Vibrations of Structures

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Lecture No # 07

Model Analysis - II

So, let us continue our discussions on modal analysis, that we are started in the previous lecture.

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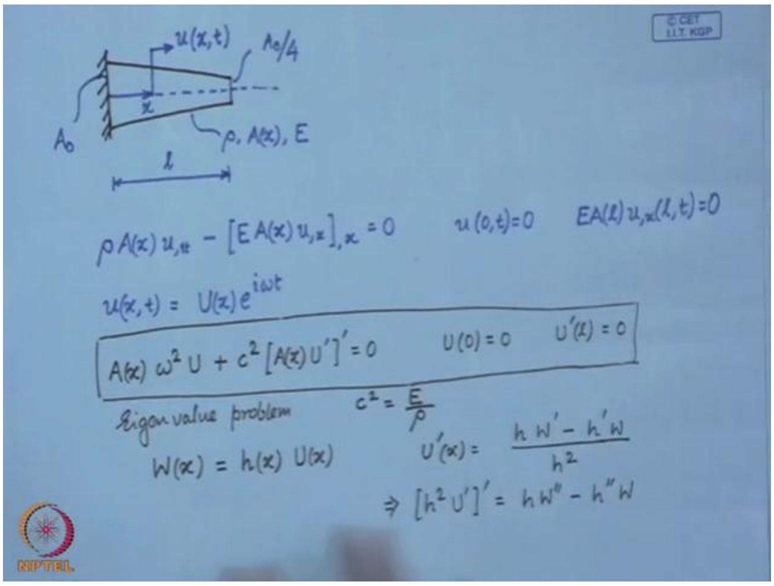

So, today we are going to take yet another example of system consisting of a bar of varying cross section. So, this the bar of varying cross section, consider that A_0 is the area of cross section at the fixed, and at the free end, it is something like $A_0/4$. The field variable is represented by U(x, t), which represents the actual displacement at any point x at any time t. We assume that rho is the density of the material of the bar; and of course, A as the function of x is the area of cross section and young’s modulus is E and the length of the bar is l. So, the equation of motion of actual vibrations of a bar of variable cross section may be written as… this. The boundary conditions for the system that we have considered are given by the displacement is zero at x equal to 0, and at the free end we have a dynamic boundary condition; this is the no force boundary condition, which can be written as… this. Now once again, we assume solution form as we have discussed in the previous lecture. So, we consider a solution of the special structure and if you introduce the solution in the equation of motion then after some simplification, we arrive at the differential equation. So, this is the an ordinary differential equation obtain by substituting this solution form in the equation of motion, and correspondingly the boundary conditions for this differential equation are given by… So this is the Eigen value problem for our system.

So, we have to solve this Eigen value problem in order to find out the Eigen, the circular Eigen frequencies or circular characteristic frequencies, omega and the corresponding modes of vibration, which are given by the Eigen functions of this the Eigen value problem. So, here of course c is… E the young's modulus divided by rho that is c square. Now, for a general variation of the area of cross section this may not be solvable analytically. So, what we are going to attempt here, today is to try to find a class of systems class of variation of cross sectional area for which this this problem, might be solvable analytically; so to see that how to find that the class. Let us make variable transformation let us consider a new variable W(x), which is expressed as some unknown function h of x in to our amplitude function U of x.

Now if you differentiate. So, U prime of x can be written as… So, this implies… Now, if you identify this h square, this quantity h square with the variation the area, then you can eliminate or replace this term with this expression; and if you make the substitution in in this Eigen value problem, the differential equation of the Eigen value problem, then you can where easily see that this will turn out to be… So if you substitute this U in terms of W in the differential equation and make some rearrangements then you can write the differential equation of the Eigen value problem in terms of in this form, and the corresponding… So, here this is the differential equations, in terms of the new variable W. Now for a special choice of this function h, this differential equations can be written in or can be expressed in a very familiar form or very simple form, if h double prime over h is the constant, let us say alpha; where alpha could be a positive or a negative constant. So, for such a class of systems our differential equation of the Eigen value problem can be rewritten as… Now the corresponding boundary conditions can be obtained similarly and which turn out to be… So, this is our new Eigen value problem in terms of the variable W.

Now the… So, we are looking at a class of systems for which this function h double prime over h, which is constant alpha which may positive or negative. So, this class of system is characterized by variation of h, which may be or which is hyperbolic for alpha greater than zero, it is harmonic for alpha less than zero, and its quadratic h is the quadratic function of x, if alpha is 0. So, let us consider certain particular case that I had shown in this figure, here the radius is reducing linearly, and the area goes from A_0 to $A_0/4$. So in that case, the variation of the cross sectional area may be expressed as A_0; remember that this is h square; so this is h square, then h…

So, h is linear in x and for this situation, if you substitute this expression here, then you will find that alpha for this special case is zero. So, if that is zero, then this simplifies further, the differential equation simplifies further. And the solution can be written as… So, the general solution of this differential equation is given here.

Now, when you use the boundary conditions, so W(0) is 0 would imply H is 0; and the second boundary condition at the free end gives us the characteristic equation; so which means here; so if H is 0, then W reduces to… Now if you substitute this expression here at x equal to l, so omega prime is given by minus of 1 over 2 l under root A_0 into minus of 1 over 2 l and so this expression becomes just minus 1. So, from here we obtain… Here of course 1 over l will remain, so what we obtain by applying these two boundary conditions is the characteristic equation of our system. So, this is the characteristic equation for fixed free bar with cross sectional area varying in this form. Now this is the transcendental equation which has to be solved numerically. Now a good way to visualize the solution of the transcendental equation is to make a graphical plot.

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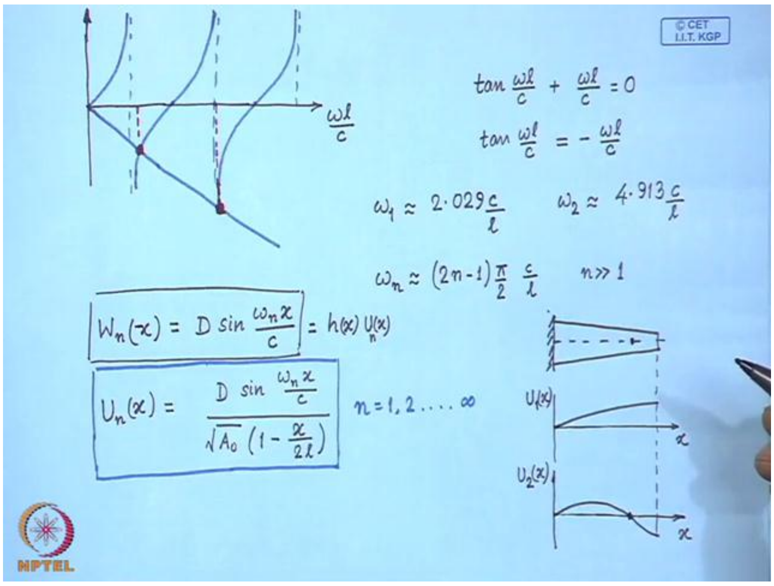

So… So, on the x axis I have omega l over c; so our characteristic equation is… So, I will plot, so I will rewrite this as… So, the tangent of omega l over c looks roughly like this, and minus of omega l over c is the 45 degree line, minus 45 degree line. So, these two functions are equal at these points, which represent the solution of the transcendental… the solutions of the transcendental equation; so these solutions are obtained, so the first this point, the first intersection gives us omega 1 this is 2.029 c over l; similarly omega 2 and so on. So, you can realize that there will be infinitely many intersections, which are discrete.

So, there will be countably infinite solutions of this transcendental equation, and for higher intersections, you have an approximate solution… for n, for high values of n. So, once we have these Eigen values or the circular natural frequencies of the system, we can find out the corresponding Eigen functions, which describe the modes of the vibration of the system. So, these are also now indexed and are given by… So, these are in terms of the new variable W; now we can go back to our original variable U and write the Eigen functions…

So, this from the structure of W that we had selected, so W was nothing but… So for our original problem, the Eigen functions turn out to be these; corresponding to the Eigen values or the circular natural frequencies given here. Now these Eigen functions may be drawn approximately… Here the amplitude function is or it represents the axial displacement of the bar. So, this is the first mode of vibration with the circular natural frequency given here. The second mode looks something like this. These things can be very easily plotted on the computer and visualized. So, here we find an antinodes, the node; this is the node at which the solution of the bars, this is the point is the second node, this point does not move in the axial, it always remains in the solutions. So, this is the node for the second mode; there is one node in the second mode; and no nodes in the fundamental or for the Eigen function U_1. So, this node have we discussed in the previous lecture is the point on the bar which remains stationary at all times.

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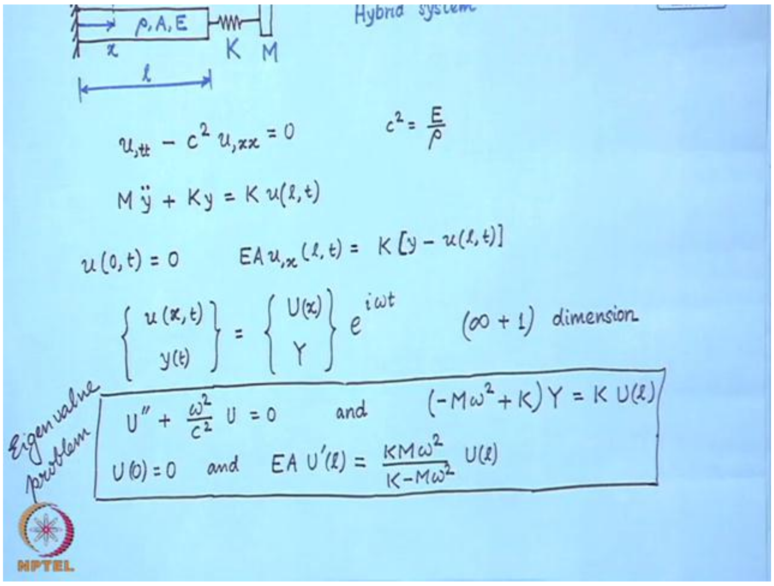

Next let us consider the system, a continuous system is interacting with discrete system. So, as an example, we consider a uniform bar fixed at one end and attached to a simple harmonic accelerator in this manner. So, here we have a discrete mass represented by capital M and a spring of stiffness capital K, which is attached to a bar of length l. These kind of systems are quite common when we have to put observers, for examples, on a vibrating continuity systems or a vibrating structure.

So, this example is one such system, in which we have a continuous system, which is bar in axial vibration, with discrete oscillator attached. So we will called them as hybrid system; because we have continuous as well as discrete system in this examples. So, the equation of motions are now the equations of motion, because we have a bar and a oscillator. So, we have two equations of motions. For the bar, the equation of motions can be written directly in this form, where c square is E over the rho; for the oscillator, the equation of motion can be easily written. So, y measures the displacement of mass M from its equilibrium position. So, as you can realize, we have two dependent variables; one is the field variable u, function of x and time, and the coordinate of the discrete mass M given by y.

Now the boundary conditions for this bar, then we easily written, so u at 0 for all times must be 0, is the fixed end. On the right end of the bar, we have this oscillator. So, we have the dynamic boundary condition; so this must be the forces exerted by the spring at this end. So, these are the two boundary conditions for the bar; now as I mentioned, this system now has a field variable for the bar and the coordinate of this discrete mass M. So you can represent these variables as a vector, and search for solutions of the form this as we have done before. Now it may be mentioned that, this vector that we are representing, it represents the configuration of a system in a dimension, which is infinity plus 1, infinity because of the bar as we already known and plus 1, because of this discrete system. So, the modal space is of dimension infinity plus 1. So, if you consider a solutions structure like this and substitute in the equation of motions, then you can immediately obtain… So, as with the structure of solution that we have been assuming, we are searching for the solutions of this form, we have synchronous motion of the bar and the discrete mass, for all points of the bar end will treat as mass.

Now, so these are the equations that we obtain after substituting the solutions in the differential equations; and the boundary conditions tell us U at 0, capital U the amplitude function at 0, must be 0, and if you substitute this structure here and simplify… we obtain the conditions at the right boundary in this form. So, here I have used this equation to simplify the structure of the boundary conditions at the right end of the bar. So, our Eigen value problem now is described completely by these equations and the boundary conditions. So, this is what we have to now solve.

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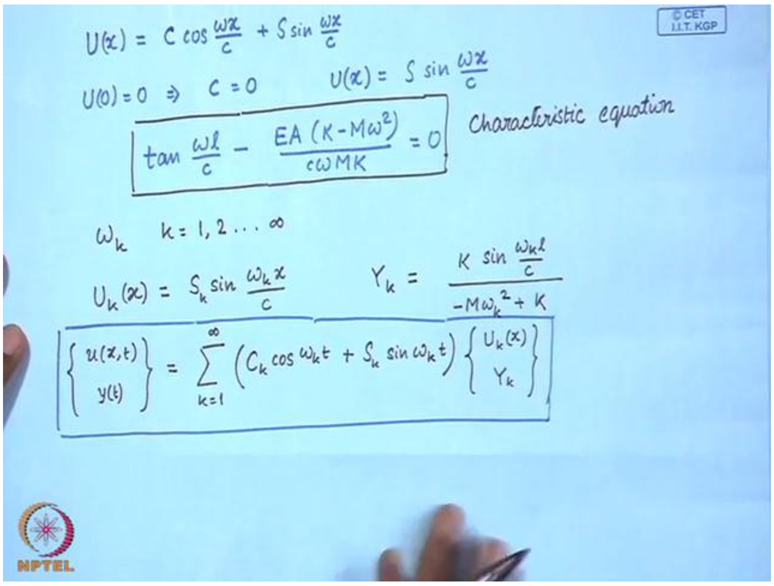

Now the solution of these differential equations may be represented in this form and if you use the boundary conditions, so the first boundary conditions for example, directly implies… C, capital C is equal to 0. So, if you… Now, this therefore, becomes simply S times sine of omega x over c. Now if you substitute this in the second boundary condition and simplify, then it can be checked that we obtain this condition, which is the characteristic equation of our system. Now again this is a transcendental equation, which has to be solved numerically for the Eigen values omega, so you will have discrete solution of this transcendental equations, but infinitely many solution exists.

So, you have countably infinitely many circular natural frequencies of this system obtained by solving this transcendental equation; and corresponding to this Eigen values or circular natural frequencies, you have the Eigen functions, the corresponding Eigen functions for the bar, and corresponding to these Eigen functions, you can now find the amplitude function or the amplitude of the discrete mass. So, this is the amplitude function for the bar, and this is the amplitude, the corresponding at the k^{th} mode for the discrete mass. Therefore, the general solution may be represented by superposing all these solutions of this form. So, this is the general solution for the system. So, you can see that the motion of the system is taking place in a modal space, which is of dimension infinity plus 1 and as we had visualized in case of strength, for example this is the motion of this bar with discrete oscillator is nothing but the motion of a point in this infinity plus one-dimensional model space or configuration space of the system.

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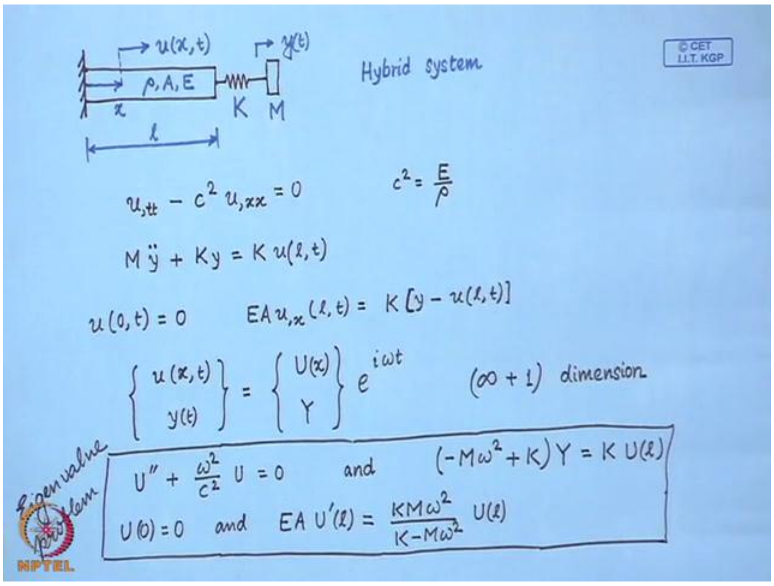

Now we can have two special cases, which follow immediately from the analysis that we are performed. One is, when the stiffness of the spring connecting the bar and the discrete mass tends to infinity, which means the discrete mass is rigidly attached to the bar. In this case, so it immediately follows from these characteristic equations by taking K tending to infinity, the characteristic equation simplifies to this.... and of course, so you can find out the circular natural frequencies from this characteristic equation and the corresponding Eigen functions now only of the bar is given by this… So, the discrete coordinate, the coordinate of this discrete mass, becomes same as u, so y is nothing but u at l. The second special case is when this mass becomes infinity, M goes to infinity. So in that case, the system simplifies to this. So, this is the end of this bar is connected to a spring, which is attached to a rigid wall. So in this case, the characteristic equation simplifies to this form, and the Eigen functions, corresponding Eigen functions are again of the same form. In this case of course, the motion of the mass varnishes, so y(t) becomes zero. Now, if you look back in this example, what we have discussed and if you see this Eigen value problem, you see this boundary condition here is dependent on the circular natural frequencies or the Eigen value itself. So, this system, in this system, the boundary condition is the function of the Eigen value.

So, to summarize, we have discussed today two further examples for which we have to perform the modal analysis by solving the Eigen value problems, and we have considered a bar with varying cross section; and we have found, we have solved a class of problems for which we have obtained the analytical solutions, which we will compare against solutions obtain by other methods later in this course. The other thing that we have discussed today is, continuous system interacting with the discrete vibrating system. So, we will continue this discussion further in the next lecture. So, this completes today's lecture.

Keywords: modal analysis, axial vibration modes, continuous system interacting with discrete elements.