Introduction to Fluid Mechanics and Fluid Engineering
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Lecture – 22
Integral Forms of Control Volume Conservation Equations (Reynolds Transport Theorem) (Contd.)

Last time we were discussing about the integral forms of the conservation equations and as an example, we looked into the integral form of the mass conservation and its corresponding differential form also we revisited and we found out that it is possible to convert one form to the other. Now, let us look into some more examples of the use of the integral form of the mass conservation equation.

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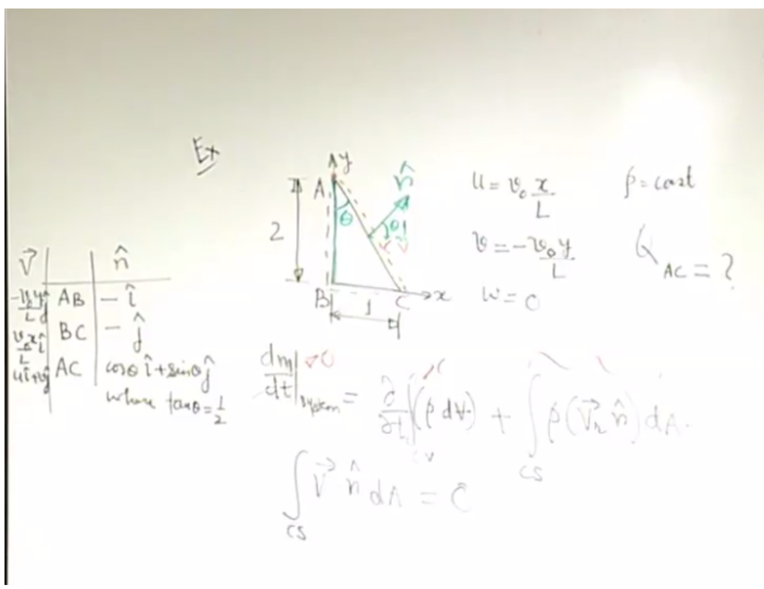

Let us say that you have wedge-shaped element like this with the axis oriented along x and y and the velocity field is a 2 dimensional velocity field is given by say u is given by this. V is given by this one and 2 dimensional W is 0. Let us give some names to the faces of the elements here. The objective is to find out that what is the volume flow rate through AC? The dimensions are given. Let us say this is just 1 unit all that given in some consistent units. See this is 2, these are given.

So what we also are assuming that v0 is a constant, it is not a function of any other that we are looking for. So how should we proceed with this problem? Let us say that we are interested to use the integral form of the mass conservation and that is one of the natural things that we should use here, because there are 3 phases across which fluid will enter and leave. So the net rate of transport should be given by the integral form of the mass conservation that is you have dm/dt for the system + this for the control volume + integral of rho.

The left hand side is 0 because no matter whatever system you are considering it is by definition of fixed mass. The right hand side because we are assuming that rho is not changing with time. Neither the control volume is changing with time. What is our control volume? Let us say that this triangular shaped element is our control volume. So whenever we are making a control volume analysis, it is important to identify what is the control volume that we are taking, because you may take different control volumes of course for this problem.

This is an obvious choice of the control volume, but there could be problems where there could be many different choices of the control volume. So the equation that you are writing should be pertinent to a particular control volume that you have chosen and that should be clearly mentions. So this triangular-shaped thing is a fixed control volume. The volume of that is not changing with time, so it is not a deformable control volume; neither the density is changing with time, so this term goes to 0.

So what remains is this may turn which is nothing by basically the net rate of outflow - inflow of mass = 0. So if rho is a constant let us make a further simplification that rho is a constant. then this will whirl down to basically integral of v dot n dA over the control surface = 0 and when we say v, here the relative velocity and absolute velocity are the same because it is a stationary control volume, it is not a moving control volume.

So this now we may break it up into 3 parts because, the control volume has 3 different distinct oriented surfaces. So you can write this as the sum of the effects of AB, BC, and AC. So for AB when you write, what should be this corresponding expression? So let us identify that where is that AB so you are looking for this phase. You are interested to find out what is v dot n, integral of that over the entire area. So one important thing is here n is not a variable.

It is like a line oriented along y axis so n is a constant. So for AB what is n? So let us try to identify what are the normal directions for the different ages. Here the ages are all straight lines. So they have unique normal directions. So let us say AB, what is the n for that? - i then BC and AC? So for AC let us say that this is the unit vector normal so this will have its components, let us say this angle is theta.

So if the angle between the normal and the horizontal is theta then the angle between the A and the vertical is also theta that means you can say that n is definitely cos theta I + sine theta j where you have tan theta = 1/2. So what will this imply? This implies that like you have identified the direction normals for all the phases. Only thing you require is the velocity. So the velocities for the phase is for AB what is V?

Let us write in the same table let us try to write what is the velocity v cap. So for AB what is v cap. For AB x is 0. So u is not there. There is some v. So it is - v0 y/Lj. For BC similarly you have y = 0 so it is v0x/Li and for AC it is the real the sum of the 2 components because here x and y are both nonzero. So we are not writing it because it is like let us just write it as in general ui + vj. Now to get the integral you have to keep in mind that if the velocity varies along that length then you have to integrate it over the length to get the total flow rate.

When you come to the let us start with AB. you can of course evaluate this. Try to evaluate this, but let us not do that bull work. You see that the velocity along AB is like it is oriented along AB. So there is no normal component of that. So there is no net flux or influx or out flux of flow across AB because there is no normal component. So if you make a dot product of this, that you can clearly make out and so that we will not give rise to any net flow.

Same might be true for BC. So the choice of the axis here has been such that this if this is the velocity field then that is the case, but the same is not the case for AC. **“Professor - student conversation starts”** So for AC how you will find out what is the total or the net rate of flow? So if there is nothing that enters here, there is nothing enters or leaves here you expect that there is nothing enters or leaves here.

How could you verify by doing the integral that it should be 0. That is what like this is QAB, this is QBC, this is QAC like if you evaluate by doing the dot products. Will the dot product automatically give it? It is expected that the dot product integral over that should automatically give it. **“Professor - student conversation ends”** So that you should check so that it gives you a confidence of how to calculate that because here.

I have given a special type of velocity field so that actually this problem solution is not necessary. I am trying to go through this through a formal route to give an idea of what should be done in a case when the problem solution deserves that, but here actually does not deserve. So this is a more intuitive case when nothing is entering and nothing therefore is expected to leave. If by chance you get something which is leaving through this and nothing is entering that will really valid well off mass conservation so that one has to be careful of.

Now let us look into some other problem which is not as trivial as this one so another example.

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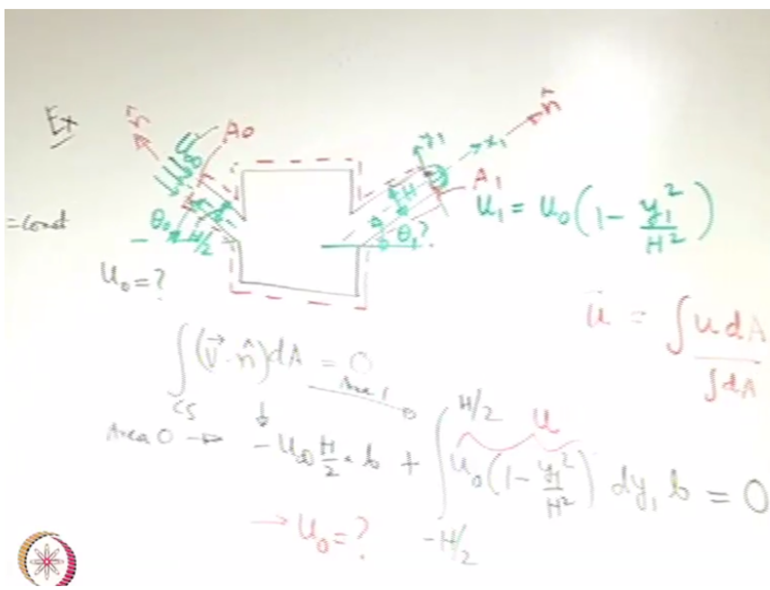

Let us say that there is a tank like this. The velocity profile at the exit of the tank is through this pipe is given by this one and in terms of a local coordinate system this let us say that the local coordinate system is x1 y1. It is given as u1 = some u0 * 1 - y1 square/h square, where y1 is the transverse coordinate and H is this height and the fluid entering here, here the again in terms of the local coordinates you can specify it.

But that specification may not be necessary because it is given that it is a uniform velocity profile here with a velocity u infinity which is uniform and let us say this is h/2. The directions of the axis are not given that means it is not given that what is this angle theta. So theta 1 it is not given that what is this angle say theta 0. These are not given. What is given it is given that the density is a constant.

You have to find out what is u0. Given the width of this figure perpendicular to the plane where it is drawn is uniform. So u infinity is given, h is given, these 2 things are given. So how will you go about it? Again looks like a situation where mass conservation should be applied and if you want to apply the integral form of the mass conservation let us say that we consider a control volume what should be a good choice of the control volume.

So a good choice of the control volume is something where across the surface we are totally confident about the velocity field. So let us say that we make a choice of the control volume something like this. So with this choice of the control volume you can write the law of mass conservation. If you want to write that then like it is because rho is a constant again it will whirl down to a case very similar to the previous one where eventually it will be integral of v dot n dA over the control surface = 0 the remaining terms will not be relevant.

So this is the only term that is relevant. Out of the surfaces that you have only you have one inflow and one outflow surface across the other surfaces fluid is not flowing. So those surfaces are not relevant, so you may break it up into 2 integrals one for this inflow and another for the outflow. One is very straight forward let us do that. So when you have the v dot n dA see this v is uniform over the area over which it is flowing over the inflow.

So you can take away this v out of the integral not only that a dot product also because n is also a constant here. So v dot n entire thing you can take out of the integral. So what it will become, it will become v dot n * area of the phase over which it is coming. So let us call that this area of the phase is here A0 here the area of the phase is say A1. So when you right v dot n you have to keep in mind that what is the direction of n along this surface so the direction of n is opposite to v.

So v dot n will give the minus of the magnitude of v. For the area 0 or area o it will become this term will become what? - u infinity * h/2 * the width? Let us say b is the width perpendicular to the plane of the figure. Then the other area for A1 we cannot have the same consideration because of velocity varies over A1. So to see how the velocity varies with A1, along A1 we are already given that with respect to the local transverse coordinate how it varies, but you have to keep in mind that forget about that functional dependence on Y1, u1 is going out of the area and what is n, n is also oriented out of the area.

This means that if you just take the vector sense v dot n that will give you the magnitude of v * 1 because the dot products of 2 vectors in the same sense that will be leading to that conclusion, but because it is varying with y1 now you have really do the integration. So when you first have v dot n that will become u0 * 1 - y1 square/h square that will be v dot n and dA, dA is what? So you take a small element on the axis Y1 so this is the small element of the area.

So what is that small element of the area say at a height y1 from the central line see the coordinate system is from the centre line so, at a height y1 from the center line so you have taken a small area of width dy1. So the elemental area which is like the dA given symbolically it is dy1 * b. If you integrate that from - h/2 to h/2 then that will represent that what happens for the area 1. So sum of these 2 should be = 0 really from the - sign of the first term.

You can make that it is inflow and the + sign of the second term means that it is outflow and it is possible to complete this integration in a very simple way we are not going into that just to save sometime, but important thing is that this is the variation of u over the section, so you can write it equivalently as integral of udA in the fundamental form in a scalar form so this is not like a vector form so it is just like because eventually with the dot product it has become a scalar.

You have to keep in mind this u is nothing but the component of velocity which is normal to the area. Here fortunately all components normal to the area. There is no component which is cross that. So this you may express through some quantity which is called as average velocity. So average velocity is this divided by the area, that means if this velocity was uniform, but the same flow rate was there. See if it was uniform then that uniform velocity time the area will give you the flow rate that is what the first term has told us and if it is not uniform obviously you have to integrate it to get the flow.

So if the flow rate were uniform in a hypothetical case if the velocity profile was uniform in a hypothetical case, but the flow rate becoming still the same as it is in the real case then if we equate those 2 flow rates then the equivalent hypothetical uniform velocity, this is called average velocity. So it is like an equivalent uniform velocity that would have prevailed across the second satisfying the same volume flow rate a it is there in a real case.

So therefore this is like as good as some u1 average * A1 and this is just like u0 * A0. So it is just like A1V1 = A2V2 you have to keep in mind again that what is V1 and what is V2 these are very important. Again I am repeating these are fundamentally the average velocities over the sections 1 and 2. So when we are writing here 1 and 2 area may be 0 and 1 or whatever subscripts there is a fundamental difference from what we will load for using the Bernoulli's equation.

Those we wrote for points 1 and 2. Now we are writing for sections I and II. So may be subscript wise they look very similar, but Bernoulli's entirely different. When it is uniform it does not matter it is as good as writing for a point because velocity does not vary from one point to the other, but you do not have anything called as area at a point, so it is basically when you write A1 no matter in the context of what we write what we have written earlier for 2 points when you use Bernoulli's equation.

We should keep in mind that there are also A1 was for the section that contained the point 1. So it is not that area of a point 1 or something like that, but there we use velocity at the point 1 the reason is that in the Bernoulli's equation we use velocity at a point. So we had to link it with velocity at a point. Now it is like we are linking things to velocity over an area. So if you complete this problem you will get what is u0 because all other things are known. Now let us look into some other examples where when we will look into different case may be on steady case.

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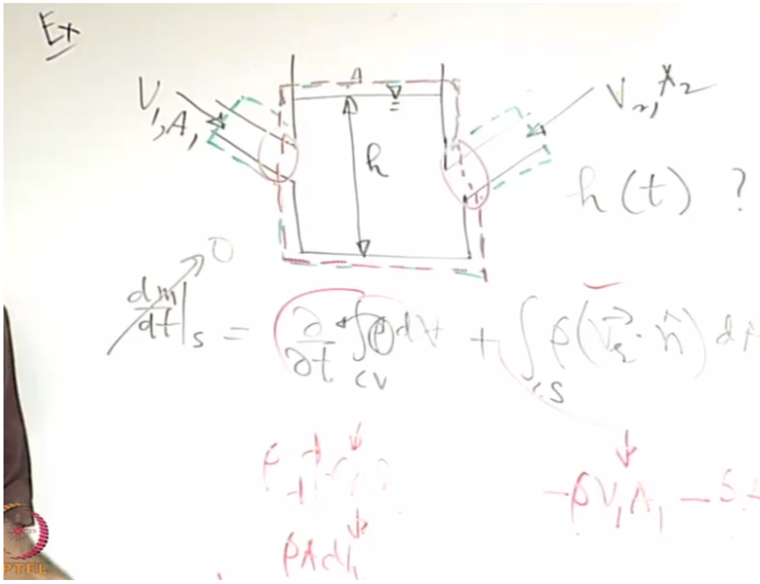

Let us say a similar problem where you have tank like this and there is a free surface here. So in the previous problem we assume that there is no change in level of the surface in the tank, but in reality that does not happen so that is something which is a bit hypothetical in reality it may happen that approximately the change in this level is 0 because, this is such a large area that no matter whatever is entering and leaving, it is not changing the height and these types of tanks are called as constant head tanks.

So maintain constant head, because with respect to the inflow and outflow the change in height of this is so small because of maybe this is a large reservoir. So the area is so large that it does not change any level, but the more realistic version of the previous problem is that is the level of the water will also change. So let us say that the height of the free surface from the bottom of the tank is h just to simply the situation.

We now go back to a case of a uniform velocity profile because you have already seen that if it is non-uniform that is not a very difficult thing we have to just integrate the velocity profile over the section. With that understanding let us say that you have a situation like this. Let us say it is uniform so you have a velocity V1 here and let us say area of cross section A1 you have a velocity V2 area of cross section A2.

When we are talking about the previous problem see we could get rid of the situation of change of height of the tank even in a real case by some approximation that is the rate at which the water is entering is the same at which the water is leaving. So it is not changing the height of the level of the tank, but when both are entering that is not the case. So here, the only chance of the height of the tank or the level of the water in the tank not changing with time may be only for the consideration.

That the area of cross section of the tank A is so large as compared to the others that the corresponding change in height is very small otherwise here there will always be a change in height. Smallness or largeness depends on the area. In the previous problem you could cleverly come up with the situation where there is 0 change in height, by making sure that the rate at which it enters is exactly the same at which it leaves.

So it appears to be bit hypothetical, but it is not that hypothetical if it is really doing that. So if you have a reservoir and water is entering and leaving at the same rate why should it change the level of the reservoir it will not, but here both are entering. So here our objective is to find out how the height is changing with time given these velocities areas and the information that these velocities are uniform over the area. So let us take a control volume. Again write the equation.

So dm/dt for the system = this one left hand side 0, right hand side the first term. Let us assume that the density is a constant. Let us say that is given. So the density is a constant. It will come out of the integral, but the volume is not a constant so the volume within the control volume is what/ It is A * h. A is the cross sectional area of the tank and A is the height. So this effectively boils down to rho * dt of Ah. Area of cross section is a constant. So this is as good as rho * A dh/dt.

See h is a function of time only and nothing else so this partial derivative becomes an ordinary derivative here and what about this term? So it has now effect of the 2 areas 1 and 2 so because we have seen that formally what is the consequence of the dot product and all those things we will now try to write it directly without going through that route. So for the surface I what will be -rho v1 A1 and for the surface II - rho A2V2.

So sum of these should be equal to 0. So from here you will get what is dh/dt that is very straight forward. Again you can make an observation that if dh/dt = 0 that is not a possibility because these 2 terms of the same sign cannot cancel that and only way it can cancel is v1 and v2 are of opposite sense that is if one is entering the other is leaving and then it again boils down to A1V1 = A2V2 so that is like a previous case that we have considered.

The other important thing that we might discuss in this context is that can we try to choose a different control volume? Let us say that we choose a control volume which is not the previous one, but say that new dotted line that I am drawing. Fundamentally there is nothing wrong, but it will not help us solving the problem why because now the control volume is cutting across some flow surfaces across which you do not know the velocity profile or you do not know how the velocity is.

So how will you find out this integral term? You have to have those locations where at least you have when idea of all the velocities varying or what is the velocity. So the choice of the control volume is not that. It is only something which is unique and u cannot choose anything else, but if there are many alternatives you have to find out that what are the unknowns and known involved with that alternative and it is wise to choose a control volume which gives a very easy situation by reducing the unknowns and the initial choice that we made is an obvious choice towards that.

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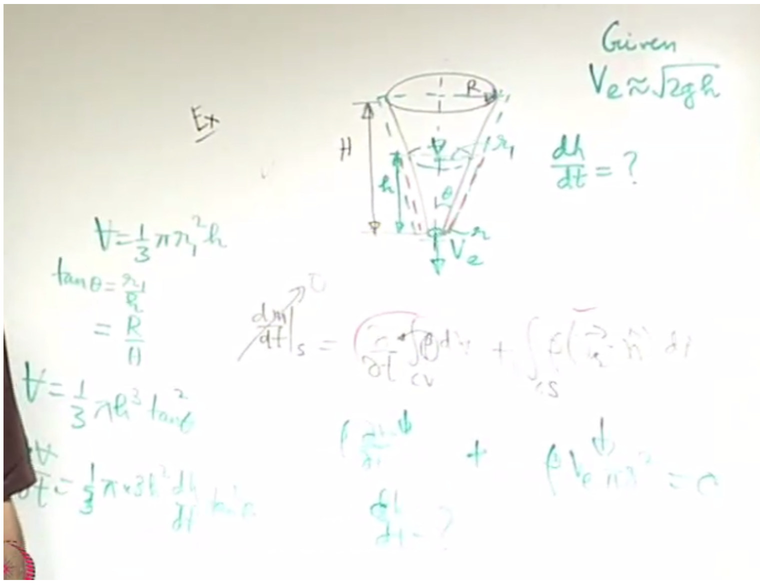

Let us take another example of similar type may be a bit different. Let us say that you have a conical tank just for a change of radius R, height H, there is a small hole at the bottom of the cone through which water is coming out with a velocity Ve and because of this leaving of the water the height of the water in this conical tank is reducing may be initially it was the full height H, but because water is leaving at some instant of time say the height is like h.

So this h is changing with time, because water is continuously leaving and let us say that is leaving through this small hole with radius of r both like this is a circular hole and cone is of course a circular cross section. Then you have to find out what is dh/dt. It is given that you may approximate ve/root 2 gh, it is given. We have seen earlier that this is not actually a very correct estimation, but it gives a sort of approximate situation under certain simplified assumptions.

We have discussed those assumptions in details when we are discussing about the Bernoulli's equation. So this problem is fundamentally not very much different from the previous one except that the geometry is such that you have variable cross section area nothing bit more complex than that. So let us try to use the conservation of mass here. So the first term what it will be? Again assume rho is a constant then if rho is a constant so the bad thing that we have approach is like this that we have tried to solve the problem without identifying the control volume.

I am trying to do it in the same way in which you are habituate to do just like straight away going to an equation and solving a problem see it is ridiculous. We are trying to apply an equation for a control volume, but we do not know what is the control volume? and that perhaps we were trying to do? So let us identify a control volume we will identify one type of control volume and I will leave it on you as an exercise to identify a different type of control volume and come to the same answer at the end and that will give you a good idea.

Let us say that we identify a control volume like this. Let us say with respect to that control volume we are writing this term. So with respect to this control volume when we are writing the term let us say that we neglect the density of the air which is there at the top of the water in comparison to that of the water. It is not a very bad engineering assumption because the reason is that air is much much lighter than that of water typically 1/1000.

In fact, that is what we did also in our previous problems where if you have a tank may be in the tank some part is water, but the remaining is air, but when we wrote this term, we did not write the term corresponding to air. So that was an inherent assumption that we were making and keeping in mind, but not explicitly stating. So then like if you take rho as a constant and out of the integral, what will become this integral of dv so this term will become basically rho derivative of v with respect to time.

Now the volume wise, the volume of the water so at this instant that we are considering the volume of the free surface is located here. So the volume of the water is one third I local r square so let us say that is be it a name say r1, one third pi r1 square h and from the semi-vertical angle of the cone let us say theta you have tan theta = ri/h which is same as R by H. So it is possible to write the whole volume in terms of h so the whole volume becomes one third pi in place of r1 it is h tan theta, so h cube tan square theta.

Then the other term, what will be this one? How many flow boundaries are there in the control volume? Only one flow boundary. Only one exit boundary so that is that? So in this boundary v and the normal vector are located, oriented similarly so the dot product will give a positive term.

So it will be rho then what. So v dot n v is uniform over the area let us assume that so v dot n will come out of the integral, rho will come out of the integral, and integral of dA will become A.

So it will become like rho ve * Ae. So it will become rho * ve * pi r square. So then it is very straight forward. This 2 together is 0 and you can different v with respect to time. It is an ordinary derivative because h is just a function of time so one third pi * 3 h square dh/dt * tan square theta. So if you substitute that here you can find out that what is dh/dt at given instant when the height is h?

Now I will leave on you solution of the same problem but with a control volume choice like this say you have a control volume which is adapting itself with a movement of the water so this control volume is a moving or a deformable control volume in a way that if the water level is coming down, this is also coming down with it. So this type of control volume and then with respect to that type of control volume you make the same analysis and try to come up with the same answer. Let us work out another problem.

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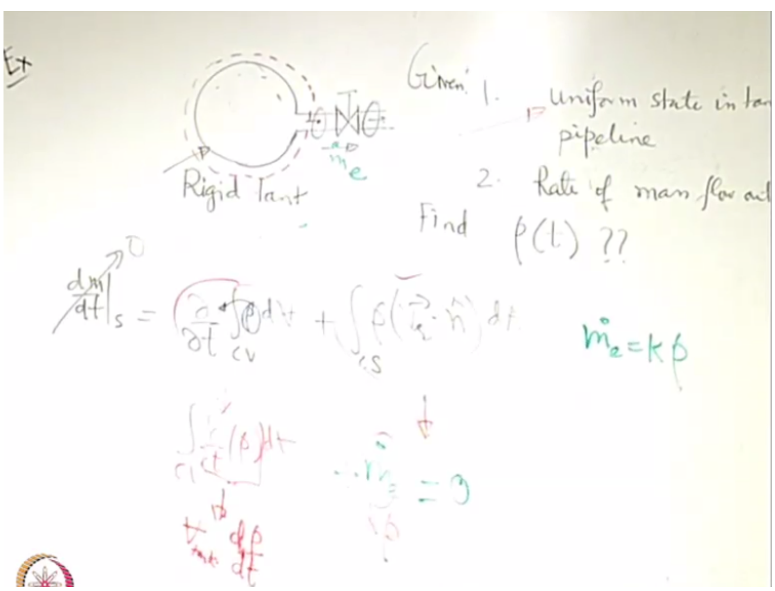

We have a rigid spherical tank which contains air and originally there was a valve located here which was preventing the air inside to go outside. Now the valve is opened and once the valve is opened, the air will go out of this pipe line which is connected to the tank. It is given that number 1 the state is uniform within the tank in tank and pipeline that means the properties are the same at a given instant of time everywhere that we are considering and number 2 is that the rate of mass flow out is proportional to the density at that instant.

This is given. You have to find out that how the density is changing with time. It is expected that the density will change with time because it is a rigid tank because it is a rigid tank if you take away air from it, its density will fall because now less mass is occupying the same volume so that we have to find out that how the density is changing with time. Again let us start with the choice of a control volume on which we want to have our analysis.

So the control volume let us say that which is the control volume like this? If you choose a control volume like this, it does not matter whether the valve is opened or closed right. There will always be a flow across it, so now if you are asked that what happens after the valve is opened you should take it of course the pipe line does not end here so the pipe line continuous beyond the valve.

So if you take it like where the effect will be apparent when the valve is opened, but again in this particular case and only if our objective is to look for a mass conservation, it makes no difference as such because even if you take a control volume like a surface which is to the right of the valve and to the left like a t any instant whatever is the mass flow rate here the same is the mass flow rate along this pipeline in terms of the mass flow rate, but that may not be the case if the density in the pipeline itself is a function of time.

So if the density in the pipe line it is a function of time, and if there is a possibility of like see variable density cases are very typical cases. **“Professor - student conversation starts”** So for variable density cases it is not so trivial to say that like the mass flow rate is the same in all cases, but here no matter how the density varies you can say that these 2 mass flow rates will be the same why?

Area of cross section is same, so density is like whatever it is even if you forget about the density like just fundamental think if you have a mass flow rate that goes out of this. If the same mass flow rate does not go out of this where does that mass go? Is there any mass like accumulated in between no. **“Professor - student conversation ends”** Velocity may be different if the densities are different, but if the density is uniform then that may be ruled out provided the time dependence of the density is not creating any big change.

So if you see that like what happens within this pipe line in terms of the velocity or density something we do not have enough information really to talk about that, but one important information we have that whatever it happens individually to velocity density, but the mass flow rate what goes through this is same as the mass flow rate that goes out because where otherwise it will go, it cannot sit on the valve that mass.

So we are assuming that there is no accumulation. It is steady flow system. So if it is a steady flow like whatever like enters here the same leaves here then it does not matter really whether your control surface goes through these or through these are whatever. Now let us look into the different terms in this expression so if you look into the first term. So here it is a rigid tank in the previous class we tried to solve a similar problem which is like a flexible balloon.

So there the volume could change with time that is may be if it were a flexible balloon you could have expected that by taking away air out of it the balloon will try to shrink, but here it is a rigid tank, so it cannot respond to that. It can only respond to that change by having a change in density inside, but not its change of its own volume. So the volume of the tank, the density of the fluid in the tank is definitely varying with respect to time but the volume is not varying with time.

So that means you can the first thing that you can do is you can take the time derivative inside because the volume of the control volume is not a function of time anymore and the next thing is that how this density changes with respect to time that does not change from one point to the other within the control volume why because it is given that is uniform state in the tank and in the pipe line that means the density may change with time, but at a given instant of time.

The density is same everywhere within the control volume, so that means this is like an isolated term which does not depend on the volume so this you can take out of the integral so that means eventually it will become the volume of the tank * the rate of change of density with respect to time in the tank. So it is not so difficult to come up with this expression, but we have to keep in mind that what are the important assumptions that are leading to this type of a simple proposition.

If the density was varying within the tank itself, then we could not write it, then we had to integrate it by keeping in mind that rho is a function of both position and time. Here the dependence on position we have chosen we have assumed that that is not there then for the next term so if you see that this now you have only one outflow boundary for the control surface. When the valve is opened, the air is leaving here so you have an n dot exit which is the rate of mass flow rate out.

So the rate of see this is eventually giving what. This is giving a rate of mass flow rate outflow - inflow. There is no inflow so only outflow so this is as good as + m dot e this = 0. What is this m dot e. m dot e is proportional to the density. So let us say that m dot e is = k * rho. So this rho is an instantaneous density that at that time whatever is the density in the system proportional to that the mass is coming out so this you can write some k * rho where k is a constant.

So you can the next work is very easy. All of you enjoy doing it, very simple integration. So d rho/rho = - k vtank dt. So you can integrate it from say time = t1 to time = t2 say the density changes from rho = rho 1 to rho = rho 2. So you can find out how the density changes with time. So important is not the solution of the problem, of course a solution is quite easy but solution is definitely having some importance, but to my understanding the greater importance is what are the assumption that are leading to the solution.

Because in reality the problems are not so simple as many of this one so this might not be very, very common as our analysis equations because if you see this equation that we have written these are really on the basis of such simplified conditions or assumptions which might not prevail in practice. So the final answer may be like interesting in terms of solving a particular problem, but the reliability of the answer may not be so strong because you might have a strong variation of density within the tank itself.

But at least by having a simplified assumption it is giving us a fair idea of like how the analysis should involve the conservation of mass. So when we have been discussing about the conservation of mass, we have discussed certain types of problems. What types of problems we have discussed? We have discussed about one case when it is totally steady that is you have like the density first of all the density is not changing with time, volume of the control volume is also not changing with time so that this term is not there.

So it is just a balance between the rate of outflow and inflow and physically that represents a condition where data of outflow = rate of inflow. For that we have to keep 2 things in mind one is that what is the sense of the velocity, velocity component normal to the area is it opposite to the area vector or is it along the area vector. The second point is the velocity uniform over the cross section or is it non-uniform accordingly we might need to integrate or not and when it comes to a unsteady case there are 2 types of possibilities.

One is the density is changing with time, another is the volume of the control volume is changing with time and may be a third case when both are changing with time, but like we have perhaps not considered that case, but that is like it is a combination of the cases that we have considered. So all these cases have given us found footing or understanding of how to use the integral forms of conservation equations.

What is the advantage of use of the integral form of conservation equation? See when you are using an integral form the important advantage is that you are being abstracted from how things vary within the control volume. You are only bothered about a gross manifestation in terms of what is entering, what is leaving so what is happening inside you are just representing it in an integral sense or an overall sense.

You are not really representing it a point by point variation. So we have to keep in mind a very important thing what is the difference between an integral equation and a differential equation physically. Differential equation gives you the variation at a point, whereas integral equation gives you the variation over a domain. Of course the domain is constituted of many such points, but the differential equation is valid only at an identified point in the domain.

So the integral form when you are writing you may get that the differential form from that, but at the same time you are not forced to track or to keep in mind that what is happening as a point by point variation. So that differential nature of variation you may not be interested in. So in such cases where you are not really interested in that it may be convenient to use the integral form. So one has to keep in mind or when should we use the integral form and when should we use the differential form. It depends on the physical sense of the problem that we are trying to solve.