Vibrations of Structures

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Lecture No - 09

Modal Analysis Approximation Methods - I

Last three lectures, we have been discussing about modal analysis of continuous systems. Now, we have solved the problem of modal analysis, which is nothing but solving an Eigen value problem, as we have seen analytically, which means that we are exactly solving the natural frequencies, and the modes of vibration, which are characterized by the Eigen functions. Now, doing an analytical solution is always preferable, because you can find out the effects of various parameters in the system on the modes of vibration and the modal frequency etc. So, analytical solution is always preferable. However, we have seen that even in very simple systems, the solution of the modal analysis problem requires solving transcendental equations, possibly which are, which might be quite cumbersome even numerically.

So, in general, analytical solutions though preferable are sometimes cumbersome and computation intensive. So, it is of interest to know if numerical methods of modal analysis or approximate methods of modal analysis are possible. So, in this lecture and the next lecture, we are going to look at, few techniques for solving the modal analysis problem or the Eigen value problem approximately numerically.

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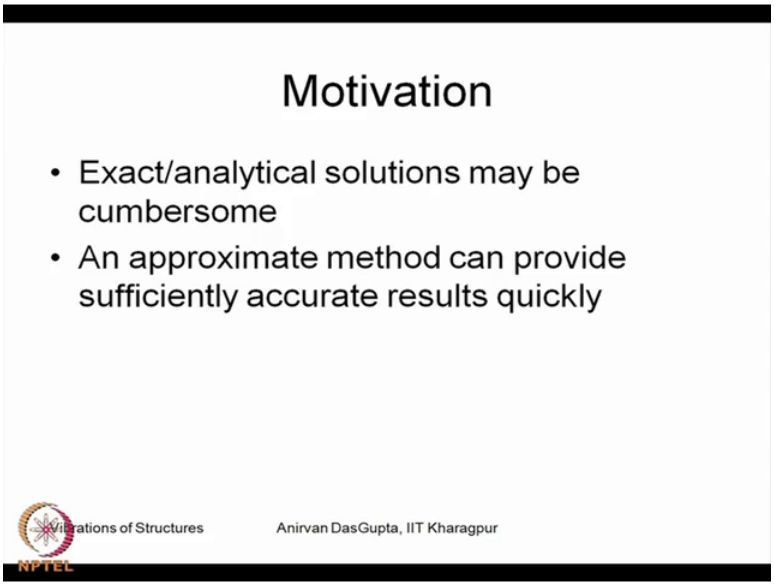

So, what is our motivation for studying the approximate solutions those. So, the first motivation is that the analytical solutions may be cumbersome. The other thing is an approximate method can provide a quick solution to the modal analysis problem, which may be sufficiently accurate for our purposes.

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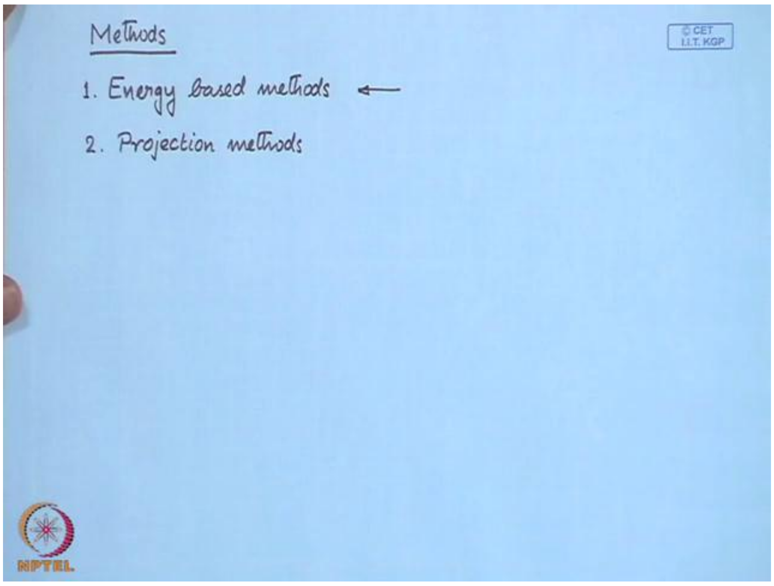

So, let us look at the methods that are available to us for approximate modal analysis. So, we are going to discuss in this course two methods and two broad methods, which are the energy based methods, which will be the topic of discussion in today's lecture; and in the next lecture, we are going to look at projection methods. So, as the name suggests, this energy based methods will use the kinetic and potential energy of the system to determine the modes of vibration or the natural frequencies; while the projection methods as we will see very soon, they use the governing equation of motion, directly to solve the modal analysis problem.

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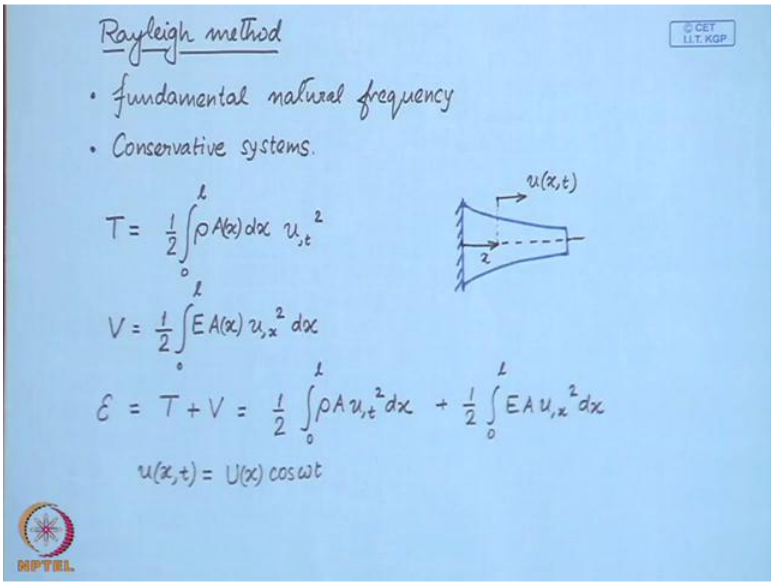

So, the first method that we are going to look at is the Rayleigh method. So, this Rayleigh method is typically used to determine the fundamental frequency of a continuous system, and this method is used for conservative systems. So, we use Rayleigh method for conservative systems.

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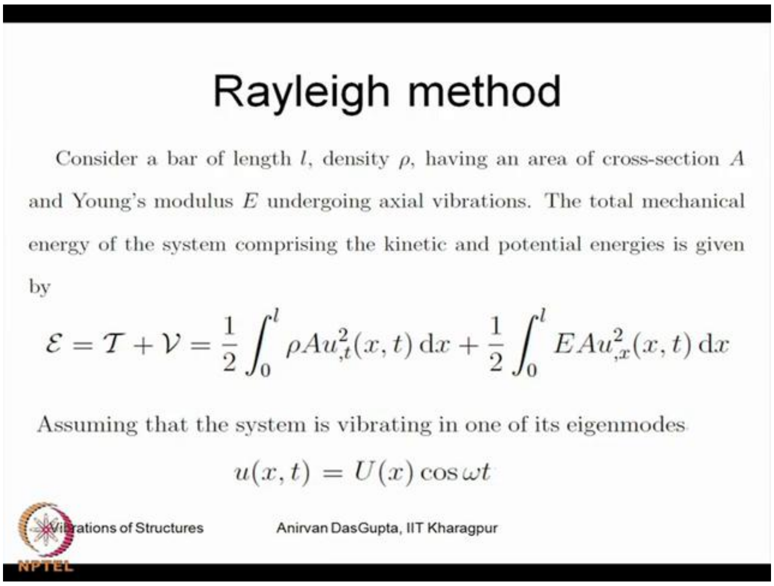

So, how does this method work? So, let us understand this with the example of a bar. So, let us consider a tapered bar in axial vibration. So, we write the kinetic energy as one half the density of the material times the area of cross-section times an elemental length times the velocity's square; now this integrated over the domain of the bar. The potential energy is one half the Young's modulus times the spatial derivative of the field variable's square dx and integrated over the domain of the bar. Now this system, this bar in axial vibration, as we know, is a conservative system, which means the total energy of this bar is constant. So, the total mechanical energy… this is a constant. Now, let us suppose that this system is vibrating in one of its modes. So, as we have already discussed that when a system is vibrating in one of its modes, the field variable can be written as a separable function of space and time in this form. So, this special solution form is valid for a system, for this bar, vibrating in one of its mode. Now we will substitute this expression in the total energy of the bar, and once we do that what we obtain… So, this is the expression of total energy after we substitute the solution structure in the total energy expression. Now, if this total mechanical energy is to be constant then what it would require is that it should be independent of time; and this is possible only when the coefficient of sine square omega t and cos square omega t is same, which means the energy is constant would imply these two, the coefficients of sine square omega t and cos square omega t, they must be equal. So, therefore we obtain this ratio which is defined as the Rayleigh quotient; and this Rayleigh quotient is the key concept in Rayleigh method. So what we have done is, we have considered the modal solution, substitute it in the total energy expression, total mechanical energy expression; and finally, we force the energy to be independent of time by matching the two coefficients of the sine square omega t and cos square omega t terms, to obtain omega square as a ratio which is known as the Rayleigh quotient. Now, sometimes this Rayleigh quotient is expressed as ratio of the maximum potential energy divided by the maximum kinetic energy. So, the amplitude, the maximum potential energy would be the amplitude of the cos square omega t term, whereas the maximum kinetic energy would be the amplitude of the sine square omega t term; and of course this omega is solved in terms of this.

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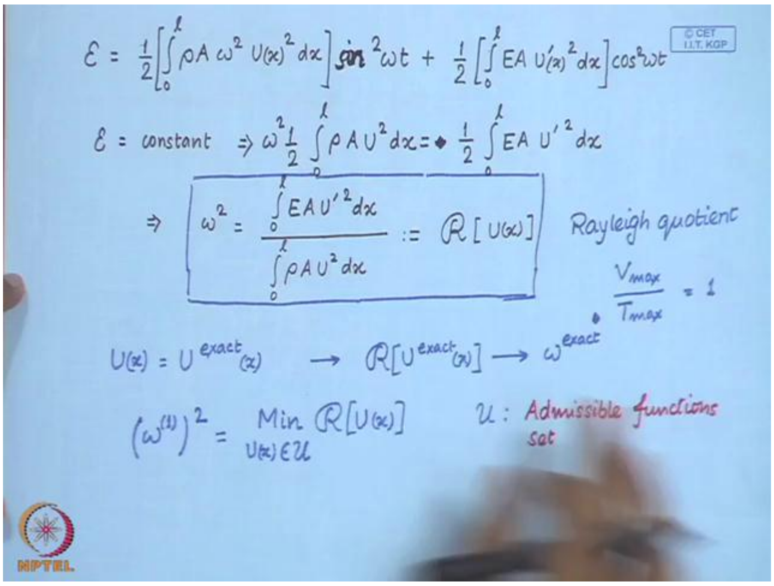

So, this is one; so, from here, we solve for omega square as we have done here. Now this, in this Rayleigh quotient, if you know the exact Eigen function; so, if you substitute, where I am using this exact super script to indicate that u exact is the exact Eigen function for a particular mode; in that case, if you put this in the Rayleigh quotient, what we will get is the exact circular Eigen frequency corresponding to that mode. But the problem comes, because we do not know the exact Eigen function. So, in that case, how do we use this Rayleigh quotient? So, usually, what is done is, we try to minimize this Rayleigh quotient, so, and by minimizing the Rayleigh quotient, we obtain the fundamental frequency.

So the fundamental frequency's square is obtained by minimizin the Rayleigh quotient over a space of possible Eigen functions U of x. Now there is a restriction on how you can choose U of x in this minimization problem. The restriction is, U must be a member of the set of what are known as admissible functions. So this u is a set of admissible function. So, you must choose the possible Eigen function from the set of admissible functions. So, what are admissible functions? This we will come to very soon. So ,let me write for this problem.

So, the fundamental frequency square is minimization over the set of Admissible functions of the Rayleigh quotient. Now, we come to these admissible functions. So, what are Admissible functions? These are functions, which satisfy the following two properties; the first property is, it is differentiable at least… So, these functions are differentiable at least upto the highest order of spatial derivative in the energy expression. Of course these functions are special functions; so they must be differentiable at least upto the highest order of the spatial derivative in the energy expression. So, in this example we are considering, the highest order of space derivative is one. So the set of admissible function should be differentiable up to first order. The second important property that it should satisfy is that it should satisfy all the geometric boundary conditions of the problem. So, admissible functions must satisfy all the geometric or essential boundary conditions of the problem. So, they satisfy all the geometric or essential boundary conditions of the problem. So, these two properties, the functions that satisfy these two properties are known as admissible functions. Now such functions can be constructed using polynomials, trigonometric functions, and other such elementary functions.

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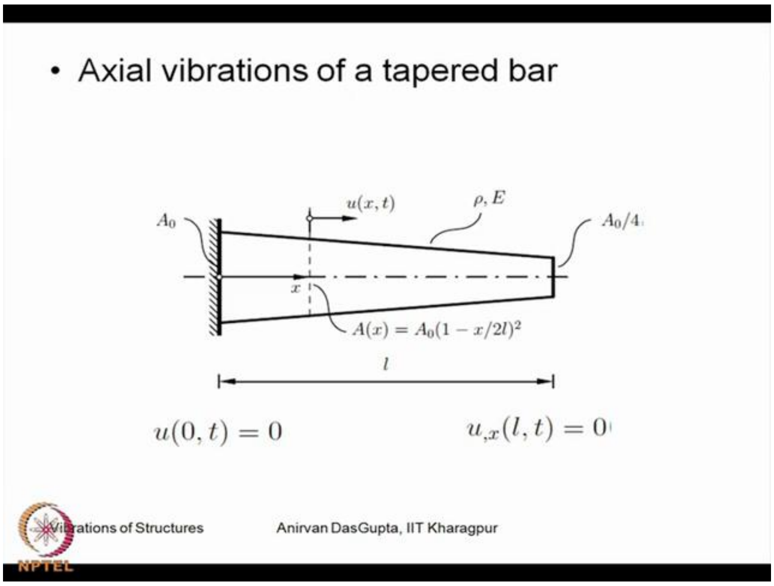

Now let us, look at this example of the tapered bar. So, as you can see, it is fixed at x is equal to zero and it is free at x is equal to l. So, the geometric or essential boundary condition is on the left boundary, where the bar is fixed. So, we must choose functions, Admissible functions, which satisfy the boundary condition at the left.

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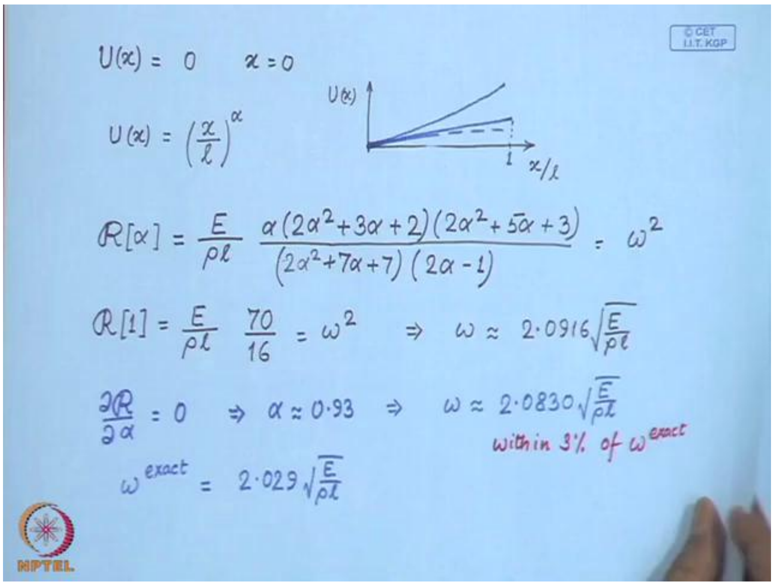

So, let us look at such functions. So, we must have... So, we have to choose Admissible functions for this problem such that it is zero at x equal to zero. This is the minimum condition that is required. So, the possible choice, you may write it like this x over l. But we can have a class of these functions, by raising it to a power alpha. So, we actually have functions, which depending on alpha, for example if alpha is 1 then this is linear, for other powers, it can go like this or it can go like this. Now, so, this alpha we will initially, keep it arbitrary, and we are going to substitute this in the Rayleigh quotient, and we are going to look at, what is the Rayleigh quotient with this expression of admissible function. Now one thing that you can see that we now have not one function, but we have a class of functions; and we can adjust alpha and see, which function minimizes the Rayleigh quotients? So, this alpha provides as with a handle to solve a minimization problem as we formulated. So, if you calculate the Rayleigh quotient with this admissible function, then the Rayleigh quotient finally turns out to be function of alpha, which is unknown as yet and this expression turns out to be, this can be solve very easily and what we obtain is this expression of the Rayleigh quotient. Now, so this term has the properties of the bar, the geometric as well as the material properties of the bar, while this coefficient, which is a function of alpha, determines the Rayleigh quotient. Now this alpha is unknown as yet. So we can put various values of alpha or we can minimize the Rayleigh quotient with respect to alpha and determine alpha. Let us see what happens with alpha is one and this, remember, this is omega square. So this turns out to be 70 over 16. So this is an estimate of omega, the first fundamental circular frequency and this turns out to be… Now let us see what happens if we minimize with respect to alpha, which means… So if you do this minimization, this gives… and corresponding to this… So, you see that this value is lower than this. So, this is a better estimate of the circular natural frequency. Now we have solved this problem of the tapered bar, in a previous lecture analytically and the exact circular natural frequency that we determine was this. So, this is still lower, as you can see that this is higher than the exact. But they are quite close. This is within three percent of the exact. So we have fairly accurately estimated the fundamental natural frequency of a tapered bar using a very simple method. But this, of course, this gives the best estimate based on the form of structure of the admissible function that we have chosen. But remember that when we go on to calculate stress, since this alpha is less than one for our best estimate, the stress that we will calculate at x equals to zero will be infinity. So, it will have some, I mean, it will give some unrealistic estimates of stress in bar. However, the frequency estimate is fairly accurate. Now here using Rayleigh method we have estimated the fundamental frequency. Can we now go on to find out the higher frequencies? Now, that is possible by using what is known as the Rayleigh-Ritz method.

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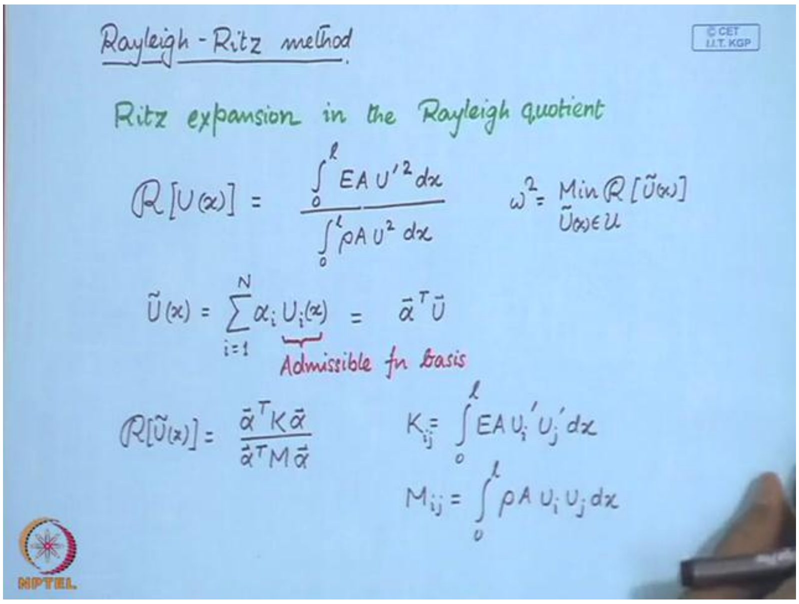

So, what we additionally have in this method is the Ritz expansion. So, we make use of the Ritz expansion of the amplitude functions in the Rayleigh quotient. So, what I mean, so if you see this Rayleigh quotient, let us say for our problem of the tapered bar, so, we will minimize this… and U has to be chosen from the set of Admissible functions. Now, this U, the amplitude function, we will expand and put this tilde symbol to distinguish this function from the basis function that we are using. So, like in the previous example of Rayleigh's method, we have kept an unknown parameter alpha; here we will expand the amplitude function in terms of admissible basis functions and certain unknown coefficients alpha. So, this is a linear combination of these basis functions, admissible functions. So these are all admissible function; and we can take any number of terms. Now, when we substitute this kind of an expansion, the Rayleigh quotient can be written as, so this I can write as a vector multiplication. So, alpha is a column vector and U is also a column vector. So, the dot product will represent this scalar function U tilde. When I substitute this expression in the Rayleigh quotient, I can write in this form where K, the matrix K, so, the ij^{th} element is expressed this form and similarly, for the ij^{th} the element of this matrix M is expression in this form. Now we this, remember this alpha, this alpha vector is unknown. So, we have to minimize with respect to this vector alpha.

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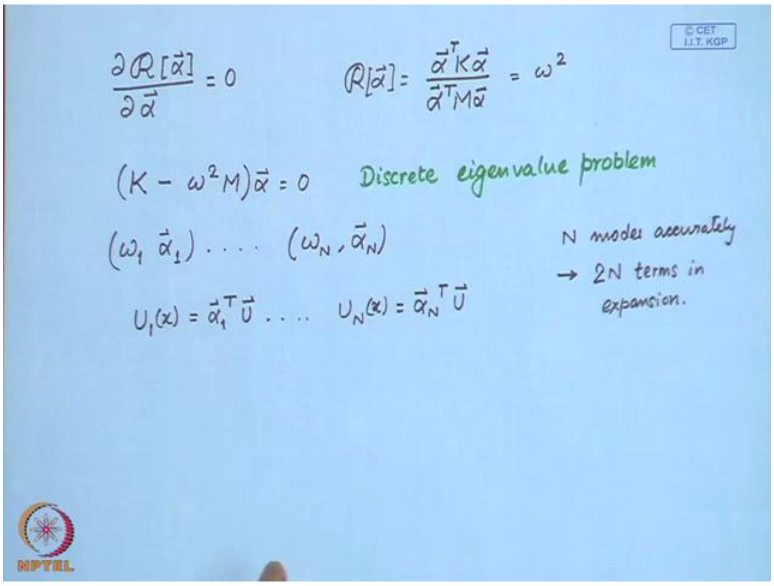

So, which implies that, now this Rayleigh quotient is function of this vector alpha, and this must be put to zero. So, the derivative must vanish for extremization. Now this derivative implies that this derivative has to be taken with each element of this vector alpha. So, we have, if there are capital N elements in this vector alpha, then there will be N equations N unknowns. So, if you consider this expression of the Rayleigh quotient, and if you perform this derivative, which is straight forward; finally what we will arrive at… Now here how will omega enter? We have used this expression here. So, if we perform the derivative, this coefficient actually turns out to be this ratio, which I am replacing by omega square. Now this is a discrete Eigen value problem, which can be solved very easily to determine omega and this vector alpha, the Eigen vectors alpha and finally once we have the circular Eigen frequencies and the corresponding vectors alpha i, these can be used to determine the corresponding Eigen functions using those basis function vectors U. So using Rayleigh-Ritz method, we can find out not only the fundamental but the higher modes of vibration. Now as a thumb rule, if we want, because the accuracy of these various modes will be different, so as a rule of thumb if we want N modes accurately, we must take 2N terms in the expansion. I mean this is rough estimate, I mean, rough estimating how many terms you must have in the expansion; and this may or may not work always; but this is the good way to start. So, if you want N modes accurately, reasonably accurately, then you must have double number of terms in your expansion.

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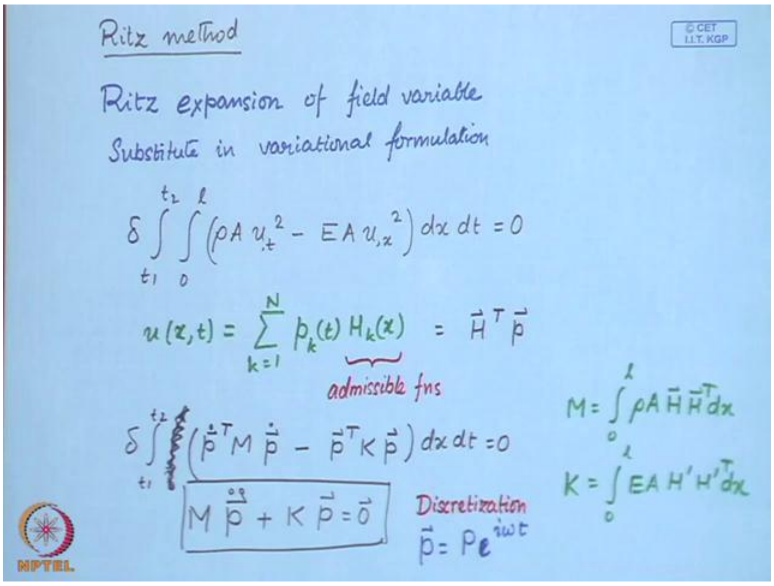

Now there exists another method, which is quite powerful in this class of method, which is known as the Ritz method. Now in the Ritz method, we use the idea of Ritz expansion of the field variable, and substitute this expansion directly in the variational formulation. So this method works with a variational formulation of dynamics. So, let us see, once again, for the tapered bar, we know that the variational formulation for this tapered bar is... Now, here, we use this expansion… again in terms of admissible functions… Now once you substitute this in the variational formulation and simplify, we obtain this where… where this is integration over the space, as already been performed. So, we substitute this here and since these functions, admissible functions are known, these bases are known to us, we can perform this spatial integration and obtain this variation of this discretized problem. So, here these matrices M and K are given by this expression and we know that and this is now the Lagrangian of a discredited system and the equation of motion can be immediately written… So, in this Ritz method, we have essentially discretized our problem. Now, once we have discretized, we can search for solutions, as we do for discrete systems. So, we search for modal solutions of this form and we solve the Eigen value problem. So after this, the things are very standard. So, let us look at the axial vibration of the tapered bar once again. Here, I have written out the admissible functions that we have chosen. So, H of x for j 1 and 2, we have taken two functions and discretized with respect to these two functions. The discretized equation of motion is shown below. So, once you have the discretized equations of motion then the standard procedure follows, which means that, you assume the solution structure as shown here, you come to the Eigen value problem and finally the characteristic equation. Now, if you solve this characteristic equation, you will obtain the Eigen frequencies of the system. Now, by solving this we obtain the circular Eigen frequencies as you can see omega 1 superscript R is calculated using the Ritz method and similarly omega 2 superscript R; and they are compared with the exact circular Eigen frequencies and you can see the fundamental, we have taken two terms; the fundamental circular frequency matches quiet well with the exact while there is some error in the second circular natural frequency.

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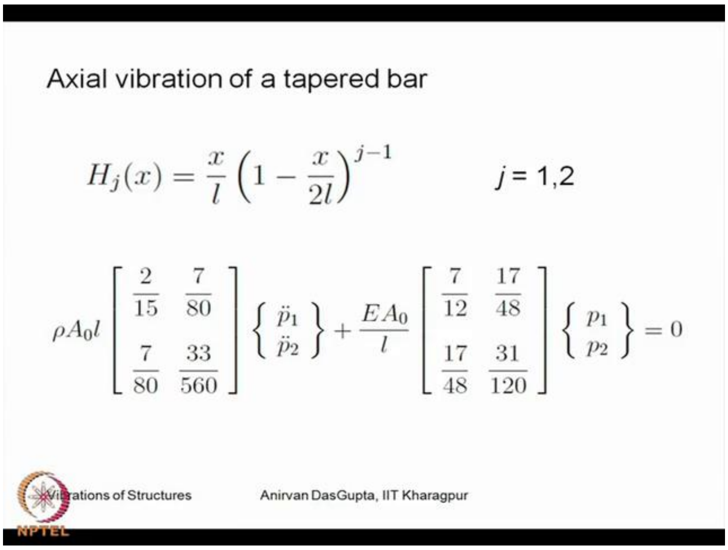

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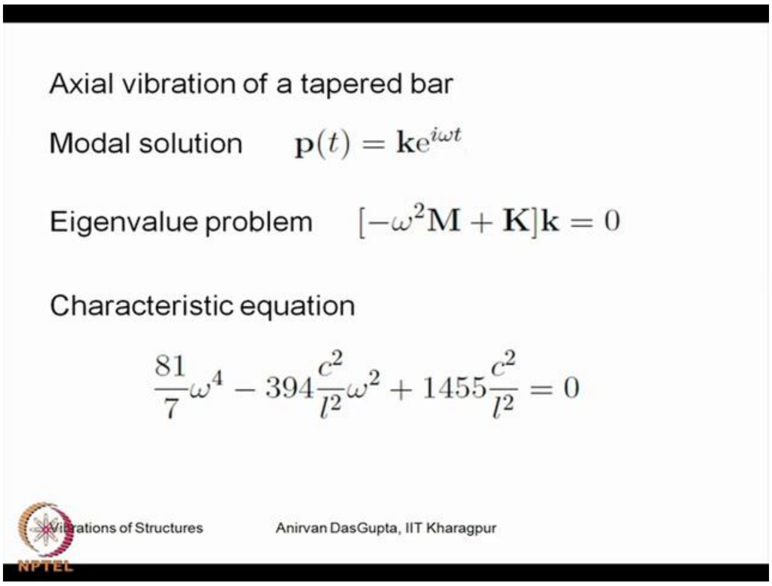

Now, this Ritz method, even Rayleigh's method have an upper bound property which means that the natural frequency calculated from these approximate methods is always greater than the exact; which means that this gives an Upper bound, the actual natural frequency, the actual natural frequency of the system is lower than what you calculate using the using these approximate methods.

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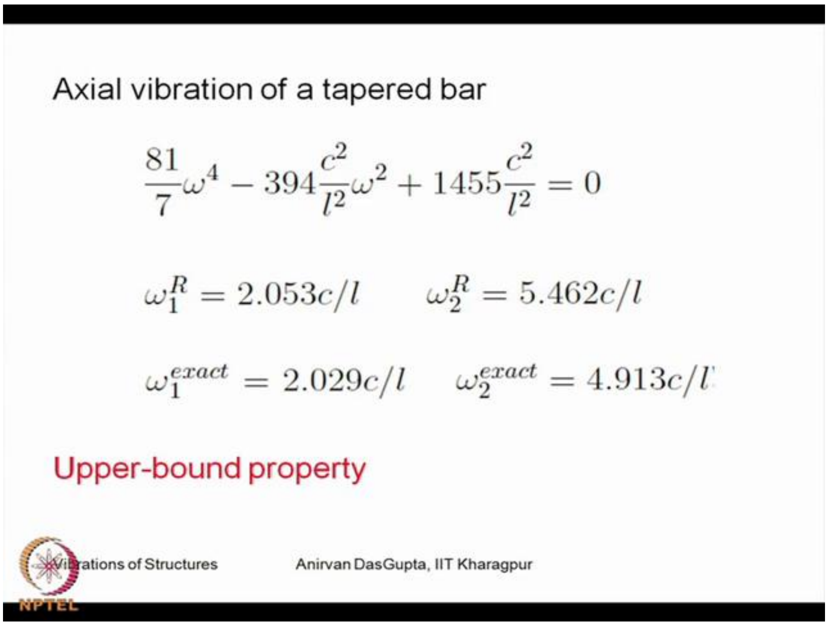

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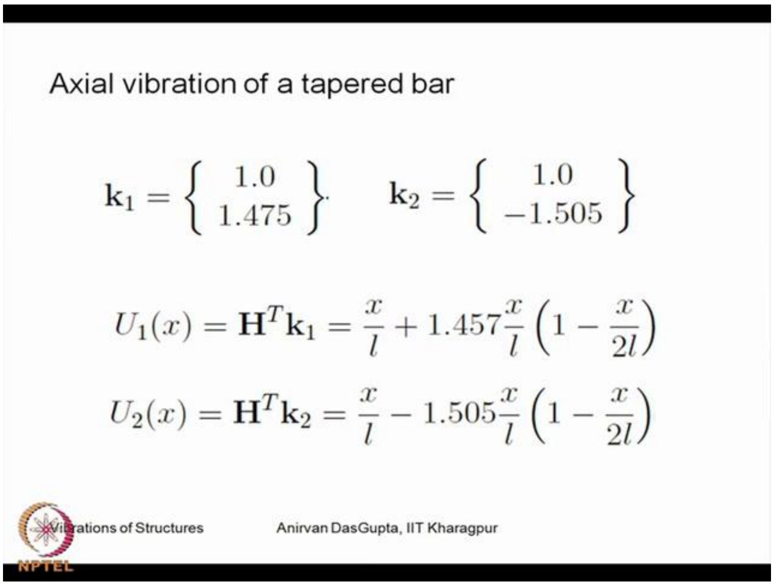

Now here you can see the Eigen vectors k_1 and k_2, and the corresponding Eigen frequencies that have been determined by using the Ritz method.

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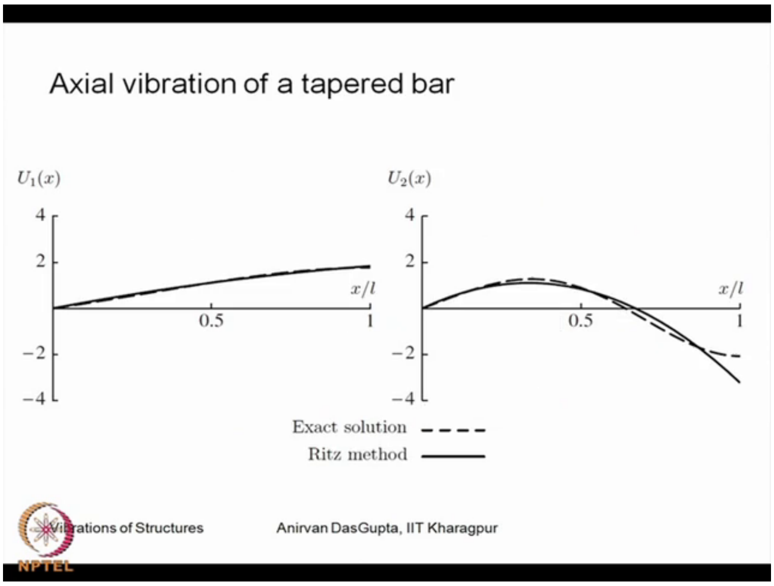

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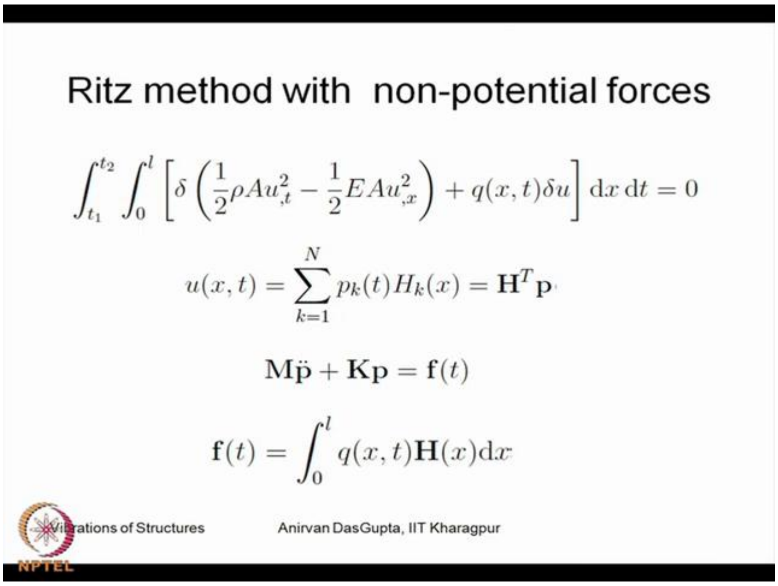

And once you plot these Eigen functions, so this shows the comparison of the Eigen functions calculated by the Ritz method and those obtain from the exact solution that we had discussed previously. So, again you can see that the fundamental Eigen frequency matches quite well with the exact while, that of the second mode is an error especially at x over l equal to 1. Since we are considering only admissible functions with satisfy the geometric boundary condition, which is the, at x is equal to zero; while the natural boundary condition is not satisfied with two terms; you have to take more and more terms, and then there is a convergence to the exact solution.

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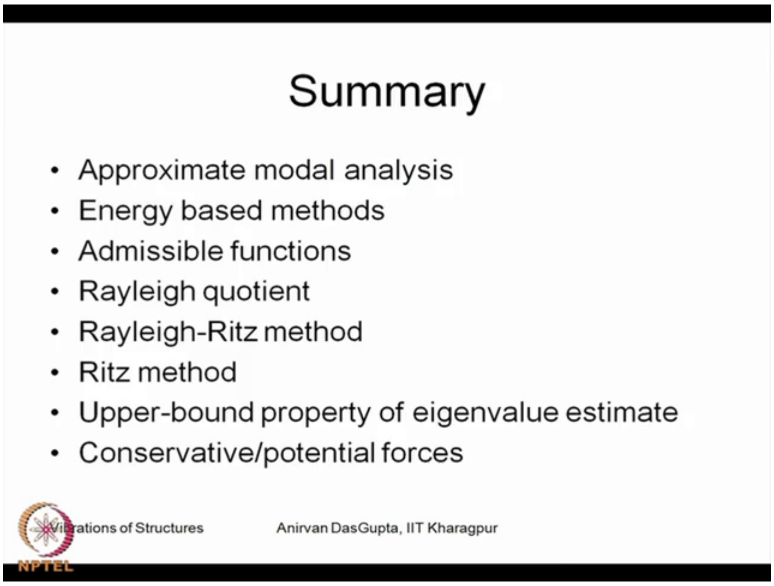

Now we summarize this lecture. So, we have considered approximate modal analysis, based on energy methods, which uses admissible functions. We have looked at three methods: Rayleigh quotient, Rayleigh Ritz method and Ritz method; and these methods have an upper-bound property of the Eigen value estimate; and these methods work for conservative systems with potential forces. With that we conclude this lecture.

Keyword: Rayleigh quotient, Rayleigh-Ritz method, Ritz method, upper-bound property.