Vibrations of Structures
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Lecture No # 08
Properties of Eigen Value Problem

In the last two lectures, we started discussions on the modal analysis of continuous systems. Now, this was the performance of modal analysis was found to be essentially solving an Eigen value problem. Now, today we are going to look at some properties of this Eigen value problem that comes up while we perform modal analysis of continuous systems.

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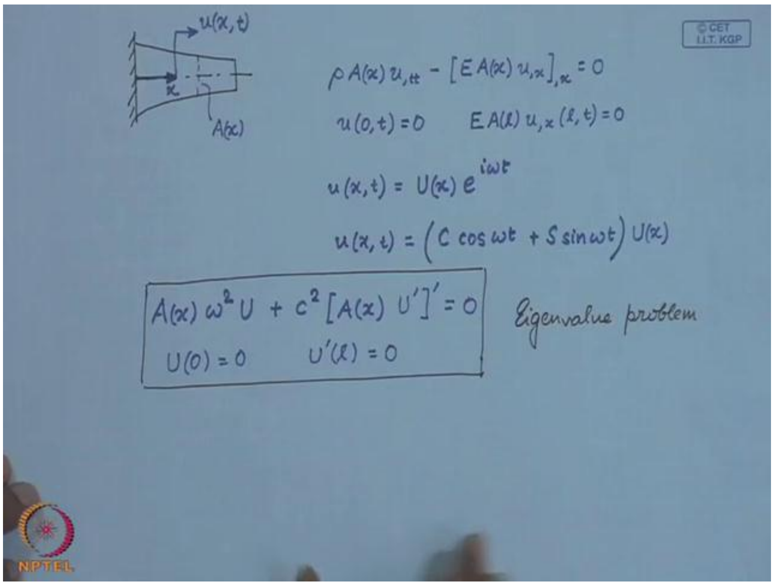

So, let us start by re-visiting the modal analysis problem. So, in our last lecture, we had discussed the problem of a bar with varying cross section. The equation of motion of the system was given by this and the relevant boundary conditions for this problem were given by U at zero is equal to zero for all time and on the right boundary we have natural boundary condition and in the order to do the modal analysis we were searching for solutions of the special form or structure.

So, the field variable is expressed as a product of an amplitude function which is the function of x, and harmonically varying time function. Now, we discuss the properties of this solution and we found that the solution is actually separable in space and time. So, when we write the actual solution in real form… it appears in this structure. So, it was separable in space and time. The other observation is all points, therefore, vibrate at the same frequency omega, the same circular frequency omega. Thirdly, all points of the system pass through the equilibrium point at the same time instant. The time instant when this temporal function is zero, the whole solution is zero, which means the bar is in its equilibrium state; so, all points will pass through the equilibrium point at the same time.

Then we observed that phase difference between any two points and the bar is either zero or pi. And finally, we observe the existence of modes that means points at which U, the amplitude function capital U of x is zero. So, the points, so, the properties of the modal solution are known to us. So, once we substitute the solution of this structure in to the equation of motion, we obtain the differential equation in terms of this amplitude function and the corresponding boundary conditions. This forms the Eigen value problem for the system, so, the differential equation along with the boundary conditions. Now, we will represent this in a slightly abstract form in this manner… where, our equation of motion was, can be written like this…

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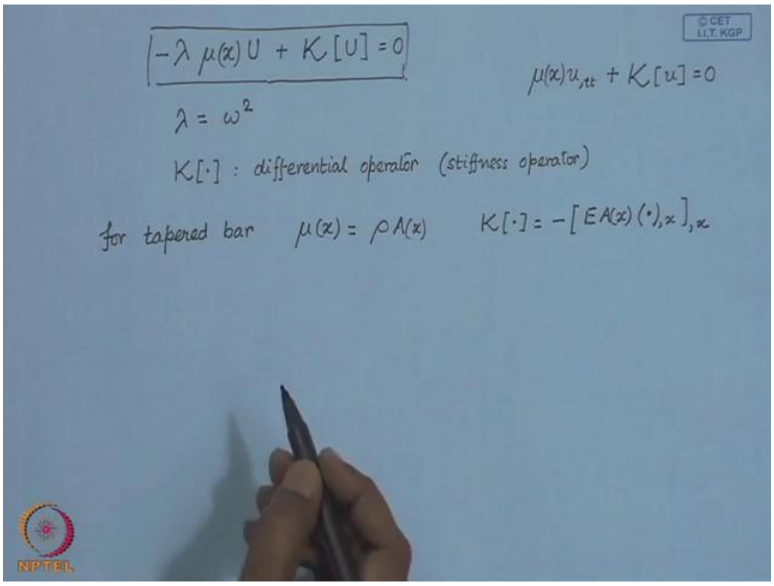

So, if you write a general equation of motion of a string or a bar in this form, then the differential equation of the Eigen value problem may be represented in this manner, where lambda is omega square. So, this actually is plus. So, this is the differential equation and the corresponding differential equation for the Eigen value problem is given in this form, where lambda is omega square and this K is the differential operator.

So, for example, in the case of the tapered bar, mu(x) is rho times the area and the differential operator K, which is also known as the stiffness operator, is the spatial derivative of this quantity, E A in the derivative of the argument. So, this is the structure of the differential equation of our Eigen value problem. Now here, as I mentioned, here this is known as the stiffness operator, because this term comes from the potential energy in the Lagrangian formulation, while this term mu(x) is the kinetic energy operator, because it comes from the kinetic energy in the Lagrangian formulation.

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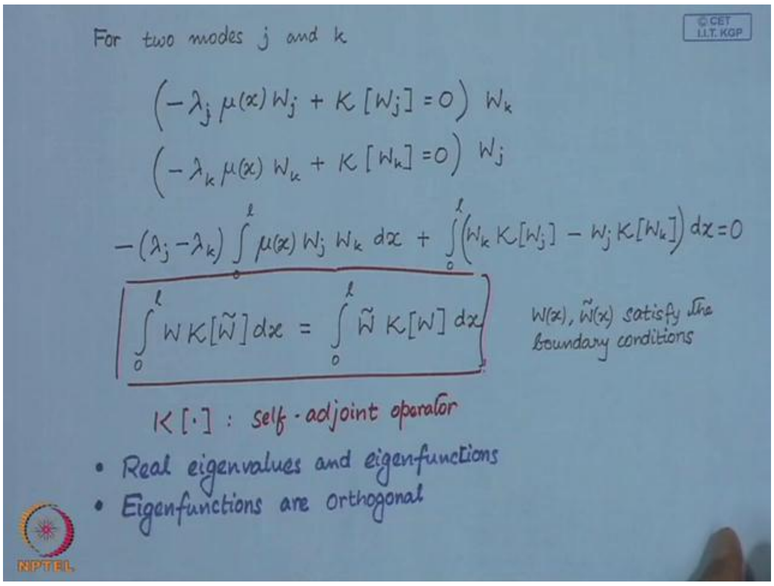

So, suppose for two modes j and k we can write this differential equation. So, for the j^{th} mode let us say, we can write the differential equation of the Eigen value problem like this, while for the k^{th} mode, the differential equation becomes this. Now, we are going to, the objective of this analysis is to determine certain properties of the Eigen value problem. So, let me multiply this first equation with W_k and the second equation with W_j, and subtract one from the other, then after some rearrangement, I can write. So, I am also integrating over the domain of the system.

So, I multiplied the first equation with W_k and the second equation with W_j subtracted one from the other and integrated over the domain of the problem; and this is what I obtain upon rearrangement. Now, suppose that this integral vanishes. So, let us consider the situation when this property holds where this W and W tilde are functions that satisfy the boundary conditions of the problem. If this property holds, then this operator K is known as self-adjoint. So, this property is satisfied by the stiffness operator then it is known as a self-adjoint operator.

Now, this self adjointness of an operator is connected to symmetry. So, as you know that the stiffness operator has a corresponding matrix, for example, in vibrations of discrete systems you come across stiffness matrix. The self adjointness of the stiffness operator is nothing but the symmetry of the stiffness matrix, the corresponding stiffness matrix. So, what are the consequences of this symmetry? As we know that when matrices are symmetric, the Eigen values are real and the Eigen functions are also real, and the Eigen vectors are orthogonal. So, in a similar manner, we have these properties which can be shown very easily that the Eigen values are real, Eigen values and Eigen functions are real, whenever the stiffness operator is self-adjoint. Secondly, the Eigen functions are orthogonal with respect to an inner product that we will find out in the course of this lecture.

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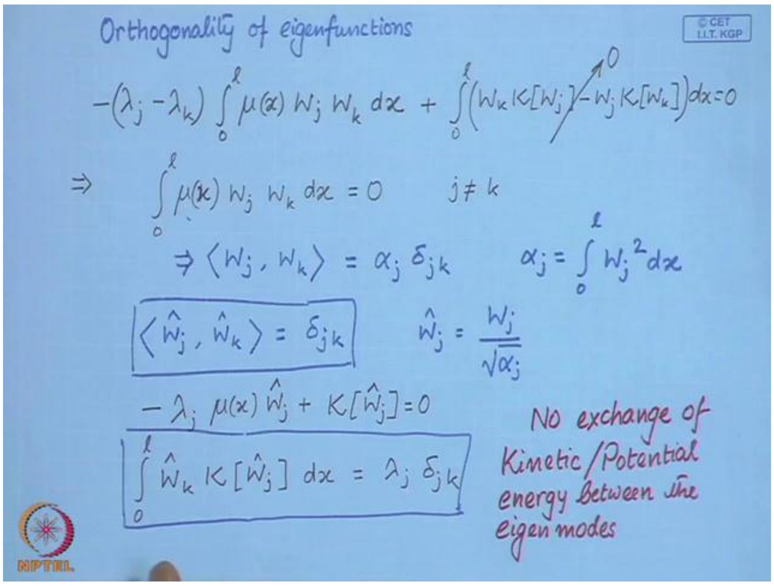

So, we are, we will be now discussing this orthogonality property. So, re-call that we have this equation. So, if the operator K is self-adjoint then this term vanishes. So, this implies this integral must vanish, whenever j is not equal to k. So, which means if I take two distinct amplitude functions modes, Eigen functions W_j and W_k, then this satisfy this property that this integral must vanish; and this, we define as the inner product of these two Eigen functions. And in a compact form, we will write this, the inner product of two Eigen functions can be written in this form where alpha j is given by this integral.

Now, one may normalize this property by appropriately scaling the Eigen functions, because as we know that, any scaled form of this Eigen function is also an Eigen function. So, we can scale appropriately, to have ortho-normality of the Eigen functions with respect to this inner product that we have defined. So, here this W_j hat is W_j over square root of alpha j. So, here we have orthogonality with respect to the inertia operator. So, if you consider that this mu of x represents the inertia operator, then this orthogonality is with respect to the inertia operator and correspondingly we have we can write… So, for the j^{th} mode, this is the differential equation. So, if I multiply this equation with… So, this can be written also for the hat Eigen function, the normalized Eigen function and if I multiply this with W_k hat and integrate… So, this shows that the Eigen functions are orthogonal also with respect to the stiffness operator K. Now, what is, what are the implications of this, so, what are what is the physical implication of this orthogonality with respect to the inertia and the stiffness operators. So, the implication is that, there is no exchange of kinetic or potential energy between the Eigen modes. The physical implication is that there is no exchange of kinetic or potential energy between the Eigen modes. And this orthogonality property is also very useful as we will see in due course for solving initial value problems or other problems related to continuous systems. And this orthogonality we have already come across when we discussed about vibrations of modal analysis of strings.

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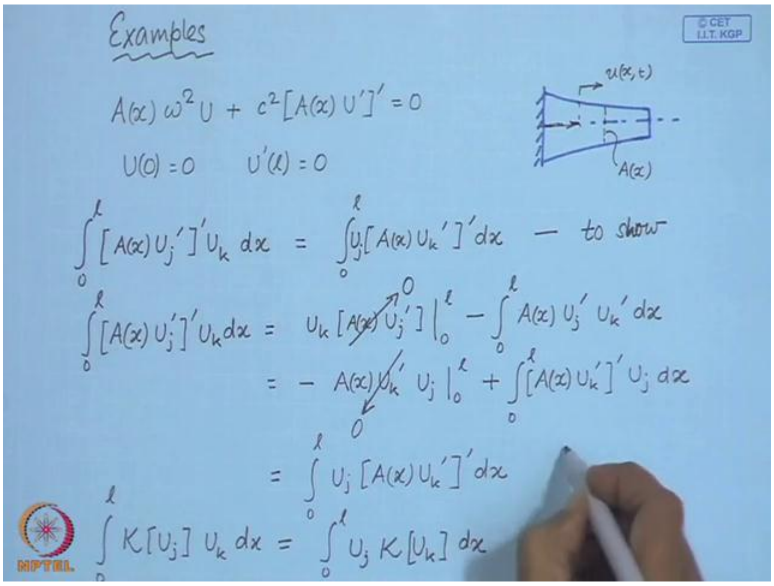

Next, we discuss some examples and we will determine the orthogonality relations for these examples. So, we once again go back to this bar of varying cross-section and follow the steps that we have done in detail. So, our Eigen value problem… Now, let us check that this stiffness operator, that we have here, is really self-adjoint. So, what we have to show… So, this we have to show. So, you have to show that these two are equal where U_k and U_j are two Eigen functions of this Eigen value problem. So, we start integrating by parts, let see from the left hand side. So, we take this as the first function and this as the second function. This is what we obtain. And here I will integrate by parts this term once again and here I will use the boundary conditions. So, the boundary terms here that I have, so, this term will be evaluated at l and at 0. Now, at l U prime at l must be 0. So, this term must vanish at l and U(0) is 0. So, U_k at 0 must be 0 becomes these are Eigen functions and satisfy the boundary conditions of the Eigen value problems. So, this term is actually zero. So, we are left with only this term and this I will integrate by parts once again.

Now, the same reasoning as we had here. This term, this boundary terms must also varnish. So, we are left with which is nothing but the right hand side of this equation. So, we have shown that this operator acting on… So, we have shown this self adjointness of the stiffness operator of that equal to bar. So, we can write this orthogonality in terms of the inner product as we were defining for the tapered bar.

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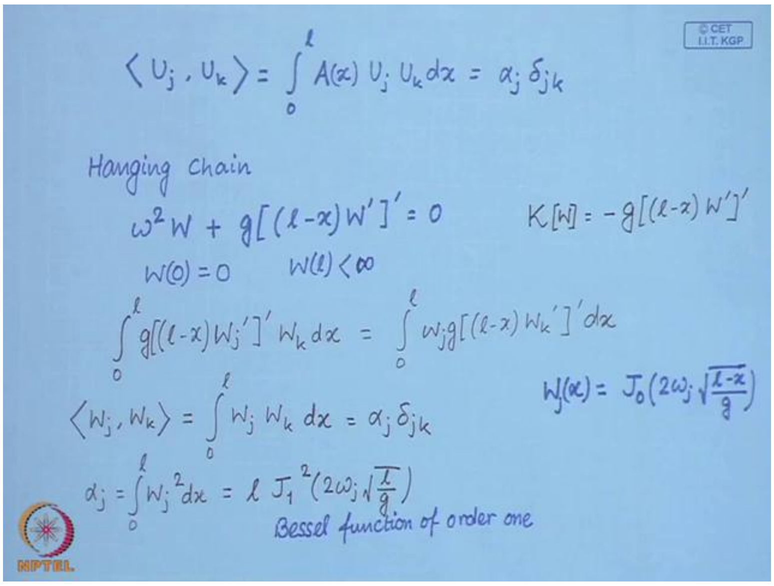

Next, we are going to look at the hanging string or hanging chain. So, for the hanging chain the Eigen value problem write this. So, in this case, the stiffness operator is given by this term and in a similar manner you can check that this self adjointness property holds for the stiffness operator of a hanging chain. And once you, I mean, used this property, you can derive the inner product of the Eigen functions of the hanging chain with respect to which the Eigen functions are orthogonal. So, let me just write down this Eigen function that we have already derived in a previous lecture. So… So, this is the structure of the Eigen functions of a hanging chain and they satisfy the orthogonality relation in this form with... So, this is alpha j is the square of the Eigen say that jth Eigen function and integrated over 0 to l, and this turns out to be… However, this J_1 is the Basel function of order 1.

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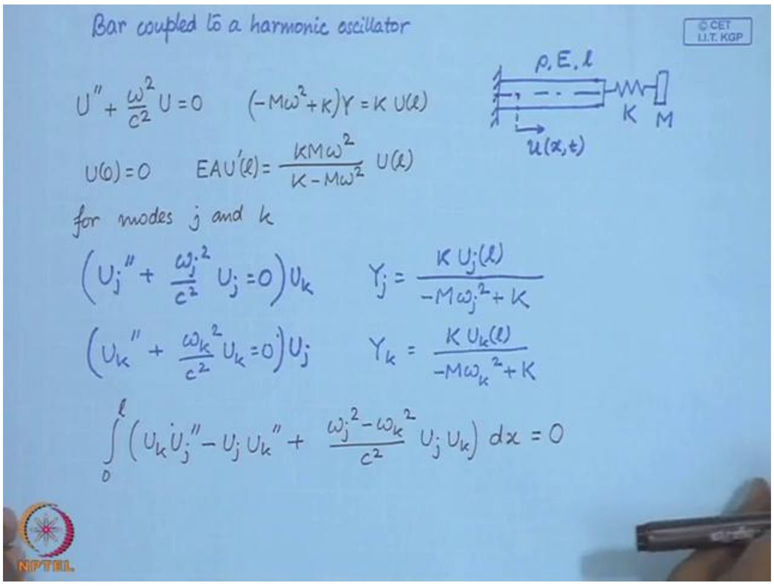

Now, let us consider the example of a uniform bar which is coupled to a harmonic oscillator, which we have discuss in our previous lecture. So, the Eigen value problem for this system was written as… as obtain in our previous lecture. So, once again for mode j and mode k, we can… So, this is the two differential equations and this is the boundary conditions for the bar. So, for the mode j and k, we can write…

Now, we will once again multiply this with the first equations for the bar with U_k and the second with U_j subtract and integrate over the domain of the bar, and upon rearrangement, you can very easily obtain… So, there are few standard steps. So, to obtain from here to here, that we can easily perform and come to this condition. Now, if you integrate by parts, let say this first term. So, integrate by parts this first term two times and used the boundary conditions for the boundary terms that you generate, then you can check that the expression reduces … So, we are integrating this term by parts two times. So at the end of the integration by parts, this term will be exactly same as this; so that two cancel off, but we will generate two boundary terms with single prime and that we have to replace that, we have to use this boundary condition. Once you use that you ultimately come to this expression. Now when j is not equal to k, and considering that omega j is not equal to omega k, there are no repeated Eigen frequencies, then this bracketed quantity must vanish; and this if you check, this can be written as…

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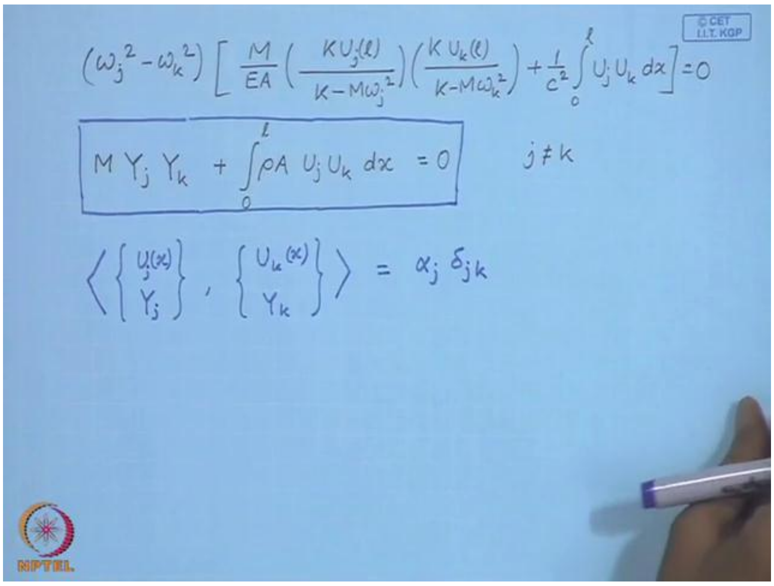

Here I have replaced these quantities by Y_j and Y_k which we have obtained this expression before. So, this is our inner product. Remember that in the case of a discrete, of a hybrid system in which we have a continuous and discrete system, we had this Eigen function vector which we have discussed in the previous lecture.

So, here we would say, we will write the inner product in this form, where the inner product is now defined in this form. So, you see this is not a trivial or a simple inner product that we obtain for other systems. So here we, so, this procedure you have to follow in order to determine this structure of the inner products, how this inner product is calculated based on the Eigen functions. So, let us summarize what we have studied today. So, we had…

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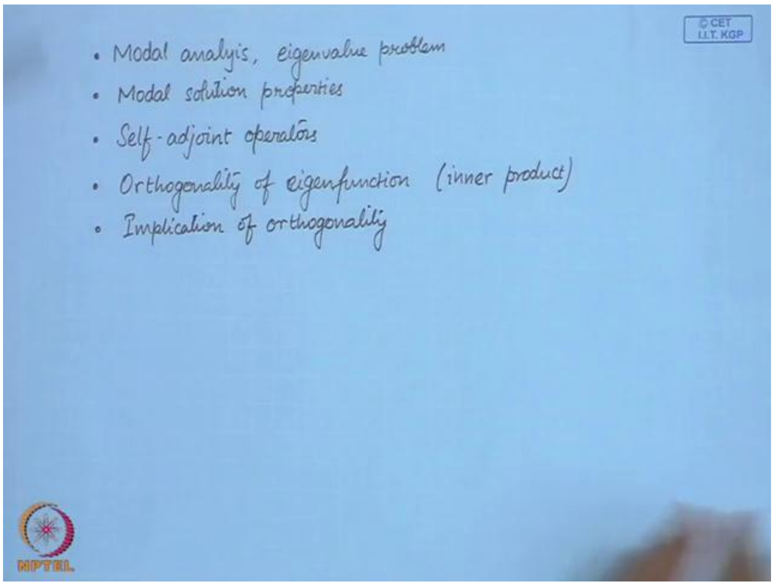

So, we have revisited this modal analysis problem and the Eigen value problem. Then we looked at the properties of the modal solution. Then we discussed about self-adjoint operators and the consequences of the stiffness operator being self-adjoint, these are real Eigen values and real Eigen functions. Then we have discussed about the orthogonality property of Eigen functions and we have determined the inner product. We have outlined steps to determine the inner product with respect to which this orthogonality property holds and we have looked at the implications of the orthogonality property of the Eigen functions. So, if the Eigen functions are orthogonal that implies that there is no exchange of energy, kinetic or potential, between the Eigen modes or the Eigen functions. So, with that we conclude this lecture.

Ketword: Eigen value problem, self-adjointness, orthogonality of Eigen functions, inner product.