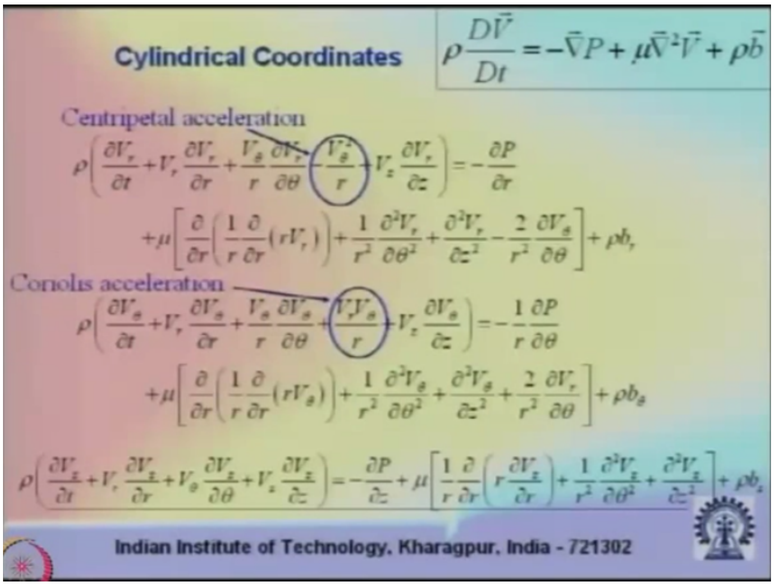

So in the momentum equation let us look into the theta component of the momentum equation. So theta component of the momentum equation has all terms involving the derivatives with respect to theta right, in fact all. So the theta component of the momentum equation will have only one term which does not involve derivative with respect to theta. What is that term? So if you look into the theta component, so in the theta component the first term is dv theta/dt=0.

Second term vr dv theta/dr=0 because v theta is 0 that term is 0, third term has vr and v theta so when v theta is 0 that is=0, fourth term also has v theta, so left hand side is 0, right hand side no p variation with theta, so that term is 0, next term you have a v theta so although the r derivative is there but because v theta is 0 that is 0. So if it is a swirl flow that term is not 0 but when there is no swirl that term is 0.

Remaining terms have either v theta or derivatives with respect to theta, either of those are 0 so it is like 0=0. So theta momentum equation does not give us anything for this problem. Then the r momentum equation, so let us look into the r momentum equation. So for the r momentum equation you see the first term. First term is dvr/dt that is because of steady flow the first term is 0.

Second term vr=0 identically that you have to keep in mind because of fully developed flow. So that means the second term is 0 and the third term is 0 either way either vr is 0 or the theta gradient is 0 either way it is 0, fourth term you have a v theta square/r, v theta square/r is like the centripetal acceleration term and because the swirl component of the rotational component is not there the fourth term is 0.

Then the fifth term vr is 0, so that term is also 0, right hand side –dp/dr now this term is there. So at least we have found out one term which is there. So –dp/dr then next term, next term is again 0 because vr is 0. Then the remaining term so d2 vr d theta 2 that is 0, d2vr dz2=0, dv theta d theta=0. You have a rho br, so rho br is like if you have a body force along the r direction.

So if you have a body force along the r direction then that is what is very much possible. So you see that there is only one component, which is vz and there is no vz term in this.

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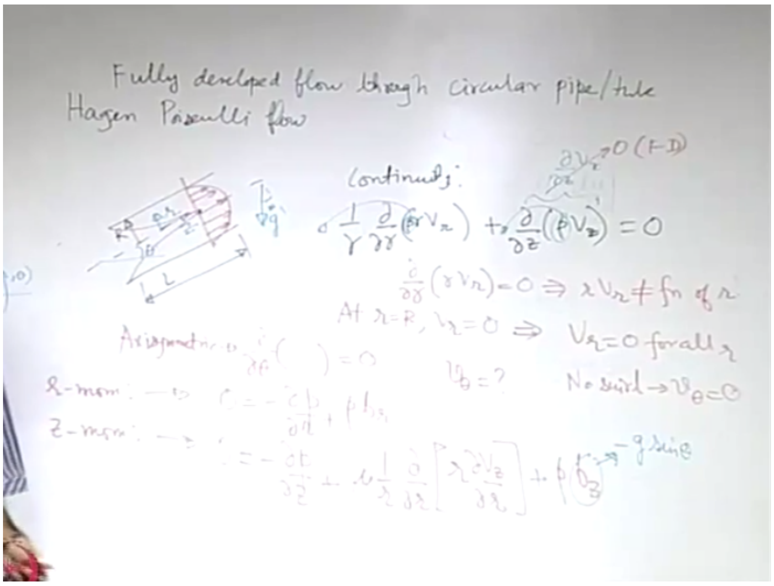

So you have basically the r momentum equation giving what 0=-dp/dr+rho*vr okay. Now we come to the z momentum equation, which is going to be the most important equation for our velocity profile solution because our velocity profile is vz. So look into the terms. First term so we are talking about the last equation, which is there in the slide. So first term the unsteady term it is 0.

Then next term is vr is there, so that is 0, third term v theta is there that is 0, fourth term v dvz/ dz is there which is 0 for fully developed flow. So the left hand side has become 0, right hand side you have –dp/dz which is there. Next term, the first term is very much there because vz as a function of r is what is reflected in that term and second term in the square bracket vz is not a function of theta.

So that is 0 and the third term is 0 by fully developed flow. So we are left with the z momentum as 0=-this one+mu*1/r this*r+okay and what will be bz here, so if you have the acceleration due to gravity like this, so let us say that this axis of the pipe is inclined at an angle theta with the horizontal. So you have this angle as theta and this angle as 90-theta. So you have a br as -g cos theta and bz as -g sin theta, so this is –g sin theta okay.

So the forms of the equations that where we have arrived are something which are very much similar to what we have arrived for parallel plate channels, only because of the cylindrical symmetry you have this 1/r d/dr this type of term so but if you write it in the divergence form, in the vector form or here the Laplacian operator form, then that is basically identical. So vector form is identical just because of the shift of the coordinate system.

This is what the second derivative of vz with respect to r looks like, whatever was there in the Cartesian system equivalent in this cylindrical system okay. So we stop here now and we will continue with this in the next class. Thank you.