Introduction to Fluid Mechanics and Fluid Engineering
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Lecture - 28
Dynamics of Viscous Flows: Navier Stokes Equation

Today, we will start with our discussion on the dynamics of viscous flows. As you remember, we were discussing about the transformation of the integral forms of the differential equation, integral forms of the conservation equations to the corresponding differential forms and from that we came up with the differential form for the momentum conservation. And we will start with that differential form.

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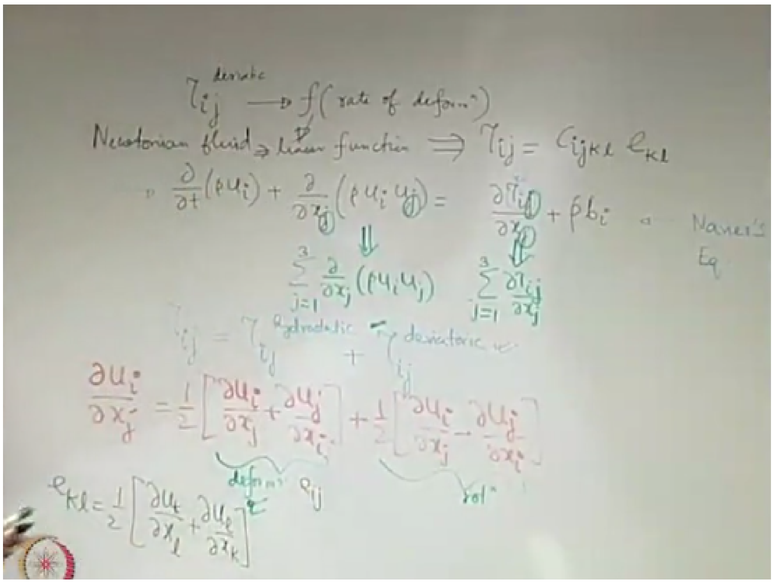

So, what was the differential form? Let us write it once more. This was the differential form that we derive. We should keep in mind that this, here everything is written in the index notation. That means there are certain terms where there is an invisible summation. In which term there are summations? There are 2 terms in the left hand side and 2 terms in the right hand side. So, out of these terms in which terms you have summations?

Second term in the left hand side because you have this repeated index, right. So, this is actually summation of this one with j running from 1 to 3. That means wherever there is such a repeated index this is like a dummy index in place of j you could right k, l, m whatever because that is not the index that is like free for you. I mean it is summed up. Here also, it is j = 1 to 3 because it is a repeated index.

So, whatever index is not a repeated index where i is not a repeated index, so i were free to choose. So, i = 1 will give the x component of the momentum conservation equation i = 2 will give the y component i = 3 will give the c component. That is the basic way of looking into it. So, it is also possible therefore to write in a way in which you write the velocity components u, v, w.

So, alternative notations if you write in terms of u, v, w, you write, let us say you write i = 1, so it is u 1 which is equivalent to u the x component of the velocity. So, what is this? Then such summations, so. This is the equation written for motion along x. You see the advantage with the index notation is so compact and this when you have expanded it, it does not look that compact.

Obviously, it means the same whenever we will be looking for analytical solutions we will be expanding in this form. But at the same time for the convenience of writing and understanding the index notation many times is quite suitable. Now, we will start with this equation, we will use the index notation for deriving the rest of the things. But whatever is our final form again we will come back to the alternative notations so that you may use those forms for solving the problems.

Now, we will start with this form with a question what are the assumptions under which this is valid because we may proceed only from that. So, we need to understand that what are the assumptions which are inbuilt with this equation? What kind of assumptions where there? Will continuum, yes because these are continuum conservation equations. Any more assumption? The reference frame is stationery that was the assumption, right.

So, if you have a moving reference frame it would be easy and straight forward to derive the form because if you recall how did these terms appear. So, you had a resultant force in the left hand side of the statement of the Reynolds transport theorem. There you had surface forces and body forces. This term came from surface forces this term came from body forces and in the right hand side there was a transient term from which this came.

And there was an advection term because of the transport of the fluid across the surface of the control volume from that this term came by converting it into a volume integral. So, it is important to keep in mind that what these are. We will see later on that it is equivalent to an equation written in a description where the left hand side is like acceleration part you must say and right hand side is force.

So, it is as good as like almost like Newton's second law of motion. Now, the other question is that which we partially address that what are the physical significances of these terms. And if you look for these terms we will recognize that these right hand side we had recognized the left hand side we will see that it will come as some function of acceleration or it will be directly related to the acceleration.

So, acceleration of the flow is related somehow with the transients that is the unsteady effect that is at a particular location if things change with time and if there is a change because of the change in position from one point to another point where the particle finds a new flow field. That is the advective effect. And the combination of that is the total derivative we will be able to express this in terms of the total derivative that we will do subsequently.

If you want to write this for a non-inertial reference frame you have to just keep in mind that the left hand side for a non-inertial reference frame is corrected with a correction factor – equivalent to like a D'Alembert's force like – mass * acceleration that type of term where that is like a relative acceleration. In that relative acceleration you have terms related to the angular velocity of the reference frame or angular acceleration of the reference frame like that.

So, you can incorporate those correction terms and that is itself a volume integral and derive it in the same way you will see that you will get some extra terms because of the transformation of coordinates. Otherwise this is quite straight forward and the advantage or the like the elegance of this form is that it is independent of the nature of the fluid that you are looking for.

Till now we have not committed ourselves to whether the fluid is Newtonian, non-Newtonian or whatever. It is just for any generic fluid. So, this equation is known as Navier's equation of equilibrium. This much we discussed in the last class but this is just a brief recapitulation so that we may start with this. Now our objective will be to see that what equations we have with us. See, what is our objective?

Our objective is to have sufficient equations, so that we may solve for the unknown variables. So, we have to identify what are the unknown variables. See, if you look into this equation what would be the unknowns. See definitely if you have a forcing on a system you know what is the body force because if you do not know what is the force to which it is subjected you cannot find out how the how the system would under the action of the force.

So, you know the force. What are the things which are unknown here? The velocity components are unknown here and also the stress tensor components. These are not known to you. So, the stress tensor components in turn may be expressed in terms of the velocities and that we have to see that is it in terms of the velocities directly or in terms of the gradients of velocities.

So, we can qualitatively understand from our studies in kinematics of flows that any deformation for a fluid is actually the not important in an absolute sense but the rate of deformation is what is important in an absolute sense. So, when we say rate of deformation we mean linear deformation angular deformation. Linear deformation may give rise to a volumetric change.

So, all these are expressed in terms of rates for fluids because the understanding is fluids are anyway continuously deforming. So, if allow more and more time deformation will be more and more that is not a quantification of anything. Quantification is the rate at which that deformation is taking place or the rate of strain so to say. Now that rate of strain should be related to this one.

But we have to keep in minds that there is a foreseen condition, there is a surface forcing condition which should not depend on the rate of deformation. What is that surface foreseen condition? So, if you have a force on a surface, see this somehow is a representative of what is the foreseen condition on a surface of a fluid element. Now there is a part of that foreseen condition which should not be dependent on the rate of deformation.

And that we should isolate, what is that? That should be the pressure distribution because we have seen that what is the hallmark of a fluid at rest? If there is a fluid at rest that means it is not deforming and we know that fluids are in continuous deformation subjected to shear forces. So, that means when fluid is at rest there are 2 important things which are occurring. One is there is no shear force.

Because if there is a shear force it will start deforming except for some special fluids, which we have discussed as some special non-Newtonian fluids. But in general, if you apply a force the fluid will start its deformation. So, if it is at rest that means there is no shear force. There is only normal force but a special type of normal force that is not dependent on its deformation.

Because you can also have a normal force which may be related to deformations may not be angular deformation but maybe volumetric deformation. So, if you have an element you may be able to stretch it or compress it by a normal force. So, it is not just true that if you have a normal force then that is not related to deformation. Normal force also may be related to deformation.

But pressure is going to be such a normal force which is not related to the deformation because it is equally applicable for fluids at rest which are not under deformation. That does not mean that when a fluid is moving it does not have pressure. Simply we can isolate that effect and say that whatever is the remaining effect that is related to the deformation of the fluid.

So, when a fluid is moving it is equally subjected to pressure but we may isolate that part and say, yes. The effect due to pressure was something which was not responsible for its deformation and anything else would be responsible for its deformation. Accordingly, what we may do is we may decompose this tau ij into 2 parts. One part we say is a hydrostatic component and the other part we call as a deviatoric component.

So, hydrostatic component is something from the name it is clear that as if it talks about a static condition. So, it is not related to therefore deformation. So, we can broadly say that we are decomposing it into 2 components. One component is not related to deformation and other component is related to deformation. That is the whole idea. And when the component that is related to deformation that component should be only a function of pressure.

Because when a fluid is at rest you can only have the description of the equilibrium through the pressure. Now, coming to the deviatoric component which will be more interesting for fluid under motion which is under our discussion at the moment. So, this deviatoric component should be a function of what? It should be a function of the rate of deformation.

So, when you write the rate of deformation let us just recall that the rate of deformation in general, in the most general form this could be a gradient of velocity where i and j could be different indices. In a special case i and j could be the same index. So, we have seen earlier which is quite trivial that you can write this as and this writing is nothing very special it is just writing an identity. And the whole understanding is that this is like a tensor.

You may write it in a metrics form also because you have 2 indices i and j and basically you are decomposing it into a symmetric and a skew symmetric tensor. So, the first one is symmetric one, next is the skew symmetric one. Physically this decomposition is very important mathematically maybe very, very trivial but physical importance is that this is related to the deformation and this is related to the rotation.

So, when you have a rotational sense you do not directly associated that with deformation because an element may have rotation without deformation. That is like a rigid body rotation. That is the special case. So, if you want to relate the deviatoric stress tensor component with something what should be that something? I would say that the deformation which is appearing from here should be naturally related to the deviatoric stress tensor component, right.

So, that is the first understanding. So, we have to first understand that what depends on what because we are trying to develop something which is called as a constitutive relationship or constitutive behavior. What is the constitutive behavior? That is if you have think about say Hooke's law in linear elasticity? So, you say that stress is linearly proportional to strain within proportional limit of an elastic material.

So, if you say that stress is linearly proportional to strain that means you are somehow relating a cause with an effect and that comes from the behavior of the material because this you cannot generalize. It is specific to different materials and that material constant will appear in form of a modulus of elasticity. So, modulus of elasticity is the gross response of the material against the applied loading condition.

Say it is a tensile load and its starts getting elongated. Now the manner in which it is going to get elongated depends on its modulus of elasticity which comes from the inner characteristics of the material. So, it is a gross consequence of the material characteristics and that is where it will be different for different materials. So it comes from the constitution of the material. That is, it is intrinsic to the behavior of the material and that we call as a constitutive behavior.

So, just like you have for solids constitutive behavior in that way. For fluids also you have constitutive behavior in that way and that is how we have Newtonian fluids, non-Newtonian fluids like that because different fluids have different constitutive behaviors. So, we are interested to have a constitutive behavior or have a functional form of the constitutive behavior.

So, the functional form in a very qualitative way should be from our discussion that tau ij deviatoric should be a function of the rate of deformation. That much we should appreciate qualitatively, okay. Now the question is this function maybe of whatever complex nature and important thing is it will depend on what? It will depend on a deformation, rate of deformation which is also a second order tensor.

So, now we are interested to write a function where the dependent variable so to say is the second order tensor. This is the second order tensor and the variable in which or the function on which it is going to depend that is the independent function so to say is also a second order tensor. And we are interested about the functional relationship between these 2. This is not a very simple task.

So, we will not look into the most general way of characterizing these. There is no very general way of characterizing this. So, we have to think about special cases. And the special cases are the cases when these are linearly related which is like the Newton's law of viscosity. So, we will now consider a Newtonian fluid. So, if you have a Newtonian fluid then this will be a linear function.

That means the deviatoric stress tensor will be a linear function of the rate of deformation. Just like Hooke's law for linearly elastic material, solid material. So, the question is now how we are going to write it? We have to keep in mind that we are now interested to write a transformation that maps a second order tensor on to a second order tensor. Earlier we wrote a transformation which maps a vector on to a vector.

If you recall the Cauchy's theorem, the traction vector was mapped in terms of the direction normal a direction cosines of the area which was chosen to evaluate the stress tensor the tractor vector components and that mapping was successful through a second order tensor which is the stress tensor. So the second order tensor was such that it maps a vector on to a vector. That much of mathematical understanding we had.

Now we are in a situation where we have to map a second order tensor on to a second order tensor. So there should be some intermediate transformation and that transformation is a 4th order tensor. So 4th order tensor is something which maps up second order tensor on to another second order tensor. So this linear function is written in this form tau ij = a 4th order tensor say Cijkl and a second order tensor.

So what is ekl? ekl is like, okay. So, this is ekl, so e is like the rate of deformation just this form. Just i and j are replaced by k and l. So, this is like eij whatever is written here, okay. Just a symbolic way so that we do not have to write the big thing again and again that is just a symbolic way of writing it. Now you look into these. So, what are the repeating indices here? So, you can specify you can fix up i and j whatever you want.

But you cannot fix up k and l whatever you want because it is summed up over that because these are repeated indices. So, you have basically summation over k and summation over l each running from 1 to 3. But the more important thing is that so k and l could be from 1 to 3 i and j also could be from 1 to 3. Those are free but you could have i = 1, 2 or 3 or j = 1, 2 or 3. So, how many combinations with Cijkl are possible in the most general case?

3 * 3 * 3*3, 3 to the power 4, so you could have 81 components of Cijkl and that is the most general constitutive behavior for a Newtonian fluid. That means if it has no other special effect they require 81 independent constants to specifies constitutive behavior. That is not

again a very simple thing. So, of course we are not going into that what would be the picture if we had 81 constants.

We will see that those will be reduced to again fewer numbers of constants even in a more general case because we know tau ij = taj ji. So, from that symmetry we can at least reduce a few in number. So, we will not go through that route but we will at least keep in mind that mathematically speaking if you do not use the symmetry of tau ij you have 81, if you use the symmetry of tau ij, 36.

So, I will leave it on you as an exercise to show that if you use tau ij = tau ji the 81 constants maybe reduced to 36 constants. But the thing is that is also not something which is very, very tractable. Think about a problem where you are a practical problem where you required 36 material constant properties to describe a flow. I mean, in a very general situation that might be the case but that is not a very common case in engineering or even in science it is not a very common case.

So, we will do in this course or we will address in this course something which is perhaps the simplest of all these cases but the most commonly encountered one. That is, we will consider a very special case of a homogeneous and isotropic fluid. So, the special case that we consider is homogeneous and isotropic fluid.

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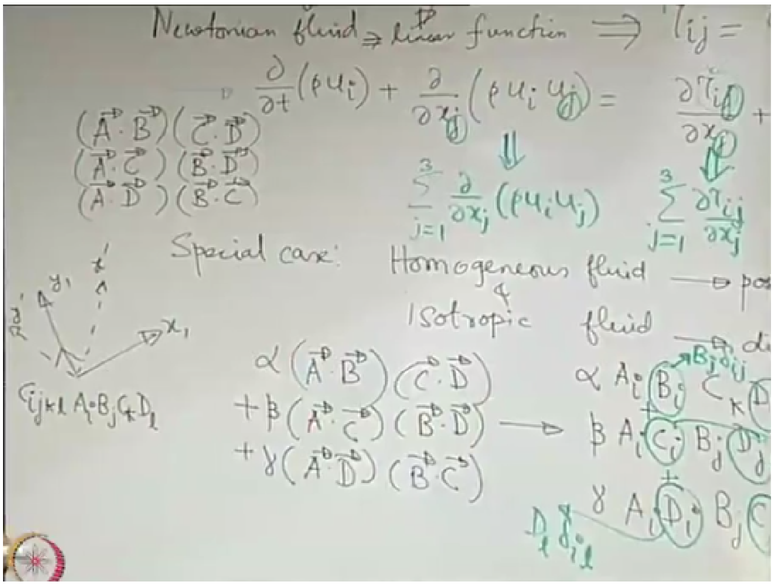

Homogeneous fluid and isotropic fluid. So till now what assumptions we have made? We have made assumptions first of all stationary reference frame that is which we started our equation then a Newtonian fluid special case, homogeneous and isotropic Newtonian Fluid, okay. So, now the question is that when you have a homogeneous and isotropic fluid what is the special characteristic that you have?

So, what should be the dictating factor for a homogeneous and isotropic? So, what is homogeneity? So, homogeneity means it is same as at all points in the flow fluid that any property is same or positioned independence at all. So this is related to position independence. So the constitutive behavior does not vary from one point to another point and isotropic means direction independence.

What is the hallmark of that? Say if you have coordinate direction say x1 and y1 and if you measure or if you try to assess a constitute behavior on the basis of that. Now if you have a transformed coordinate system still orthogonal but let us say x prime and y prime with a rotation. This does not disturb the constitutive behavior, although the orientations of the axis that you have considered for describing the behavior those have changed.

But in terms of that new axis the constitutive behavior still remains the same. The description still remains the same. That is called as an isotropic. So, we are going to think about a homogeneous and isotropic fluid. So, if you have a homogeneous fluid the thing that we will immediately conclude is that Cijkl should be independent of position. Otherwise, if Cijkl is a function of position it loses its homogeneity.

What would be the consequence of isotropic? That is something which we need to look into more details mathematically. So, for isotropic to say that you want to form something like you have the 4th order tensor Cijkl. How could you convert this into a scalar? Because see, let us say our objective is to convert or transform this one into an isotropic scalar, okay. Let us say there could be many ways of doing it.

Let us say you are, what you had given are vectors. So, using vector how can you convert this into a scalar? Forget about isotropic. How can you use vectors to convert it to a scalar? Let us say you have how many vectors would you require? First you say. This is just from common index notation understanding you can say. 4, because if somehow you can sum it up over these indices then it will have no free indices and that then it will become a scalar.

So, if you have say 4 vectors, A, B, C, and D you know vector is characterized by 1 index. So, this say it has Ai just for its index notations say this is Bj, this is Ck this is Dl. Just mathematically now if you have say this Ai, Bj, Ck, Dl then and if you write it in this way that means there is an invisible summation over ijkl. So, at the end whatever comes out does not have any index and it is a scalar, right.

So, you can employee 4 vectors, 4 independent vectors to convert a 4th order tensor to a scalar that we can understand. But we are interested for a special type of scalar which is an isotropic scalar. Now let us see that what can lead to isotropy if you are dealing with 4 vectors. So, how can you say just look into this example? Here we had 2 vectors.

When we talked about this rotation actual we talked about a vectors say maybe 1 vector oriented along x another vector oriented along y and this vector combination was rotated. And after rotating this vector combination the property or the constitutive behavior whatever is described through the new orientation that is same as whatever is the description to the old orientation. That is the implication of isotropic.

So, what is the meaning? The meaning is very interesting. If you do not change the angle between the 2 vectors, then it does not change. So, what is rotation? Rotation is something which preserves the angle between whatever vectors were there. So as if you are rotating this vectors like a rigid body, keeping the angle between the vectors same. So, if you have vectors A, B, C and D and if you somehow preserve the angle between these 2 to taking in pairs.

Say A with B and C with D Maybe A with C and B with D and A with D and B with C. these are the possible pairings. So, with these possible pairings if somehow you keep the angle between these as unaltered then no matter whatever is there orientation in space so long as the relative angle does not get disturbed. It should give back the same behavior. That is the meaning of isotropic.

So, if you have vectors A, B, C and D, so what kind of combinations you can have taking all those 4. You can have A dot B and that corresponding combination is C dot D. Then you could have A dot C, B dot D and A dot D, B dot C. Why dot that? Because dot directly gives the angle between 2 vectors. We were talking about the angle between 2 vectors. So, if we consider the dot product that returns sort of angle between the vectors.

So, if that is preserved then isotropic is preserved. So, why such a combination because you see that this will give rise to a scalar. So, if you take these in pairs these are individually scalars. So the combinations, all combinations of these ones will form a general isotropic scalar involving the 4 vectors A, B, C, and D. So, this we can write, so let us form that scalar let us say that, that scalar is = A dot B, C dot D sum multiplier of that say alpha.

Because the scalar multiplier does not change its any characteristics. So, alpha is just an arbitrary scalar multiplier + beta * A dot C, B dot D and let us say + gamma A dot D, B dot C, okay. So, we are just made a linear combination of these 3. Our objective is to ensure that we get an isotropic scalar. So, now what we will do, we will write it in terms of the index notations. So, if you write it in terms of the index notations how we are going to write it?

So, we are going to write it in this way. This is the first term that we are writing. Just look into the corresponding term. See, A dot B may be written as Ai Bi because it is what? A1 B1 + A2 B2 + A3 B3. That is the dot product. You take the corresponding components and multiply with each other, sum it up. That is how you get the dot product of 2 vectors, right. So, there is an invisible summation like this.

And here also it is like this. Keep in mind i and k are just dummy indices. So, in place of i you could write j in place of this you could write whatever, okay. Similarly, let us write the other terms +, okay. Now you see that we wanted to get it as an isotopic scalar what is that special isotropic scalar that we are looking for? We are looking for Cijkl * Ai Bj Ck Dl, right. that was the form that we are looking for.

Here in the form you do not have the indices A, B, C and D as differ. So, if somehow we can convert these indices A of A, B, C and D as entirely different like 1 i, another j, k and l. Then we would be able to compare this form with this form that we have for our constitutive relationship. So, our next objective will be to convert these forms like Ai Bi Ck Dk to a form like Ai Bj Ck Dl that form.

So, how do we do that? Let us say you want to convert Bi to Bj. That is our objective say because you have Ai next you want Bj. So, you want to convert Bi to Bj. So, there is something very useful in terms of index notation which is called as Kronecker delta. So, this is given by this is just a notation. So, this is defined in this way delta ij such that this is 1 if i is = j and 0 if i is not = 0, okay.

So, this is just a notation that means if you have only i = j this is 1 otherwise 0. Now let us see. Let us say we want to write Bi delta ij. What is this? See, you have to keep in mind that when j = i, so rather you may also think it as j = i or i = j whatever like when they are equal. So, think about that only when you have i = j then only this is 1. So, there is only 1 case when this is = 1 and that is when this i is j, right.

So, this becomes Bj because only when i = j then these 2 indices are = that is 1 otherwise it is 0. So, you can see that this is a very simple and elegant way of switching the indices from i to j, j to i like that. You can play with these. So, what we will do we will use that switching index here. So, these Bi we will write Bj * delta ij. We will keep Ck because that is what we want. We want to convert Dk to Dl. So, what we will do?

We will have Dl * what? Delta kl, right. Similarly, let us look into the second term Ai let us say we keep Ai, Bj is there. We want to change Ci to Ck. So, what will be the transformation? Ck * delta ik, okay. It is transformation from i to k. That is why delta ik. Then we want to change Dj to Dl. So, Dl * delta jl. Similarly, the third term you have Ai you have Bj then you want Ck. So, Cj will become Ck delta jk and Dl * delta il, right.

So, if you write in this way see what is our desperate act? The desperate act is we want to write the right hand side just in a form Ai Bj Ck Dl, okay. So, in this using this simple notation we are able to write it in this way. So, let us write that then. So, what at the end we can write?

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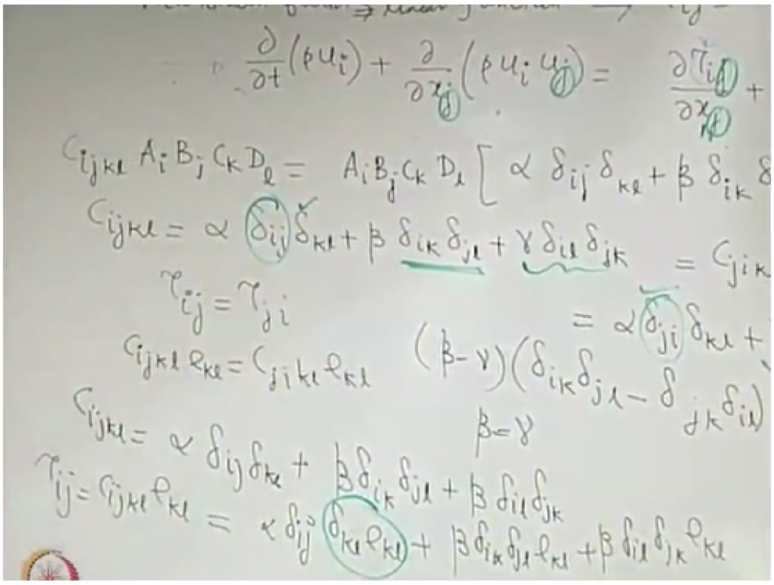

We can write that you have now the right hand side which we want to express as say the left hand side as Cijkl * Ai Bj Ck Dl. We want to make it an isotropic tensor and the right hand side we have expressed in terms of an isotropic tensor with again you have Ai Bj Ck Dl that is like they are common for all terms then you have alpha delta ij delta kl + beta delta ik delta jl and from the third term gamma delta il delta jk, right.

So, from here it follows that you should have Cijkl = alpha * delta ij delta kl + beta delta ik delta jl + gamma delta il delta jk, where delta ij is defined as above. See, with this simple exercise we have come up with a very interesting and advantageous conclusion. What is that? We were, we started with 81 independent constants in general to describe the material property. The material property is given by the Cijkl. Now how many constants you have?

3. 3 independent constants, alpha, beta and gamma. Because delta ij like these are like you know that either 1 or 0, okay. So, we can see that and these are positioned independent. So, these are really constants. If these were position dependent these could be also functions of position. So, we can say that for a homogeneous isotropic fluid we are now able to come up with 3 independent constants.

We may reduce it even to less because we have not yet exfoliated the symmetry of tau ij that we have never used. We must have tau ij = tau ji that we have seen from angular momentum conservation. So, the next step is we use the symmetry that is you have tau ij = tau ji. So, tau ij is what? Tau ij is Cijkl ekl that must be = Cjikl ekl, right. So, what it follows is that you must have Cijkl = Cjikl.

So, this must be also = Cjikl. So, how do you write Cjikl just switch i and j here. So that will be = alpha delta ji delta kl + beta delta jk delta il + gamma delta jl delta ik. We have just switched i and j, okay. Now you see first of all what is the relationship between delta ij and delta ji? They are the same. Delta is just a symmetric tensor. That means when you equate these 2, these 2 get nullified, cancelled. They are the same.

For the remaining ones you see delta ik delta jl is this term. One is coming with the coefficient beta in the left hand side in the right hand side that is coming with the coefficient gamma. That is the only change and for the other term here, this term is same as like the coefficient of gamma here is same as the coefficient of beta here. So, you can just combine that in a simple factorization and write that beta – gamma * delta ik delta jl – delta jk delta il that is = 0, okay.

Whatever is there inside the bracket in general this is not 0. Special cases are there but there are general, I mean there are certain terms which will not be 0 when the indices are =. So, this in general for all cases this is not = 0. For all cases if you want to make it satisfied only thing you must have beta = gamma, okay. Because what we understand is that the second term in the bracket in some cases it is 0, it is true but in all cases it is not 0.

But we must have a constitutive behavior that should be true for all cases. So, now you see now we can write Cijkl = alpha * delta ij delta kl + beta * delta ik delta jl + beta * delta il delta jk. So, how many independent material constants we have? 2. Keep in mind that there is an exact analogy with mechanics of solids. If you have a solid material say elastic material, then how many independent material constants you required to specify its behavior?

2 independent constants, modulus of elasticity and Poisson's ratio. You can express the modulus in terms of any of these 2. So, you require only 2 independent material constants. Here also it is totally analogues. So, you are requiring 2 independent material constants to specify its behavior. What are those material constants? That will come from the physical understanding of these terms.

So, when we want a physical understanding of these terms we will now write the expression for tau ij. That will give a physical understanding of the terms. So, tau ij is Cijkl * ekl. So, we will multiply this with ekl. So, when you multiply these with ekl, so you have alpha delta ij. Let us just write it in an expanded form delta kl ekl + beta delta ik delta jl * ekl + beta delta il delta jk * ekl, okay.

Let us now simplify these terms. The simplification is easy. It is quite simple. If you consider let us say this term delta kl ekl, you tell what should be the corresponding simplification? Let us erase this part and we will write the simplifications here.

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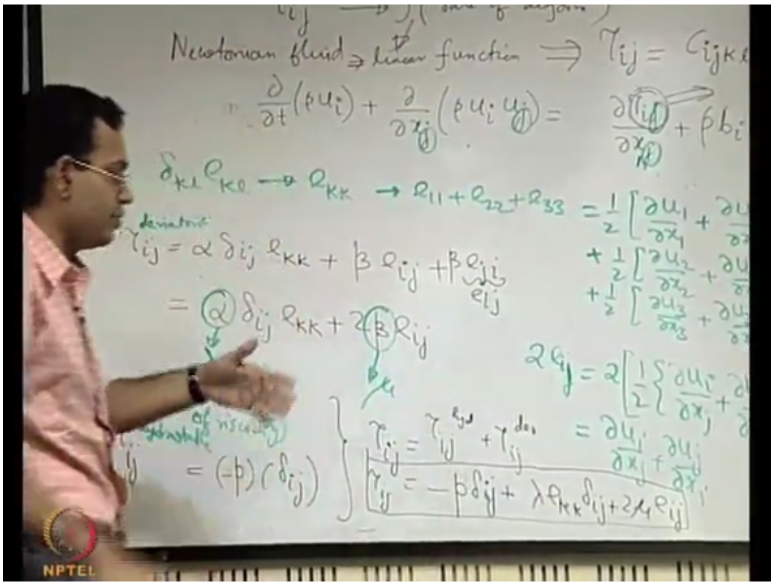

So, delta kl ekl, say we want to write it. How do you write it? We have to keep in mind that this delta is 1 only when l = k. So, this you can write as ekk or ell, right because in all other cases it is 0. What is this actually? This is again not one term. It is e11 + e22 + e33 because it is a repeated index, so you have a summation basically. So, what is e11 + e22+ e333? Let us just write it. e11 is, this is just by using definition of eij then e22 and e33, okay.

What is this? This is the divergence of the velocity vector. So, we can clearly understand that this has something to do with the compressibility of the fluid. Volumetric, rate of volumetric strain of the fluid. So, the first term we can understand that it should be related to the rate of volumetric strain of the fluid. So, tau ij's first term we have come up with, so tau ij = alpha * delta ij * ekk then next terms let us write the next terms.

Next terms you see that what are the indices that you play with here? So, what are the repeated things? You have k as repeated and l as repeated, okay. So, there are special cases of k and l when this term is non 0. What is that? When k = I and l = j, right because then only these deltas are non 0. So, the next term will be beta * eij, right. Because this i and j should be something for which delta ik and delta jl and non-zero.

And that is there when i = i becomes k or rather k becomes i and l become j because you are writing a summation over k and l. You are trying to find out those values of k and l for which these term is not 0 and that is only k = i and l = j. The next term, so you have here again you see what are the repeated things? You have k and you have l. So, again when l = i and k = j. So, + beta eji and the rate of deformation is a symmetric tensor. We have already seen.

So, eji = eij, right. That means we are able to write tau ij in a form alpha * delta ij * ekk + 2 beta eij, okay. Now the question is how physically we may relate these coefficients alpha and beta? To do that we should understand one very important thing that where from the effects of these come? First you have let us say you have beta. So, where from the effect of beta is coming?

Let us write what is 2 eij? So, what is 2 eij? You have 2 * eij is half of this one, okay. So, this is considered as very special case of a 1 dimensional flow which we did at the very beginning when we are discussing about viscosity that you have only 1 component of flow say u component of velocity other component was not there. We wrote something very loosely from physical understanding that tau = mu du dy, right.

So, when you have say 1 component only u other components say u1 only you have and say you are thinking about the radiant along y, so j = 2 and other term is not there because only 1 component of velocity is there say you can write this as du1 dx2. That is like du dy. So, you are relating that with some tau. Remember this tau is not the total but tau deviatoric, which is related to the deformation.

So, that is related to this through a coefficient and that we have already seen. Therefore, in the general notation this beta must be the viscosity of the fluid mu, right. So, see from mathematics we are converging to the physical understanding and this is very important.

Continuum mechanics is a beautiful subject. It can make you agglomerate physics and mathematics very nicely. So, this is mu.

This constant is not so well known to us still now whatever we have discussed but it is given a general notation of lambda. This is called as second coefficient of viscosity. We will see later on that what is the significance of this, physical significance. We will see in details but just a name second coefficient of viscosity. Why because already you have a viscosity like mu which already is there.

So, if you want to have a different coefficient but we have to keep in mind that this is related to sort of angular deformation of the fluid and this is related to volumetric deformation of the fluid. At least qualitatively we can understand the significance of these 2 terms. One term is related to the angular deformation one term is related to the volumetric deformation. Angular deformation means shear type of deformation.

Now, this is the deviatoric stress tensor only but what about the hydrostatic one? So, tau ij there is one tau ij hydrostatic because the total tau ij is tau ij hydrostatic + tau ij deviatoric. So, what is tau ij hydrostatic? That is considered a case when the fluid is in static equilibrium and then even there is a stress in condition because of the pressure distribution. So, how do we write it in terms of this tau ij?

See, you have to keep in mind that it is a pressure effect is like a normal force effect. So, it is there only when I = j because i = j gives the normal components. So, it should be something which is 0 if i is not = j and which should be 1 if i = j multiplied with the magnitude of that. So, it should be something multiplied with delta ij. That is clear. Because it is should not be non 0 if i is not = j.

What should be the multiplier? That is the corresponding scalar of it. So, what is that? That is because of pressure but you have to keep in mind that pressure by default is compressive in nature. Whereas our positive signs convention for normal stress is tensile. So, it should be – P, okay. So, you come up with the final form where what is the tau ij total? That is tau ij hydrostatic + tau ij deviatoric.

So, this is tau ij hydrostatic + tau ij deviatoric. So, what is that? So, you have lambda. So, first let us write the hydrostatic – P delta ij + lambda * ekk * delta ij + 2 mu eij. That is the tau ij. So, we have come up with an expression for constitutive behavior. It has what unknowns? What are the unknowns here? See, eij has unknowns as velocities because it is written in terms of radians of velocities. So, you have velocities and you have pressure.

So, in the equation of equilibrium see what is our objective at the end to solve for the unknowns? So, where we have landed up? We have landed up with an expression for this tau ij which is expressed in terms of unknowns as velocities which are already there + a new unknown has appeared which is pressure. So, in the next class we will see that first of all what are the physical significances of different terms?

What is the physical significance of lambda? How it is related to mu and then how we can come up with an equation in the final simplified form where we substitute this constitutive behavior of tau ij in this expression. That we will take up in the next class. Thank you.